

# The Trouble with Rational Expectations in Heterogeneous Agent Models: A Challenge for Macroeconomics

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# Heterogeneous agent models with aggregate risk

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- Classic papers by Krusell-Smith and Den Haan from late 90s...
- ... huge literature since then
- My argument: what we're doing “makes no sense” and the problem is rational expectations about equilibrium prices!
- Challenge = what should replace rational expectations?
  - spell out some criteria
  - discuss some promising directions

# The key problem in HA models with aggregate risk

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Key problem: **rational expectations** + general equilibrium

⇒ cross-sectional distribution enters household/firm decision problem

- true even though households/firms **do not really care about distribution** and only care about prices
- intuition: next slide
- Mean Field Games “Master equation” a.k.a. “Monster equation”

Recent work: **impressive advances** solving such models (DNNs etc)

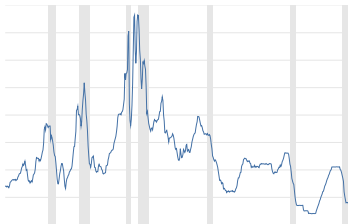
(e.g. Schaab, Bilal, Bhandari-Bourany-Evans-Golosov, Han-Yang-E, Gu-Lauriere-Merkel-Payne, Gopalakrishna-Gu-Payne, Huang, Proehl)

... but this still really **holds back HA literature**, e.g. ~~non-linearities, crises~~

Intuition: with RE households/firms **forecast prices by forecasting distributions**

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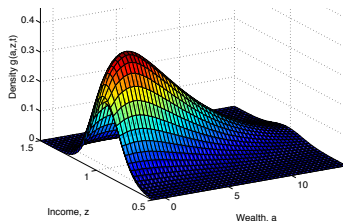
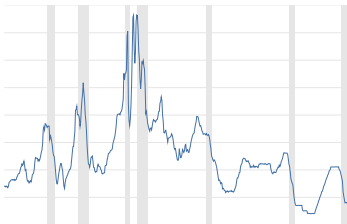
- Suppose I live in one of our models, only care about  $r$



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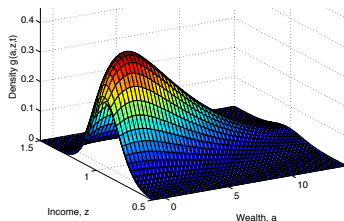
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  - I'd realize that in equilibrium  $r$  depends on distribution  $G$



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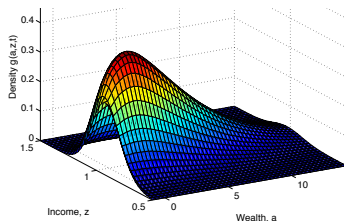
- Suppose I live in one of our models, only care about  $r$ 
  - I'd realize that in equilibrium  $r$  depends on distribution  $G$
  - RE  $\Rightarrow$  in order to forecast  $r$ , I'd **forecast entire distribution  $G$ !**



Intuition: with RE households/firms **forecast prices by forecasting distributions**

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- **Why make our lives so hard?** If best economists with best algorithms, computers cannot solve problem, stretch that real-world people do
- We're spending a lot of intellectual and computational horse power spent solving unrealistically complex problem
- $\Rightarrow$  go back to drawing board and replace RE about equilibrium prices

# Plan

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1. Back to the roots of RE: it was all about equilibrium prices
2. The trouble with rational expectations in heterogeneous agent models
3. What should replace RE?



# Back to the roots of RE

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- Back to John Muth = father of rational expectations (1961 paper)
- ... and to Lucas, Prescott, Sargent & co
- Better modeling expectations of equilibrium prices was **the** central goal in the development of RE

## 3. PRICE FLUCTUATIONS IN AN ISOLATED MARKET

We can best explain what the hypothesis is all about by starting the analysis in a rather simple setting: short-period price variations in an isolated market with a fixed production lag of a commodity which cannot be stored.<sup>5</sup> The market equations take the form

$$\begin{aligned} C_t &= -\beta p_t && \text{(Demand) ,} \\ (3.1) \quad P_t &= \gamma p_t^e + u_t , && \text{(Supply) ,} \\ P_t &= C_t && \text{(Market equilibrium) ,} \end{aligned}$$

where:  $P_t$  represents the number of units produced in a period lasting as long as the production lag,

$C_t$  is the amount consumed,

$p_t$  is the market price in the  $t$ th period,

$p_t^e$  is the market price expected to prevail during the  $t$ th period on the basis of information available through the  $(t-1)$ 'st period,

$u_t$  is an error term—representing, say, variations in yields due to weather.

