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Convergence of approximation schemes for fully nonlinear second order equations

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Abstract

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We present a simple, purely analytic method for proving the convergence of a wide class of approximation schemes to the solution of fully non linear second-order elliptic or parabolic PDE. Roughly speaking, we prove that any monotone, stable and consistent scheme converges to the correct solution provided that there exists a comparison principle for the limiting equation. This method is based on the notion of viscosity solution of Crandall and Lions and it gives completely new results concerning the convergence of numerical schemes for stochastic differential games.

0. Introduction

In this paper we show the convergence of a wide class of approximation schemes to the solution of fully nonlinear second-order elliptic or parabolic, possibly degenerate, partial differential equations. Roughly speaking, we prove that any monotone, stable and consistent scheme converges (to the correct solution) provided that there exists a comparison principle for the limiting equation. We then give several examples of concrete schemes where the result applies, in order to emphasize both the simplicity and the efficiency of this approach. The formulation of the schemes follows along the lines of Souganidis [26] where the analogous problem was studied for first-order equations but not in the present generality. The convergence result is based upon exploring a basic idea of Barles and Perthame [2,3] regarding passage to the limits in fully nonlinear second-order elliptic PDE with only L^∞ estimates. This method relies on the notion of viscosity solutions, introduced by Crandall and Lions [8] for first-order problems (see also [7,20]) and extended to second-order equation by Lions [21]. Our approach is purely analytic and does not rely on any convexity or concavity assumptions; we are thus able to present completely new results concerning convergence of numerical schemes... etc. These results include as special cases most of the results of Bardi and Falcone [1], Capuzzo-Dolcetta and Falcone [5], Capuzzo-Dolcetta and Ishii [6], Falcone [9], Kushner [18,19] and Menaldi

[23,24], who deal with convex or concave problems. Our goal in this paper is to underline the general unifying principle behind the convergence and to explain the method. Keeping this in mind, we have chosen to present several relatively simple examples (some of which have already been proved by different methods) without trying to list any optimal conditions.

In order to give a flavor of the results of the paper, we next present a special case as an example. In particular, we consider the Cauchy problem

$$u_t + F(D^2u) = 0 \quad \text{in } \mathbb{R}^N \times (0, T), \tag{0.1a}$$

$$u = u_0 \quad \text{on } \mathbb{R}^N \times \{0\}. \tag{0.1b}$$

Here u and F are continuous functions of their arguments, D^2u denotes the second derivative matrix of u with respect to x, and F is assumed to be elliptic, i.e.

$$F(M) \leqslant F(N) \quad \text{if } M \geqslant N, \tag{0.2}$$

for all M, $N \in S^N$ (the space of $n \times n$ symmetric matrices). The relation $M \geqslant N$ should be understood as the usual partial ordering of symmetric matrices. It is well known (cf. [13–17]) that if F is uniformly continuous and $u_0 \in BUC(\mathbb{R}^N)$, then (0.1) has a unique solution in $BUC(\mathbb{R}^N \times [0, T])$, where $BUC(D)$ denotes the space of bounded uniformly continuous functions defined on D.

We now construct a general scheme that is supposed to approximate (0.1). To this end, for $\rho > 0$ let $S(\rho): B(\mathbb{R}^N) \to B(\mathbb{R}^N)$ [1] be such that

$$S(\rho)u \geqslant S(\rho)v \quad \text{if } u \geqslant v, \tag{0.3}$$

$$S(\rho)(u + k) = S(\rho)u + k \quad (k \in \mathbb{R}), \tag{0.4}$$

and

$$\frac{\phi - S(\rho)\phi}{\rho} \to F(D^2\phi) \quad \text{as } \rho \to 0 \text{ for all } \phi \in C^\infty. \tag{0.5}$$

Given such an S and a positive integer M we define $u_M : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ as follows:

$$u_M(\cdot, t) = \begin{cases} S\left(t - i\frac{T}{M}\right)u_M\left(\cdot, i\frac{T}{M}\right)(\cdot) & \text{if } t \in \left(i\frac{T}{M}, (i+1)\frac{T}{M}\right], \\ u_0(\cdot) & \text{if } t = 0. \end{cases}$$

Theorem. *Assume* (0.2)–(0.5) *and* $u_0 \in BUC(\mathbb{R}^N)$. *Then* $u_M \to u$ *locally uniformly on* $\mathbb{R}^N \times [0, T]$ *as* $M \to \infty$. □

We continue with two applications of this Theorem. One comes from the theory of stochastic differential games (cf. Fleming and Souganidis [10]), the other is related to the convergence of finite difference approximations to the solution of (0.1).

