Supplement to Lecture 1: Existence of Representative Firm and Representative Consumer

Macroeconomics, EC2B1
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Section 1 spells out a condition under which there is a representative firm, specifically that there are "perfect markets" that efficiently allocate resources across heterogeneous firms. Section 2 spells out a condition under which there is a representative consumer, namely so-called "Gorman aggregation." Jointly, these conditions would justify the existence of a representative agent. As discussed in the lecture notes, we do not think that these conditions hold in practice.

The content of this supplement is quite advanced (particularly that of section 2) and therefore not examinable.

1 Existence of Representative Firm with "Perfect Markets"

Notation:

- Individual firms $i = 1, ..., I$
- Output of firm i: $y_i, i = 1, ..., I$
- Labor of firm i: $n_i, i = 1, ..., I$
- Capital of firm i: $k_i, i = 1, ..., I$
- Firm i's production function: $y_i = f_i(k_i, n_i)$

Note: I spell out the case with two factors of production, labor and capital, but the analysis generalizes to an arbitrary number of factors.

Production with perfect factor markets. Firms maximize profits

$$\pi_i = \max_{k_i, n_i} f_i(k_i, n_i) - Rk_i - Wn_i$$

where R is the rental rate of capital and W is the wage. The first-order conditions are

$$\frac{\partial f_i(k_i, n_i)}{\partial k_i} = R \quad \text{and} \quad \frac{\partial f_i(k_i, n_i)}{\partial n_i} = W \quad \text{all } i = 1, ..., I \tag{1}$$

Also note that this implies that marginal products of capital and labor are equalized across firms, i.e. the allocation of production is efficient.

Social planner's problem and implied production function of "representative firm". Now consider instead the problem of a fictitious social planner who can allocate aggregate endowments of capital and labor K and N across individuals by choosing each individual firm's k_i and $n_i, i = 1, ..., I$ subject only to the resource constraints $\sum_{i=1}^I k_i \leq K$ and $\sum_{i=1}^I n_i \leq N$.

Further assume that the social planner maximizes total production $\sum_{i=1}^I y_i = \sum_{i=1}^I f_i(k_i, n_i)$. The planner's problem is:

$$F(K, N) = \max_{\{k_i, n_i\}_{i=1}^I} \sum_{i=1}^I f_i(k_i, n_i) \quad \text{s.t.} \quad \sum_{i=1}^I k_i \leq K \quad \text{and} \quad \sum_{i=1}^I n_i \leq N \tag{2}$$

Remarks:

1. Denoting the Lagrange multipliers on the planner's resource constraints by R and W (why will become clear momentarily) the first-order conditions are given by (1), i.e. the planner's allocation is the same as in a competitive equilibrium with perfect factor markets.

2. The Lagrange multipliers R and W in the planner's problem mathematically play the same role as the prices in the competitive equilibrium. The fact that I knew this would happen was the reason I denoted the Lagrange multipliers by R and W in the first place.

3. The "value function" $F(K, N)$, i.e. the value of the maximized objective in (2) depends only on the aggregate capital and labor K and N. Note that the function F can have very different properties from the individual production functions $f_i, i = 1, ..., I$.

Hence, in the above example, perfect factor markets imply the existence of a representative firm with production function $F(K, N)$.

2 Representative Consumer with Gorman Aggregation

The situation is more complicated for consumers. In particular the argument in Section 1 cannot be adapted to the case of consumers.[1] There are also various "impossibility results" in the literature that show that, for general individual utility functions and resulting demand curves, aggregation is "impossible" in the sense that aggregating "well behaved" individual consumer demands can yield very "badly behaved" aggregate demands (these results are jointly known as the Sonnenschein-Mantel-Debreu theorem). Here we therefore consider a different argument that delivers a representative consumer by making particular assumptions on preferences (utility functions). This construction is known as "Gorman aggregation."

Indirect utility function: recall from your microeconomics course the concept of an indirect utility function. There is a chance you may not have covered this yet but I am pretty sure you will do so soon. If you haven't, please consult any intermediate microeconomics textbook (e.g. Varian). I will explain everything for the case of two goods 1 and 2 denoting consumption by c_1 and c_2 and the prices by p_1 and p_2. Consider an individual consumer who has utility $u(c_1, c_2)$ and maximizes this utility subject to a budget constraint $p_1c_1 + p_2c_2 = m$ where m is the consumer's total income (as usual this gives rise to some demand functions $c_1(p_1, p_2, m)$ and $c_2(p_1, p_2, m)$). Then the *indirect utility function* is

$$v(p_1, p_2, m) = \max_{c_1, c_2} \; u(c_1, c_2) \quad \text{s.t.} \quad p_1c_1 + p_2c_2 = m.$$

The key trick in what follows will be to make assumptions not on consumers' utility functions u but instead on these *indirect* utility functions v. Once we have found the right conditions on *indirect* utility functions v that yield a representative consumer, we can then figure out (really "backward-engineer") the right assumptions on u that deliver these conditions on v.