*All the variables used are deviations from equilibrium values.*

## In Bob Lucas' words

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“[1960s-style macroeconometric models] implied behavior of actual **equilibrium prices** and incomes that bore no relation to, and were in general grossly inconsistent with, the price expectations that the theory imputed to individual agents.” (Lucas 1995, Nobel Lecture)

“One needs a principle to reconcile the **price** distributions implied by the **market equilibrium** with the distributions used by agents to form their own views of the future. John Muth noted that [...] these distributions could not differ in a systematic way. His term for this latter hypothesis was **rational expectations**.” (Lucas 1980, “Methods and Problems in Business Cycle Theory”)

# Lucas and Prescott (1971) “Investment under Uncertainty”

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- Paper that first spells out RE the way we now understand it
- Muth: only price means consistent. Lucas-Prescott: whole distributions.

Briefly, we shall be concerned with a competitive industry in which product demand shifts randomly each period, and where factor costs remain stable. In this context, we attempt to determine the competitive equilibrium time paths of capital stock, investment rates, output, and output price for the industry as a whole and for the component firms. From the viewpoint of firms in this industry, forecasting future demand means simply forecasting future output prices. The usual way to formulate this problem is to postulate some forecasting rule for firms, which in turn generates some pattern of investment behavior, which in turn, in conjunction with industry demand, generates an actual price series.

To avoid this difficulty, we shall, in this paper, go to the opposite extreme, assuming that the actual and anticipated prices have the same probability distribution, or that price expectations are rational.<sup>4</sup> Thus we surrender, in advance, any

$$(9) \quad v(k, u) = \sup_{x \geq 0} \left\{ s(k, u) - x + \beta \int v \left[ kh \left( \frac{x}{k} \right), z \right] p(dz, u) \right\}.$$

<sup>4</sup> This term is taken from Muth [15], who applied it to the case where the expected and actual price (both random variables) have a common mean value. Since Muth's discussion of this concept applies equally well to our assumption of a common distribution for these random variables, it seems natural to adopt the term here.

# Important goal of developing RE: operational macro theories

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Lucas and Prescott (1971) “Investment under Uncertainty”:

- “[By imposing RE], we obtain an operational investment theory linking...”

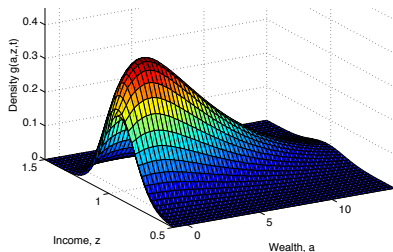
Lucas (1980) “Methods and Problems in Business Cycle Theory”:

- “Our task as I see it [...] is to write a FORTRAN program that will accept specific economic policy rules as ‘input’ and will generate as ‘output’ statistics describing the operating characteristics of time series we care about, which are predicted to result from these policies.”

The trouble with rational expectations  
in heterogeneous agent models

# Intuition: RE $\Rightarrow$ forecast prices by forecasting distributions

- Suppose I live in one of our models, only care about interest rate  $r$ 
  - I'd realize that in equilibrium  $r$  depends on distribution  $G$
  - RE  $\Rightarrow$  in order to forecast  $r$ , I'd **forecast entire distribution  $G$ !**



- Why make our lives so hard?
- Real-world people do not forecast prices by forecasting distributions
- **Next: explain this in a bit more detail using specific example**