Example 1 (cf. [10]). Suppose that

$$F(M) = \max_{y \in Y} \min_{z \in Z} \{-\text{trace } \tfrac{1}{2}a(y, z)M\} \tag{0.7}$$

[1] $B(D)$ is the space of bounded functions defined on D. We use such a notation (instead of L^∞) to point out that no measure theory is involved in this framework.

where Y, Z are compact sets and $a=\sigma\sigma'$, and define

$$S(\rho)u(x)=\min_{y\in Y}\max_{z\in Z}Eu(x+\rho^{1/2}\sigma\eta)$$

where η is a mean zero random vector whose components are independent and take values ± 1. Here Eg denotes the expected value of g. The function u_M defined by (0.6) and the S above converges as $M\to\infty$ to the lower value of the stochastic differential game with dynamics

$$\mathrm{d}X_s=\sigma(Y_s,\, Z_s)\,\mathrm{d}W_s, \qquad X_t=x$$

and payoff $Eu_0(X_T)$.

Example 2. For the sake of simplicity we assume here that $N=1$ and define

$$S(\rho)u(x)=u(x)-\rho F\left(\frac{u\left(x+(\alpha\rho)^{1/2}\right)-2u(x)+u\left(x-(\alpha\rho)^{1/2}\right)}{\alpha\rho}\right).$$

The function u_M defined by (0.6) and the above S converges as $M\to\infty$ to the unique solution of (0.1) for α sufficiently large. Such results were known but only in the case the F is either convex or concave and uniformly elliptic, neither of which are necessary for our theory.

The paper is organized as follows: in Section 1 we discuss the equations and the notion of solution we are considering in the generality needed to cover all the examples later. Section 2 is devoted to the construction of the approximation schemes and the proof of the convergence. Section 3 is devoted to examples.

1. Discontinuous viscosity solutions of fully nonlinear second-order equations

In this section we consider fully nonlinear, second-order elliptic or parabolic equations and discuss the notion of weak solutions (*viscosity solutions*) to such equations. Continuous viscosity solutions in the context of second-order equations were first introduced by Lions [21]. (For an overview of the theory we refer to Lions and Souganidis [22]). The basic ideas explained below regarding discontinuous solutions are due mostly to Ishii [12] and Barles and Perthame [2,3].

The equations we are considering are of the form

$$F(\mathrm{D}^2u,\, \mathrm{D}u,\, u,\, x)=0 \quad \text{in } \overline{\Omega}. \tag{1.1}$$

Here Ω is an open subset of $\mathbb{R}^N$, $\overline{\Omega}$ is its closure, the functions $F\colon S^N\times\mathbb{R}^N\times\mathbb{R}\times\overline{\Omega}\to\mathbb{R}$ and $u:\overline{\Omega}\to\mathbb{R}$ are locally bounded (possibly discontinuous) and, finally, $\mathrm{D}u$ and D^2u stand for the gradient vector and second derivative matrix of u.

We will say that (1.1) is *elliptic* if for all $(p,\, u,\, x)\in\mathbb{R}^N\times\mathbb{R}\times\overline{\Omega}$

$$F(M,\, p,\, u,\, x)\leqslant F(N,\, p,\, u,\, x) \quad \text{for all } M,\, N\in S^N \text{ such that } M\geqslant N, \tag{1.2}$$

where $M\geqslant N$ is the usual partial ordering of S^N.

Next we recall the notions of the upper semi-continuous (usc in short) and the lower semi-continuous (lsc in short) envelopes of a function $z:C\to\mathbb{R}^N$, where C is a closed subset of $\mathbb{R}^N$. These are

$$z^*(x)=\limsup_{\substack{y\to x\\ y\in C}} z(y), \tag{1.3}$$

and

$$z_*(x) = \liminf_{\substack{y \to x \\ y \in C}} z(y), \tag{1.4}$$

respectively. In the sequel, we will consider the cases where $z \equiv u$ with $C \equiv \bar{\Omega}$ and $z \equiv F$ with $C \equiv S^N \times \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega}$.