For future reference also note that the individual demands of the two goods can be obtained from this indirect utility function via "Roy's identity":[2]

$$\frac{\partial v(p_1, p_2, m)/\partial p_1}{\partial v(p_1, p_2, m)/\partial m} = c_1(p_1, p_2, m) \quad \text{and} \quad \frac{\partial v(p_1, p_2, m)/\partial p_1}{\partial v(p_1, p_2, m)/\partial m} = c_2(p_1, p_2, m). \tag{3}$$

The aggregation problem: now consider an economy with many different consumers indexed by $i = 1, ..., I$. Denote consumption and income of each consumer i by c_{1i}, c_{2i} and m_i and her utility function $u_i(c_{1i}, c_{2i})$. Note the i subscript on u_i and m_i meaning that the utility

[1] There is one exception to this, namely if one were to assume that households have what is known as "quasi-linear utility." For example with two goods quasi-linear utility is $u(c_1, c_2) = v(c_1) + c_2$ (note the linearity in c_2, hence the name). For reasons you can find by googling or consulting standard textbooks, a quasi-linear utility function completely shuts down income effects of price changes. But I find quasi-linear utility to be such an unpleasant assumption that I will not consider it here.

[2] See `https://en.wikipedia.org/wiki/Roy's_identity` for a derivation, in particular my preferred proof using the envelope theorem `https://en.wikipedia.org/wiki/Roy's_identity#Alternative_proof_using_the_envelope_theorem`.

function (the functional form etc) may differ across consumers as well as the i subscript on incomes m_i which may also differ. Each consumer then also has resulting optimal demands

$$c_{1i}(p_1, p_2, m_i) \quad \text{and} \quad c_{2i}(p_1, p_2, m_i) \tag{4}$$

and an indirect utility function

$$v_i(p_1, p_2, m_i). \tag{5}$$

Again note the i subscript on c_{1i}, c_{2i}, v_i meaning that demands and indirect utility functions may differ across consumers (because the underlying utility functions u_i do). Also note that there is an i subscript on income m_i but not on prices p_1 and p_2 because all consumers face the same prices.

The key question is: can we aggregate all these different consumers into a well-defined representative consumer? That is, can we obtain some *aggregate* demands for the two goods that "look like" they come from a well-defined utility maximization problem (in which case we could call this utility function the utility function of the representative consumer).

To answer these questions, we define the economy's aggregate demands of the two goods obtained by summing everyone's individual demands (c_{1i}, c_{2i}) in (4)

$$C_1(p_1, p_2, \{m_i\}_{i=1}^I) = \sum_{i=1}^I c_{1i}(p_1, p_2, m_i) \quad \text{and} \quad C_2(p_1, p_2, \{m_i\}_{i=1}^I) = \sum_{i=1}^I c_{2i}(p_1, p_2, m_i) \tag{6}$$

as well as aggregate income

$$M = \sum_{i=1}^I m_i. \tag{7}$$

Note that, in general, the aggregate demands defined in (6) don't just depend on aggregate income M; instead they depend on income of each and every individual $m_1, m_2, ..., m_I$ – this is what is meant by the notation $\{m_i\}_{i=1}^I$ in (6). Put another way, aggregate demands depend on the entire *distribution* of income. This dependence on the distribution is precisely one the key challenges with aggregation.

With this notation in hand, let us return to our question whether we can aggregate all these different consumers into a well-defined representative consumer? The answer would be "yes" if summing everyone's individual demands (c_{1i}, c_{2i}) in (4) yields aggregate demands (C_1, C_2) in (6) that are the solution to a hypothetical utility maximization problem of the form

$$V(p_1, p_2, M) = \max_{C_1, C_2} U(C_1, C_2) \quad \text{s.t.} \quad p_1 C_1 + p_2 C_2 \leq M \tag{8}$$

for *some* utility function U and where it is aggregate income M that shows up on the right-hand side of the budget constraint. When this is possible, we say that there is a representative consumer with utility function U and indirect utility function V.

If this were possible, this would immediately take care of the difficulty that, in general, aggregate demands in (6) depend on the distribution of income $\{m_i\}_{i=1}^I$. This is because aggregate demands resulting from the utility maximization problem (8) are necessarily of the form

$$C_1(p_1, p_2, M) \quad \text{and} \quad C_2(p_1, p_2, M), \tag{9}$$

i.e. they depend only on aggregate income M and not the entire distribution of income (because only aggregate income M enters the maximization problem (8)). In fact note that, if aggregate demands are the solution to a hypothetical utility maximization problem of the form (8), this is quite a lot stronger than these aggregate demands only depending on aggregate income M. In particular, this means that the aggregate demands in (9) will be "well behaved" and satisfy all the standard axioms of consumer theory.