## Example: forecasting equilibrium $w$ and $r$

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- Start with rep agent economy (RBC model) then add heterogeneity
- Preferences:

$$\mathbb{E}_0 \sum_{t=0}^T \beta^t U(c_t, n_t)$$

- Technology:

$$y_t = z_t F(k_t, \ell_t), \quad k_{t+1} = i_t + (1 - \delta)k_t$$

- Resource constraints:

$$c_t + i_t = y_t, \quad \ell_t = n_t, \quad \text{all } t$$

- Notes:

- time horizon  $T$  can be finite or  $\infty$ . Useful case: two periods  $t = 0, 1$
- aggregate productivity  $z_t$  is stochastic (Markov process)



# Representative agent case: competitive equilibrium

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Quantities and prices  $\{w_t, r_t\}$  such that

1. Households maximize

$$\max_{\{c_t, n_t, a_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^T \beta^t U(c_t, n_t) \quad \text{s.t.}$$

$$c_t + a_{t+1} = w_t n_t + (1 + r_t) a_t$$

2. Firms maximize

$$\max_{\{i_t, \ell_t, k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^T R_{0 \rightarrow t}^{-1} (z_t F(k_t, \ell_t) - w_t \ell_t - i_t) \quad \text{s.t.}$$

$$k_{t+1} = i_t + (1 - \delta) k_t \quad \text{with } R_{0 \rightarrow t} = \prod_{s=1}^t (1 + r_s)$$

3. Markets clear

$$k_t = a_t, \quad \ell_t = n_t, \quad \text{all } t$$

## Key difficulty: households, firms need to forecast $w$ and $r$

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Focus on **wages**  $\{w_t\}$  for now

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Focus on **wages**  $\{w_t\}$  for now

1. Households maximize

$$\Rightarrow \text{Labor supply} = n(w_t, a_t)$$

2. Firms maximize

$$\max_{\{i_t, \ell_t, k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^T R_{0 \rightarrow t}^{-1} (z_t F(k_t, \ell_t) - w_t \ell_t - i_t) \quad \text{s.t.}$$
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3. Markets clear

$$k_t = a_t, \quad \ell_t = n_t, \quad \text{all } t$$

$$\Rightarrow \text{Equilibrium wage} = w^*(k_t, z_t)$$

Note: RE equilibrium is pretty complicated even in this rep agent economy

# Solution methods for representative agent case

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Two (global) solution methods:

1. Tackle competitive equilibrium directly

- actually pretty hard even in rep agent case, e.g.  $(k, K)$  trick

2. Solve via planning problem

- no prices so completely sidesteps key difficulty
- frequent approach in literature (e.g. RBC model)

# Heterogeneous agents case: competitive equilibrium

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Quantities and prices  $\{w_t, r_t\}$  such that

1. Households: heterogeneous in  $(a_{it}, y_{it})$ ,  $y_{it} = \text{id. risk}$ , distribution  $G_t(a, y)$

$$\max_{\{c_{it}, n_{it}, a_{it+1}\}} \mathbb{E}_0 \sum_{t=0}^T \beta^t U(c_{it}, n_{it}) \quad \text{s.t.}$$

$$c_{it} + a_{it+1} = w_t y_{it} n_{it} + (1 + r_t) a_{it}$$

2. Firms (as before): rep firm optimally chooses  $\{\ell_t, k_t\}$  given  $\{w_t, r_t\}$

3. Markets clear

$$k_t = \int a dG_t(a, y), \quad \ell_t = \int n_t(a, y) dG_t(a, y), \quad \text{all } t$$

**Note:** households/firms do not care about dist'n  $G_t$ , only care about prices



Key difficulty: households, firms need to forecast  $w$  and  $r$

---

Focus on **wages**  $\{w_t\}$  for now

1. Households: heterogeneous in  $(a_{it}, y_{it})$ ,  $y_{it} = \text{id. risk}$ , **distribution**  $G_t(a, y)$

$\Rightarrow$  Household  $i$ 's labor supply  $= n(w_t, a_{it}, y_{it})$

2. Firms (as before)

$\Rightarrow$  Labor demand  $= \ell(w_t, k_t, z_t)$

3. Markets clear

$$k_t = \int a dG_t(a, y), \quad \ell_t = \int n_t(a, y) dG_t(a, y), \quad \text{all } t$$

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$$k_t = \int a dG_t(a, y), \quad \ell_t = \int n_t(a, y) dG_t(a, y), \quad \text{all } t$$

$\Rightarrow$  Equilibrium wage  $= w^*(G_t(a, y), z_t)$

Note: **equilibrium prices depend on entire cross-sectional distribution**  $G_t$ !