Definition 1.1. A locally bounded function $u : \bar{\Omega} \to \mathbb{R}$ is a (*viscosity*) *subsolution* (respectively *supersolution*) of (1.1), if for all $\phi \in C^2(\bar{\Omega})$ and all $x \in \bar{\Omega}$ such that $u^* - \phi$ (respectively $u_* - \phi$) has a local maximum (respectively minimum) at x, we have

$$F_*(\mathrm{D}^2\phi(x), \mathrm{D}\phi(x), u^*(x), x) \leqslant 0 \tag{1.5}$$

(respectively

$$F^*(\mathrm{D}^2\phi(x), \mathrm{D}\phi(x), u_*(x), x) \geqslant 0). \tag{1.6}$$

The function u is said to be a (*viscosity*) *solution* of (1.1), if it is both sub- and supersolution of (1.1).

Viscosity solutions turn out to be unique under very general assumptions (cf. [7,8,13–17]) and stable under passage to limits (cf. [2,3,7,8,12]). Both these facts are used strongly in the next sections. Here we want to take some time to explain the, admittedly, strange setting of (1.1) in $\bar{\Omega}$ instead of Ω. The reason is, loosely speaking, that we want to write both the equation and the boundary conditions as one expression. Indeed, let us consider for example the Dirichlet problem

$$H(\mathrm{D}^2u, \mathrm{D}u, u, x) = 0 \quad \text{in } \Omega, \tag{1.7a}$$

$$u = \varphi \quad \text{on } \partial\Omega. \tag{1.7b}$$

We can write (1.7) as in (1.1), if we consider the boundary condition only as a discontinuity of the equation. To this end, if $F: S^N \times \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ is defined by

$$F(M, p, u, x) = \begin{cases} H(M, p, u, x) & \text{if } x \in \Omega, \\ u - \varphi & \text{if } x \in \partial\Omega, \end{cases}$$

then (1.7) becomes

$$F(\mathrm{D}^2u, \mathrm{D}u, u, x) = 0 \quad \text{on } \bar{\Omega}.$$

For a locally bounded function $u : \bar{\Omega} \to \mathbb{R}$ to be a viscosity solution of (1.7), in view of Definition 1.1, we need u to satisfy in the viscosity sense

$$H(\mathrm{D}^2u, \mathrm{D}u, u, x) = 0 \quad \text{in } \Omega, \tag{1.8}$$

and

$$\max(H(\mathrm{D}^2u, \mathrm{D}u, u, x), u - \varphi) \geqslant 0 \quad \text{on } \partial\Omega, \tag{1.9a}$$

$$\min(H(\mathrm{D}^2u, \mathrm{D}u, u, x), u - \varphi) \leqslant 0 \quad \text{on } \partial\Omega. \tag{1.9b}$$

It is well known (cf. Gilbarg and Trudinger [11]) that (1.7) does not have, in general, a solution which assumes the boundary conditions continuously. Therefore (1.8) and (1.9) appear to be its "natural" replacement. On the other hand, in some "regular" cases (for example, $H(D^2u, Du, u, x) = -\Delta u - f(x)$), the boundary condition $u = \varphi$ on $\partial\Omega$ can be recovered by using a suitable (or a suitable sequence of) test function ϕ in Definition 1.1 (for example $\pm\{|x-x_0|^2/\varepsilon + M_\varepsilon \log(1 + K_\varepsilon d(x, \partial\Omega))\}$ for suitable constants M_ε and K_ε if $x_0 \in \partial\Omega$). This reflects the impossibility of the inequalities $H(D^2u, Du, u, x) \leqslant 0$ and $H(D^2u, Du, u, x) \geqslant 0$ on $\partial\Omega$ in these cases. Of course, all these remarks are valid for any type of boundary conditions (Neumann or oblique derivatives, mixed, state-constraints, etc.).

2. The convergence result

We consider approximation schemes of the form

$$S(\rho, x, u^\rho(x), u^\rho) = 0 \quad \text{in } \overline{\Omega}. \tag{2.1}$$

Here $S : \mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R} \times B(\overline{\Omega}) \to \mathbb{R}$ is locally bounded, where $\mathbb{R}^+ \equiv [0, \infty)$. We prove that as long as these schemes are monotone, stable and consistent, they converge to the solution of (1.1), provided that the latter problem admits a comparison principle.