Finally note that the functional form of U does not necessarily need to be related in any way to those of the u_i's; U just needs to satisfy the standard properties of a utility function. Similarly, the indirect utility function V does not need to be related to the v_i's and also just needs to satisfy the standard properties of an *indirect* utility function.

Gorman aggregation: Now comes the key result which we will first state and then proof afterwards.

Result: Assume that each consumer has an *indirect* utility function of the form

$$v_i(p_1, p_2, m_i) = a_i(p_1, p_2) + b(p_1, p_2)m_i, \tag{10}$$

where $a_i(p_1, p_2)$ and $b(p_1, p_2)$ are some known functions of prices. Then there is a representative consumer with indirect utility function

$$V(p_1, p_2, M) = \sum_{i=1}^{I} a_i(p_1, p_2) + b(p_1, p_2)M. \tag{11}$$

Proof: applying Roy's identity (3) to the individual indirect utility functions (10), individual demands are given by

$$\begin{aligned} c_{1i}(p_1, p_2, m_i) &= \frac{\partial v_i(p_1, p_2, m_i)/\partial p_1}{\partial v_i(p_1, p_2, m_i)/\partial m_i} = \frac{\partial a_i(p_1, p_2)/\partial p_1}{b(p_1, p_2)} + \frac{\partial b(p_1, p_2)/\partial p_1}{b(p_1, p_2)} \times m_i \\ c_{2i}(p_1, p_2, m_i) &= \frac{\partial v_i(p_1, p_2, m_i)/\partial p_2}{\partial v_i(p_1, p_2, m_i)/\partial m_i} = \frac{\partial a_i(p_1, p_2)/\partial p_2}{b(p_1, p_2)} + \frac{\partial b(p_1, p_2)/\partial p_2}{b(p_1, p_2)} \times m_i \end{aligned} \tag{12}$$

Similarly, applying Roy's identity to the representative consumer's indirect utility function (11) yields

$$\begin{aligned} C_1(p_1,p_2,M) &= \frac{\partial V(p_1,p_2,M)/\partial p_1}{\partial V(p_1,p_2,M)/\partial M} = \frac{\partial\left(\sum_{i=1}^{I} a_i(p_1,p_2)\right)/\partial p_1}{b(p_1,p_2)} + \frac{\partial b(p_1,p_2)/\partial p_1}{b(p_1,p_2)} \times M \\ C_2(p_1,p_2,M) &= \frac{\partial V(p_1,p_2,M)/\partial p_2}{\partial V(p_1,p_2,M)/\partial M} = \frac{\partial\left(\sum_{i=1}^{I} a_i(p_1,p_2)\right)/\partial p_2}{b(p_1,p_2)} + \frac{\partial b(p_1,p_2)/\partial p_2}{b(p_1,p_2)} \times M \end{aligned} \tag{13}$$

Finally, we can see that summing the individual demands (12) (that we just derived from individuals' indirect utility functions) across individuals to obtain aggregate demands yields the aggregate demands in (13) (that we just derived from the representative consumer's indirect utility fuunction). This works because the individual demands in (12) are linear in individual income m_i (and with the same slope coefficient in front of m_i). This shows that if individual indirect utility functions v_i take the form (10), then the resulting aggregate demands indeed correspond to a well-defined indirect utility function of a representative consumer V defined in (11) and concludes the proof.□

Remarks:

1. The form of the indirect utility function (10) is known as "Gorman polar form" `https://en.wikipedia.org/wiki/Gorman_polar_form`.

2. A key implication of this Gorman polar form is that individual demands in (12) are linear in individual income m_i. This is indeed precisely the reason why the aggregation across individuals works out nicely and aggregate demands depend only on aggregate income M rather than the entire income distribution. The property that demands are linear income is known as "linear Engle curves." This property is clearly quite restrictive and it's easy to think of cases where this would not be satisfied.

3. As noted at the beginning of this section, one can figure out what types of individual utility functions u_i deliver indirect utility functions that take the Gorman polar form (10) and therefore yield a representative consumer via Gorman aggregation. The prime example are so-called "homothetic" utility functions which include many of the utility functions we typically use, e.g. Cobb-Douglas, Leontief, perfect substitutes and more general CES utility functions (see Lecture 3 of this course). See `https://en.wikipedia.org/wiki/Gorman_polar_form#Examples`. At the same time, as we just noted, the fact that individual demands are linear in income (linear Engle curves) is quite restrictive and you can therefore immediately see that Gorman aggregation works only for a pretty special class of preferences (though a very useful one).

4. Nothing is special to the case of two goods we covered here. Instead you can convince yourself that everything goes through with J goods in which case individuals maximize $u(c_1,...,c_J)$ subject to $\sum_{j=1}^{J} p_j c_j = m$ and indirect utility functions take the form $v(p_1,...,p_J,m)$.