Generic feature of heterogeneous agent models:  $p_t = \mathcal{P}^*(G_t(x), z)$

## Rational expectations: forecast prices by forecasting distributions

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See this clearly in special case with two time periods  $t = 0, 1$

1. Households solve

$$V_0(a, y, G, z) = \max_{c, n, a'} U(c, n) + \beta \mathbb{E}[V_1(a', y', G', z') | y, G, z] \quad \text{s.t.}$$

$$c + a' = w_0^*(G, z) y n + (1 + r_0^*(G, z)) a$$

$$V_1(a', y', G', z') = \max_{c', n'} U(c', n') \quad \text{s.t.} \quad c' = w_1^*(G', z') y' n' + (1 + r_1^*(G', z')) a'$$

where  $G'$  = cross-sectional distribution at  $t = 1$ , satisfying  $G' = \mathcal{T}_{s_0} G$

2. Firm investment decision: similar problem featuring

- prices  $w_1^*(G', z')$  and  $r_1^*(G', z')$
- value function  $J_1(k, G', z')$

**MFG “Monster equation”, makes solution extremely hard**

**Why make our lives so hard?** Clearly people do not do this...

# Solution methods for heterogeneous agent case

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## 1. Linearization or MIT shocks: typical approach in particular in HANK literature

- certainty (equivalence) for prices so sidesteps key difficulty
- but not suitable for inflation debate, financial crises, asset pricing, ...

## 2. Krusell-Smith/DenHaan

- forecast prices by forecasting **moments** of distributions, e.g. mean:

$$\bar{a}_t = \int a dG_t(a, y) \quad \text{instead of} \quad G_t(a, y)$$

- **bounded rationality interpretation**
- **but do we think people do **that**?** I personally also don't
- exception: moment = price, more momentarily

(Gomes-Michaelides, Favilukis-Ludvigson-VanNieuwerburgh, Kaplan-Mitman-Violante, Lee-Wolpin, Lull, Storesletten-Telmer-Yaron ...)

## 3. Tackling full RE equilibrium: impressive advances in recent literature

(e.g. Schaab, Bilal, Bhandari-Bourany-Evans-Golosov, Han-Yang-E, Gu-Lauriere-Merkel-Payne, Gopalakrishna-Gu-Payne, Huang, Proehl)

- unrealistically complex: too much intellectual/computational horse power 18

# Taking stock and what next?

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Goal of Muth, Lucas & co when developing RE: **operational** macro theories

RE achieves exactly this goal in representative agent models

But **RE  $\Rightarrow$  het. agent models with aggregate risk “not operational”**

- attributes to people extreme ability to think through equilibrium
- means that people forecast prices by forecasting distributions
- thereby making solution extremely hard

**We should go back to drawing board:**

- replace RE about equilibrium prices in HA models
- existing attempts (e.g. KS 98) but we need to be more systematic
- Payoff: **kill two birds with one stone**
  1. make models operational (solution feasible)
  2. ... and more empirically realistic / more interesting

What should replace RE?

# What should replace RE?

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- I only know the problem, not the solution!
- But spell out some **criteria** that I find reasonable
- Common element: form expectations about prices directly
  - natural solution
  - different from RE
  - but how discipline prob. distributions to compute price expectations?
- Note: keep RE about non-equilibrium variables, e.g. idiosyncratic  $y_{it}$