We next formulate the precise assumptions on S. (As mentioned in the Introduction the presentation here follows the one of [26], the results, however, extend those of [26]). The first assumption is the *monotonicity*, i.e.

$$S(\rho, x, t, u) \leqslant S(\rho, x, t, v) \quad \text{if } u \geqslant v \text{ for all } \rho \geqslant 0,\ x \in \overline{\Omega},\ t \in \mathbb{R} \text{ and } u, v \in B(\overline{\Omega}). \tag{2.2}$$

The *stability* of S reads as follows:

For all $\rho > 0$, there exists a solution $u^\rho \in B(\overline{\Omega})$ of (2.1), with a bound independent of ρ. (2.3)

The scheme defined by (2.1) also has to be *consistent*, i.e. for all $x \in \overline{\Omega}$ and $\phi \in C_b^\infty(\overline{\Omega})$

$$\limsup_{\substack{\rho \to 0 \\ y \to x \\ \xi \to 0}} \frac{S(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} \leqslant F^*(D^2\phi(x), D\phi(x), \phi(x), x), \tag{2.4a}$$

and

$$\liminf_{\substack{\rho \to 0 \\ y \to x \\ \xi \to 0}} \frac{S(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} \geqslant F_*(D^2\phi(x), D\phi(x), \phi(x), x). \tag{2.4b}$$

Finally, we assume that (1.1) has the following *strong uniqueness* property:

If $u \in B(\overline{\Omega})$ is an usc solution of (1.1) and $v \in B(\overline{\Omega})$ is a lsc solution of (1.1), then $u \leqslant v$ on $\overline{\Omega}$. (2.5)

Theorem 2.1. *Assume* (2.2), (2.3), (2.4) *and* (2.5). *Then, as* $\rho \to 0$, *the solution* u^ρ *of* (2.1) *converge locally uniformly to the unique continuous viscosity solution of* (1.1).

Remark 2.1. The monotonicity assumption (2.2) is the analogue to the ellipticity condition (1.2); it ensures some maximum principle type property for the scheme. The numerical schemes considered in optimal control or differential games are often based on the dynamic programming principle (cf. [1,5,6,9,18,19,23,24]). In this case, this assumption is automatically satisfied. Finally, it can be relaxed in several ways, since in fact the inequality in (2.2) needs only to hold within up to $o(\rho)$ terms.

Remark 2.2. In the framework presented in Section 1, (2.4) stands for the natural replacement of the usual consistency requirement like (0.5). Finally, let us mention that (2.5) is the only limiting step of our method.

Proof. Let $\bar{u}$, $\underline{u} \in B(\bar{\Omega})$ be defined by

$$\bar{u}(x) \equiv \limsup_{\substack{y \to x \\ \rho \to 0}} u^\rho(y) \quad \text{and} \quad \underline{u}(x) \equiv \liminf_{\substack{y \to x \\ \rho \to 0}} u^\rho(y). \tag{2.6}$$

We claim that $\bar{u}$ and $\underline{u}$ are respectively sub- and supersolutions of (1.1). Assume for the moment that this claim is true; then, since $\bar{u}$ is usc and $\underline{u}$ is lsc, (2.5) yields $\bar{u} \leqslant \underline{u}$ on $\bar{\Omega}$. But the opposite inequality is obvious by the very definition of $\bar{u}$ and $\underline{u}$, hence

$$u \equiv \bar{u} = \underline{u}$$

is the unique continuous solution of (1.1) (again by (2.5)). This fact together with (2.6) also imply the local uniform convergence of u^ρ to u.

Next we prove the above claim. Here we only consider the $\bar{u}$ case, since the argument for $\underline{u}$ is identical. To this end, let x_0 be a local maximum of $\bar{u} - \phi$ on $\bar{\Omega}$ for some $\phi \in C_b^\infty(\bar{\Omega})$. Without any loss of generality, we may assume that x_0 is a strict local maximum, that $\bar{u}(x_0) = \phi(x_0)$ and, finally, that $\phi \geqslant 2 \sup_\rho \| u^\rho \|_\infty$ outside the ball $B(x_0, r)$, where $r > 0$ is such that

$$\bar{u}(x) - \phi(x) \leqslant 0 = \bar{u}(x_0) - \phi(x_0) \quad \text{in } B(x_0, r).$$

Then there exist sequences $\rho_n \in \mathbb{R}^+$ and $y_n \in \bar{\Omega}$ such that as $n \to \infty$

$$\begin{aligned} &\rho_n \to 0, \qquad y_n \to x_0, \qquad u^{\rho_n}(y_n) \to \bar{u}(x_0), \quad \text{and} \\ &y_n \text{ is a global maximum point of } u^{\rho_n}(\cdot) - \phi(\cdot). \end{aligned} \tag{2.7}$$

Denoting by ξ_n the quantity $u^{\rho_n}(y_n) - \phi(y_n)$, we have $\xi_n \to 0$ and $u^{\rho_n}(x) \leqslant \phi(x) + \xi_n$ for all $x \in \bar{\Omega}$.