## Natural solution: form expectations about prices directly

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In the 2-period example

$$V_0(a, y, G, z) = \max_{c, n, a'} U(c, n) + \beta \mathbb{E}[V_1(a', y', G', z') | y, G, z] \quad \text{s.t.}$$

$$c + a' = w_0^*(G, z)yn + (1 + r_0^*(G, z))a$$

$$V_1(a', y', G', z') = \max_{c', n'} U(c', n') \quad \text{s.t.} \quad c' = w_1^*(G', z')y'n' + (1 + r_1^*(G', z'))a'$$

where  $G'$  = cross-sectional distribution at  $t = 1$



## Natural solution: form expectations about prices directly

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In the 2-period example

$$V_0(a, y, w, r) = \max_{c, n, a'} U(c, n) + \beta \tilde{\mathbb{E}}[V_1(a', y', w', r') | \cdot] \quad \text{s.t.}$$

$$c + a' = w y n + (1 + r) a$$

$$V_1(a', y', w', r') = \max_{c', n'} U(c', n') \quad \text{s.t.} \quad c' = w' y' n' + (1 + r') a'$$

where **subjective** expectation  $\tilde{\mathbb{E}}$  computed using probability distribution

$$\mathbb{P}(w', r' | \cdot)$$

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where **subjective** expectation  $\tilde{\mathbb{E}}$  computed using probability distribution

$$\mathbb{P}(w', r' | \cdot)$$

Note: different from Krusell-Smith (forecast prices using moments)

- **exception: moment = price**

(Gomes-Michaelides, Favilukis-Ludvigson-VanNieuwerburgh, Kaplan-Mitman-Violante, Lee-Wolpin, Lull, Storesletten-Telmer-Yaron ...)

## Challenge: discipline price expectations $\mathbb{P}$

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Price expectations  $\tilde{\mathbb{E}}[V(x', p') | \cdot]$  computed using probability distribution

$$\mathbb{P}(p' | \cdot)$$

**Challenge:** navigating the “wilderness of non-rational expectations”

Sargent (2008) AEA Presidential Address:

- “There is such a bewildering variety of ways to imagine discrepancies between objective and subjective distributions”
- “There is an infinite number of ways to be wrong, but only one way to be correct”
- “Desire to retain discipline of RE”  $\Rightarrow$  “cautious modifications of RE”

# Three criteria for price expectations $\mathbb{P}$

Price expectations  $\tilde{\mathbb{E}}[V(x', p') | \cdot]$  computed using probability dist'n  $\mathbb{P}(p' | \cdot)$

## Three criteria for $\mathbb{P}$ :

1. Simplify solution of het. agent models (make them operational)
  - eliminates models that nest RE:  $\mathbb{P}^\theta$  with  $\mathbb{P}^{\theta=0} = \mathbb{P}^{RE}$  (e.g. diagnostic)
2. Consistency with empirical evidence
  - large literature, e.g. survey expectations  
(e.g. Manski, Armantier-et-al, Weber-DAcunto-Gorodnichenko-Coibion, DAcunto-Weber, Handbook of Economic Expectations)
  - large heterogeneity (disagreement)  $\neq$  RE “communism”  $\Rightarrow \mathbb{P}_i(p')$
3. Endogeneity of beliefs to model reality (Lucas critique)
  - 3a) Stationary environ's: subjective  $\mathbb{P}$  “not too far” from objective dist'n  
 $||\mathbb{P}(p|\cdot) - \mathbb{P}^{obj}(p|\cdot)|| < \varepsilon$
  - 3b) All environments:  $\mathbb{P}$  responds to model reality / policy (Lucas critique)

## Another requirement: compatibility with non-linear models and recursive methods

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- Existing non-RE literature: mostly linear or linearized models
- But goal here: develop HA models with aggregate non-linearities
- $\Rightarrow$  need compatibility with non-linear models
- Similarly, need compatibility with recursive methods (Bellman equations)
- Lie at heart of HA solution methods for two reasons:
  1. micro non-linearities like borrowing constraints
  2. state space  $\Rightarrow$  solve one optimization problem rather than millions

# Some promising directions

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- Temporary equilibrium and internal rationality (but only intermediate step)  
Hicks, Grandmont, Woodford, Piazzesi-Schneider, Adam-Marcet
- Survey expectations and hypothetical vignettes  
Manski, Malmendier-Nagel, Coibion-Gorodnichenko, Haaland-Roth-Wohlfart, ...
- Least-squares learning and restricted perceptions equilibrium  
Bray, Marcet-Sargent, Woodford, Evans-Honkapohja,...
- Reinforcement learning ( $\neq$  deep neural networks)  
“optimal control of incompletely-known Markov decision processes” (Sutton-Barto)
- Big world hypothesis  
“agent magnitudes smaller than environment, cannot perceive state of world and action values” (Javed-Sutton)
- Heuristics and simple models  
Tversky-Kahnemann, Molavi,...
- ...

All of these: interesting in RA models but potentially larger payoff in HA models

## 3. PRICE FLUCTUATIONS IN AN ISOLATED MARKET

We can best explain what the hypothesis is all about by starting the analysis in a rather simple setting: short-period price variations in an isolated market with a fixed production lag of a commodity which cannot be stored.<sup>5</sup> The market equations take the form

$$\begin{aligned} C_t &= -\beta p_t && \text{(Demand) ,} \\ (3.1) \quad P_t &= \gamma p_t^e + u_t , && \text{(Supply) ,} \\ P_t &= C_t && \text{(Market equilibrium) ,} \end{aligned}$$

where:  $P_t$  represents the number of units produced in a period lasting as long as the production lag,

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*All the variables used are deviations from equilibrium values.*

# Least-squares learning in Muth 1961

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- Producers estimate  $p_{t+1}^e = \hat{\mathbb{E}}_t[p_{t+1}]$  from past data on  $p_1, \dots, p_t$
- Simplest case: no serial correlation, no underlying state variable
- General case: serial correlation  $u_{t+1} = \rho u_t + \varepsilon_t$ , **perceived law of motion**

$$p_{t+1} = \theta_0 + \theta_1 u_t + \varepsilon_{t+1}, \quad u_t = \text{state variable}, \quad \varepsilon_t = \text{i.i.d}$$



# Least-squares learning in Muth 1961

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- Simplest case: no serial correlation, no underlying state variable
- Natural approach: estimate  $p_t^e$  as mean of past observations

$$\hat{p}_t^e = \frac{1}{t} \sum_{s=1}^t p_s$$

- Recursive implementation

$$\hat{p}_{t+1}^e = \hat{p}_t^e + \frac{1}{t} [p_t - \hat{p}_t^e]$$

- Converges to rational expectations as  $t \rightarrow \infty$  (Marcet-Sargent,...)

# Least-squares learning in Muth 1961

---

- Producers estimate  $p_{t+1}^e = \hat{\mathbb{E}}_t[p_{t+1}]$  from past data on  $p_1, \dots, p_t$
- Simplest case: no serial correlation, no underlying state variable
- General case: serial correlation  $u_{t+1} = \rho u_t + \varepsilon_t$ , **perceived law of motion**

$$p_{t+1} = \theta_0 + \theta_1 u_t + \varepsilon_{t+1}, \quad u_t = \text{state variable}, \quad \varepsilon_t = \text{i.i.d}$$

# Least-squares learning in Muth 1961

---

- Producers estimate  $p_{t+1}^e = \hat{\mathbb{E}}_t[p_{t+1}]$  from past data on  $p_1, \dots, p_t$
- Simplest case: no serial correlation, no underlying state variable
- General case: serial correlation  $u_{t+1} = \rho u_t + \varepsilon_t$ , **perceived law of motion**

$$p_{t+1} = x_{t+1}^T \theta + \varepsilon_{t+1}, \quad x_{t+1} = \begin{bmatrix} 1 \\ u_t \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}$$

# Least-squares learning in Muth 1961

---

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- General case: serial correlation  $u_{t+1} = \rho u_t + \varepsilon_t$ , **perceived law of motion**

$$p_{t+1} = x_{t+1}^\top \theta + \varepsilon_{t+1}, \quad x_{t+1} = \begin{bmatrix} 1 \\ u_t \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}$$

- Estimate  $\hat{\mathbb{E}}_t[p_{t+1}]$  by least squares (simplest case: intercept  $\theta_0$  only)

$$\hat{\theta}_t = \left[ \sum_{s=1}^t x_s x_s^\top \right]^{-1} \left[ \sum_{s=1}^t x_s p_s \right]$$