The definition of u^ρ, the monotonicity of S and (2.7) above yield

$$S(\rho_n, y_n, \phi(y_n) + \xi_n, \phi + \xi_n) \leqslant 0. \tag{2.8}$$

Taking limits in (2.8) and using the consistency of S ((2.4)) we get

$$\begin{aligned} 0 &\geqslant \liminf_n \frac{S(\rho_n, y_n, \phi(y_n) + \xi_n)}{\rho_n} \\ &\geqslant \liminf_{\substack{\rho \to 0 \\ y \to x_0 \\ \xi \to 0}} \frac{S(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} \\ &\geqslant F_*(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0), \end{aligned}$$

which is desired inequality, since $\bar{u}(x_0) = \phi(x_0)$. □

3. Examples

We present three examples of applications of the result of the preceding section. The first one is the convergence of the Trotter–Kato product formula. The second is motivated by the theory of stochastic differential games. Finally, the last example proves the convergence of numerical approximations to solution of (1.1). References to related results will be given, when each example is discussed. We refer, however, to Souganidis [26] for analogous results for the first-order case. Finally, we remark, that all examples below will be descripted for special cases of (1.1), but they can be extended, under suitable hypotheses, to very general equations.

Example 1 (*Trotter–Kato products*). We are interested in expressing the solution of the Cauchy problem

$$\begin{cases} u_t + (F_1 + F_2)(\mathrm{D}^2 u) = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \quad (3.1a) \\ u = u_0 & \text{on } \mathbb{R}^N \times \{0\}, \quad (3.1b) \end{cases}$$

in terms of the solution of Cauchy problems of the form

$$\begin{cases} v_{i,t} + F_i(\mathrm{D}^2 v_i) = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ v_i = v_{i,0} & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \quad (i = 1, 2). \tag{3.2}$$

Here $F_1, F_2 : S^N \to \mathbb{R}$ are uniformly continuous and satisfy (0.2). If $u_0, v_{1,0}, v_{2,0} \in BUC(\mathbb{R}^N)$, it follows ([13–17]) that (3.1) and (3.2) have unique solutions and satisfy (2.5). If we denote by $S_{F_1+F_2}$ and S_{F_i} the solution operators of (3.1) and (3.2) respectively, then (cf. [13–17])

$$S_{F_i}(\rho)u \geqslant S_{F_i}(\rho)v \quad \text{if } u \geqslant v \text{ and } \rho \geqslant 0 \quad (i = 1, 2), \tag{3.3}$$

and

$$S_{F_i}(\rho)(u + k) = S_{F_i}(\rho)u + k \quad \text{for all } k \in \mathbb{R} \quad (i = 1, 2). \tag{3.4}$$

We next construct the Trotter–Kato product which approximates $S_{F_1+F_2}(t)u_0$. To this end, let $P = \{0 = t_0 < \cdots < t_n = T\}$ be a partition of $[0, T]$ and define

$$u_P(x, t) = \begin{cases} \left[S_{F_2}(t - t_i)S_{F_1}(t - t_i)u_P(\cdot, t_i)\right](x) & \text{if } t \in (t_i, t_{i+1}], \\ u_0(x) & \text{if } t = 0. \end{cases}$$

Theorem 3.1. *As* $\| P \| \to 0$, $u_P(x, t) \to S_{F_1+F_2}(t)u_0(x)$ *locally uniformly on* $\mathbb{R}^N \times [0, T]$, *where* $\| P \| = \max_{0 \leqslant i \leqslant n-1} |t_{i+1} - t_i|$. □

Proof. For $v \in B(\mathbb{R}^N \times [0, T])$ define

$$S(\rho, x, t, v(x, t), v) = v(x, t) - \left[S_{F_2}(\rho)S_{F_1}(\rho)v(\cdot, t)\right](x).$$

In view of (3.3) and (3.4) and [13–17], S satisfies (2.2), (2.3) and (2.5). In order to apply Theorem 2.1, we need to check (2.4). This, however, follows from the definition of S and the fact that

$$\frac{\phi(x) - S_{F_2}(\rho)\phi(x)}{\rho} \xrightarrow[\rho \to 0]{} F_2(\mathrm{D}^2\phi(x))$$

and

$$\frac{\phi(x) - S_{F_1}(\rho)S_{F_2}(\rho)\phi(x)}{\rho} \xrightarrow[\rho \to 0]{} (F_1 + F_2)(D^2\phi(x))$$

locally uniformly in $\mathbb{R}^N$ for all $\phi \in C_b^\infty(\mathbb{R}^N)$. These two facts are not really immediate, but follow along the lines of the analogous results for first-order problems of [26], thus we omit their proof.