- **Recursive least squares:** compute  $\hat{\theta}_{t+1}$  from  $\hat{\theta}_t$  and new data  $(p_t, x_t)$

# Least-squares learning in Muth 1961

---

- General case: serial correlation  $u_{t+1} = \rho u_t + \varepsilon_t$ , **perceived law of motion**

$$p_{t+1} = x_{t+1}^\top \theta + \varepsilon_{t+1}, \quad x_{t+1} = \begin{bmatrix} 1 \\ u_t \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}$$

- **Recursive least squares**: compute  $\hat{\theta}_{t+1}$  from  $\hat{\theta}_t$  and new data  $(p_t, x_t)$
- Marcet-Sargent: special case of **stochastic approximation**

$$\text{NewEstimate} \leftarrow \text{OldEstimate} + \text{StepSize} [\text{Target} - \text{OldEstimate}]$$

- Stochastic approximation literature  $\Rightarrow$  **convergence** results
- Connection to Krusell-Smith/DenHaan
- Jacobson (2025) “Beliefs, Aggregate Risk, and the U.S. Housing Boom”

# Another stochastic approximation method: Reinforcement learning

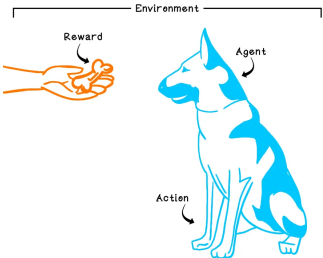
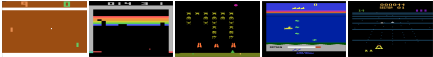


MAR 5, 2025 5:00 AM

## Playing Atari with Deep Reinforcement Learning

Volodymyr Mnih   Koray Kavukcuoglu   David Silver   Alex Graves   Ioannis Antonoglou  
Daan Wierstra   Martin Riedmiller  
DeepMind Technologies

{vlad,koray,david,alex.graves,ioannis,daan,martin.riedmiller} @ deepmind.com

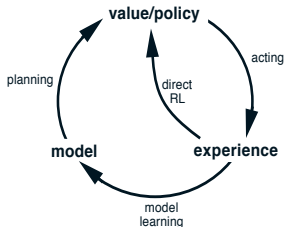


Pioneers of Reinforcement Learning Win the Turing Award

## RL: learning value functions of incompletely-known Markov decision processes

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- See Sutton-Barto textbook for great introduction
- Interesting feature: learning can be completely “model-free”



- Temporal difference learning (= stochastic approximation method)

$$V(S_t) \leftarrow V(S_t) + \alpha \underbrace{[R_t + \beta V(S_{t+1}) - V(S_t)]}_{\text{Target}}$$

- Temporal difference learning about equilibrium prices?
- Perhaps with behavioral twist so don't just converge to RE

Least-squares learning in HA models



# Least-squares learning in heterogeneous agent models

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- Spell out concrete example of approach satisfying criteria 1 and 3
- Material from “Mean Field Games without Rational Expectations” with Lenya Ryzhik
- Our version of Maggie Jacobson (2025) “Beliefs, Aggregate Risk, and the U.S. Housing Boom”

# A general heterogeneous agent model with aggregate risk

- Agents  $i$  with states  $X_{i,t} \in \mathbb{R}^n$ , “aggregate state”  $Z_t \in \mathbb{R}^k$
- State of the economy: **density**  $g_t(x)$  and  $Z_t$
- Each agent  $i$  chooses control  $\alpha_{i,t} \in A \subset \mathbb{R}^n$  to maximize

$$V_{i,0} = \max_{\alpha_i \in A} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t R(X_{i,t}, Z_t, \alpha_{i,t}, p_t) \right] \quad \text{subject to}$$

$$X_{i,t+1} \sim \mathcal{T}_x(\cdot | X_{i,t}, Z_t, \alpha_{i,t}, p_t), \quad Z_{t+1} \sim \mathcal{T}_z(\cdot | Z_t), \quad p_t = P^*(g_t, Z_t)$$