Theorem 2.1 now yields that u^ρ converge locally uniformly to the unique solution of the problem

$$\begin{aligned} &u_t + F(D^2u) = 0 && \text{in } \mathbb{R}^N \times (0, T], \\ &\max(\bar{u}_t + F(D^2\bar{u}), \bar{u} - u_0) \geqslant 0 && \text{on } \mathbb{R}^N \times \{0\}, \\ &\min(\underline{u}_t + F(D^2\underline{u}), \underline{u} - u_0) \leqslant 0 && \text{on } \mathbb{R}^N \times \{0\}, \end{aligned}$$

provided that the latter has a strong comparison principle, which is the case. We conclude by remarking that the two inequalities above yield $\bar{u}(\cdot, 0) \leqslant u_0$ in $\mathbb{R}^N$ and $\underline{u}(\cdot, 0) \geqslant u_0$ in $\mathbb{R}^N$ (cf. [4]), thus the result. □

We conclude the discussion about Trotter–Kato products by remarking that we can treat more general functions F_i $(i = 1, 2)$ as well as more general problems than the Cauchy problems.

Example 2 (*Stochastic Differential Games*). The relation between the theory of two players, zero sum, stochastic differential games and second-order nonlinear PDE's, as well as the existence of a value for the game, was developed in a paper by Fleming and Souganidis [10] ([10] studies only the finite horizon case, but the results extend easily to other situations). Here, we only review the PDE-aspects of the theory for infinite horizon game set in an open bounded set Ω. The game, which we do not describe here, has a lower and an upper value u^- and u^+ respectively, which are solutions of the equations

$$F^-(D^2u^+, D^2u^+, u^+, x) = 0 \quad \text{in } \Omega, \tag{3.7a}$$

$$u^+ = \varphi \quad \text{on } \partial\Omega, \tag{3.7b}$$

and

$$F^+(D^2u^-, Du^-, u^-, x) = 0 \quad \text{in } \Omega, \tag{3.8a}$$

$$u^- = \varphi \quad \text{on } \partial\Omega, \tag{3.8b}$$

where F^+ and F^- are defined by

$$F^+(M, p, t, x) \equiv \min_{\alpha \in A} \max_{\beta \in B} \left\{ -\mathrm{Tr}\left\{ \sigma_{\alpha\beta}(x)\sigma_{\alpha\beta}^{\mathrm{T}}(x)M \right\} - b_{\alpha\beta}(x)p + c_{\alpha\beta}(x)t - f_{\alpha\beta}(x) \right\}, \tag{3.9}$$

and

$$F^-(M, p, t, x) \equiv \max_{\beta \in B} \min_{\alpha \in A} \left\{ -\mathrm{Tr}\left\{ \sigma_{\alpha\beta}(x)\sigma_{\alpha\beta}^{\mathrm{T}}(x)M \right\} - b_{\alpha\beta}(x)p + c_{\alpha\beta}(x)t - f_{\alpha\beta}(x) \right\}. \tag{3.10}$$

Here A and B are given compact sets, $\sigma_{\alpha\beta} \in M(N, m)$ (the set of $N \times m$ matrices), $b_{\alpha\beta} \in \mathbb{R}^N$, $\sigma_{\alpha\beta}$, $b_{\alpha\beta}$, $C_{\alpha\beta}$ and $f_{\alpha\beta}$ are given uniformly bounded and Lipschitz continuous functions in x uniformly with respect to $(\alpha, \beta) \in A \times B$ and $\sigma_{\alpha\beta}^{\mathrm{T}}$ denotes the adjoint matrix of $\sigma_{\alpha\beta}$. Finally, $\varphi \in C(\partial\Omega)$ and there exists a constant $c_0 > 0$ such that $c_{\alpha\beta}(x) \geqslant c_0$ for all $(\alpha, \beta, x) \in A \times B \times \bar{\Omega}$.