- $V$ : value function,  $R$ : reward function,  $0 < \beta \leq 1$ : discount factor
- $\mathcal{T}_x$ : transition prob. of  $X_{i,t}$ , independent across  $i$
- $\mathcal{T}_z$ : transition prob. of  $Z_t$
- Key: reward  $R$  depends on density  $g_t$  through **low-dim. price functional**

$$p_t = P^*(g_t, Z_t), \quad P^* : \mathcal{P}(\mathbb{R}^n) \times \mathbb{R}^k \rightarrow \mathbb{R}^\ell$$

- Given stochastic policy  $\alpha_{i,t} \sim \pi_t(\cdot | X_{i,t})$

$$g_{t+1}(x) = \sum_{\tilde{x}, \tilde{\alpha}} g_t(\tilde{x}) \pi_t(\tilde{\alpha} | \tilde{x}) \mathcal{T}_x(x | \tilde{x}, Z_t, \tilde{\alpha}, p_t) \quad \text{or} \quad g_{t+1} = \mathbf{A}_{\pi_t, Z_t}^\top g_t$$

# Least-squares learning about prices

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- Perceived law of motion (PLM) = Markov process

$$\hat{p}_{s+1} \sim \hat{\mathcal{T}}_p(\cdot | \hat{p}_s, Z_s, \theta), \quad s \geq t, \quad \hat{p}_t = p_t,$$

where  $\theta \in \mathbb{R}^d$  = parameter vector

- Example: VAR for prices (and possibly GDP,...)
- Learn  $\theta$  over time from past observations of  $p_t$
- Agents form estimate  $\hat{\theta}_t$  of  $\theta$ , update it using learning rule

$$\hat{\theta}_{t+1} = L(p_t, \hat{\theta}_t)$$

- Example: recursive least squares estimator

$$\hat{\theta}_t = \left[ \sum_{s=1}^t x_s x_s^T \right]^{-1} \left[ \sum_{s=1}^t x_s p_s \right]$$

# Least-squares learning in heterogeneous agent models

- Given current estimate  $\hat{\theta}_t = \theta$ : solve **finite-dimensional** Bellman equation

$$\hat{V}(x, z, p; \theta) = \max_{\alpha} R(x, z, \alpha, p) + \beta \mathbb{E}_{x', z', p'} [\hat{V}(x', z', p'; \theta) | x, z, p] \quad \text{s.t.}$$

$$x' \sim \mathcal{T}_x(\cdot | x, z, \alpha, p), \quad z' \sim \mathcal{T}_z(\cdot | z), \quad p' \sim \hat{\mathcal{T}}_p(\cdot | p, z, \theta).$$

- $\Rightarrow$  perceived policy  $\hat{\pi}_t(x, z, p, \theta)$ . Actual policy at time  $t$ :

$$\pi_t(x, Z_t) := \hat{\pi}_t(x, Z_t, p_t; \hat{\theta}_t)$$

- Then solve forward-in-time system:

$$g_{t+1} = \mathbf{A}_{\pi_t, Z_t}^T g_t$$

$$Z_{t+1} \sim \mathcal{T}_z(\cdot | Z_t)$$

$$\hat{\theta}_{t+1} = L(p_t, \hat{\theta}_t)$$

- Note: Least-squares learning  $\Rightarrow$  completely **sidestep** “**Monster equation**”
- Difference to KS with prices: “**in one sweep**” rather than inner+outer loop
- Satisfies criteria 1 & 3 (computation & Lucas critique) but not 2 (empirics)

## Summary: the trouble with RE in het. agent models

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- Too much intellectual and computational horsepower solving unrealistically complex problem  $\Rightarrow$  we should drop RE about equilibrium prices
- Open question: what should replace RE?
- ... how discipline  $\mathbb{P}(p' | \cdot)$  to compute price expectations  $\tilde{\mathbb{E}}[V(x', p') | \cdot]$ ?
- Spelled out three criteria for  $\mathbb{P}$
- Discussed some promising directions

Thanks!