If the Isaacs' condition holds, i.e. if

$$F^+(M, p, t, x) = F^-(M, p, t, x) \quad \text{for all } (M, p, t, x),$$

and if $u^{\pm}$ are continuous on $\partial\Omega$ with $u^{\pm} = \varphi$ on $\partial\Omega$, then the uniqueness results ([13–17]) regarding (3.7) and (3.8) yield

$$u^+ = u^- \quad \text{on } \bar{\Omega},$$

in which case we say that the game has a value. A condition which guarantees the continuity of $u^{\pm}$ on $\partial\Omega$ is

$$\min_{\alpha \in A} \max_{\beta \in B} \left| \sigma_{\alpha\beta}^{\mathrm{T}}(x) n(x) \right| \geqslant \gamma > 0 \quad (x \in \partial\Omega), \tag{3.11}$$

where n denotes the unit outward normal of $\partial\Omega$. Such a condition was already used by Oleinik [25] to treat degenerate elliptic *PDE*'s. We will explain later why such a condition yields the desired continuity.

Now we introduce our approximation scheme. We will concentrate on (3.7); everything that follows, however, can be routinely changed to apply to (3.8). We define

$$\begin{aligned} &S(\rho, x, u(x), u) \\ &\quad \equiv \inf_{\alpha \in A} \sup_{\beta \in B} \Big\{ \Big[E\Big\{ \big[u(x) - \rho f_{\alpha\beta}(x) - u(X_\rho)\, \mathrm{e}^{-c_{\alpha\beta}(x)\rho} \big] \chi_{\{X_\rho \in \Omega\}} \\ &\qquad + \mathrm{e}^{-c_{\alpha\beta}(x)\rho} \big[u(x) - \tilde{\varphi}(x) \big] \chi_{\{X_\rho \notin \Omega\}} \Big\} \Big\} \quad (x \in \bar{\Omega}), \end{aligned}$$

where $X_\rho \equiv x + b_{\alpha\beta}(x)\rho + \sigma_{\alpha\beta}(x) W_\rho$, W_ρ is a standard Brownian motion in $\mathbb{R}^N$, $\tilde{\varphi}$ is a continuous extension of φ to $\tilde{\Omega}$, χ_D denotes the indicator function of the set D and E stands for the expected value.

Our result is:

Theorem 3.2. *Let $b_{\alpha\beta}$, $c_{\alpha\beta}$, $f_{\alpha\beta}$ and φ be as above and assume* (3.11). *Then the problem*

$$S(\rho, x, u^\rho(x), u^\rho) = 0 \quad \text{in } \bar{\Omega}$$

has a unique solution u^ρ such that $u^\rho \underset{\rho \to 0}{\to} u$ locally uniformly, u being the unique solution of (3.7).

Proof. Equation (3.12) can be rewritten as

$$\begin{aligned} u^\rho(x) = \sup_{\alpha \in A} \inf_{\beta \in B} E\Big\{ &\big[\rho f_{\alpha\beta}(x) + u^\rho(X_\rho)\, \mathrm{e}^{c_{\alpha\beta}(x)\rho} \big] \chi_{\{X_\rho \in \Omega\}} \\ &+ \big[\big(\rho - \mathrm{e}^{-c_{\alpha\beta}(x)\rho}\big) u^\rho(x) + \mathrm{e}^{-c_{\alpha\beta}(x)\rho} \tilde{\varphi}(x) \big] \chi_{\{X_\rho \notin \Omega\}} \Big\}; \end{aligned}$$

hence

$$u^\rho = T_\rho u^\rho, \tag{3.13}$$

where, in view of (3.11), T_ρ is a strict contraction. By the classical fixed point theorem, (3.13) has a unique solution $u^\rho \in C(\bar{\Omega})$. Moreover, it is easy to check that

$$\| u^\rho \|_\infty \leqslant \max\Big\{ \| \tilde{\varphi} \|_\infty,\ c_0^{-1} \max_{\alpha,\beta} \| f_{\alpha\beta} \|_\infty \Big\}.$$

The monotonicity of S is obvious; the consistency follows from standard stochastic calculus arguments via Ito's formula. We only need to check (2.5). To this end, let u and v be respectively an usc subsolution and a lsc subsolution of (3.7) and recall that the Dirichlet condition on $\partial\Omega$ means

$$\min\big(F^+(\mathrm{D}^2 u, \mathrm{D}u, u, x),\ u-\varphi\big) \leqslant 0 \quad \text{on } \partial\Omega,$$

$$\max\big(F^+(\mathrm{D}^2 v, \mathrm{D}v, v, x),\ v-\varphi\big) \geqslant 0 \quad \text{on } \partial\Omega.$$

We claim that in fact we have $u \leqslant \varphi$ and $v \geqslant \varphi$ on $\partial\Omega$. If this is true, then (2.5) follows from standard uniqueness results ([13–17]). Here we only check that $u \leqslant \varphi$ on $\partial\Omega$. The other inequality follows in exactly the same way. Let $x_0 \in \partial\Omega$. Since $u \in B(\bar{\Omega})$, the function

$$u(x) - \frac{|x-x_0|^2}{\varepsilon^2} - \left[\frac{d(x)}{\varepsilon} - \frac{Kd^2(x)}{\varepsilon^2}\right] M$$

has a maximum point in $\{x \in \bar{\Omega} \mid d(x) \leqslant \varepsilon/2K\}$ for M large enough, where d denotes the distance from $\partial\Omega$. Moreover, $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. It then follows that, for K sufficiently large and ε small, $x_\varepsilon \in \partial\Omega$ and $u(x_\varepsilon) \leqslant \varphi(x_\varepsilon)$. Indeed, if not, then

$$F^+\big(\mathrm{D}^2\phi_\varepsilon(x_\varepsilon), \mathrm{D}\phi_\varepsilon(x_\varepsilon), u(x_\varepsilon), x_\varepsilon\big) \leqslant 0,$$

where $\phi_\varepsilon(x) = |x-x_0|^2/\varepsilon^2 + [d(x)/\varepsilon - Kd^2(x)/\varepsilon^2]M$. Using (3.9) we obtain

$$\inf_{\alpha\in A} \sup_{\beta\in B} \Big\{ -\operatorname{trace}\big(\sigma_{\alpha\beta}(x_\varepsilon)\sigma_{\alpha\beta}^{\mathrm{T}}(x_\varepsilon)\big) + MK\big|\sigma_{\alpha\beta}^{\mathrm{T}}(x_\varepsilon)n(x_\varepsilon)\big|^2 + \mathrm{o}(1)\big) \leqslant 0,$$

where $n(x_\varepsilon) = -\mathrm{D}d(x_\varepsilon)$. If K and M are large enough, the above inequality yields a contradiction for ε small, in view of (3.11) and boundedness of $\sigma_{\alpha\beta}$.

Finally, since x_ε is a maximum point of $u - \phi_\varepsilon$,

$$u(x_0) \leqslant u(x_\varepsilon) - \phi_\varepsilon(x_\varepsilon) \leqslant u(x_\varepsilon).$$

This inequality together with the uppersemicontinuity of u, yields $u(x_\varepsilon) \to u(x_0)$ as $\varepsilon \to 0$ and, therefore,

$$u(x_0) \leqslant \varphi(x_0). \qquad \square$$

Example 3 (*Numerical Approximations*). We are now interested in a numerical scheme approximating (0.1). We present two examples.

For the first example, in order to simplify the presentation, we consider the case when $N=1$ and u_0 is 1-periodic, which in turn yields that u is 1-periodic in x. To compute the solution, we consider a grid in space and time of mesh size Δx and Δt respectively, and denote by M and P

the integers such that $M\,\Delta x = 1$ and $P\,\Delta t = T$. Finally, we write u_j^n for the quantity $u(j\,\Delta x,\ n\,\Delta t)$. One of the simplest scheme approximating u is given by

$$u_j^{n+1} = u_j^n - \Delta t F\left(\frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{(\Delta x)^2}\right), \tag{3.14a}$$

$$u_j^0 = u_0(j\,\Delta x), \tag{3.14b}$$

for $0 \leqslant n < P$ and for $j \in Z$ with the convention that $u_{j+M}^n = u_j^n$. This implies in particular that the domain of (real) computation is bounded. Finally, let us denote by $u_{\Delta x, \Delta t}$ the function defined by

$$u_{\Delta x,\Delta t}(x,\ t) = u_j^n \quad \text{if } x \in \left[\left(j - \tfrac{1}{2}\right)\Delta x,\ \left(j + \tfrac{1}{2}\right)\Delta x\right) \text{ and } t \in \left[\left(n - \tfrac{1}{2}\right)\Delta t,\ \left(n + \tfrac{1}{2}\right)\Delta t\right),$$

where u_j^n is defined by (3.14).

Theorem 3.3. *Assume* (0.2), $u_0 \in C(\mathbb{R})$ *and* $(\Delta t/(\Delta x)^2)\,\|F'\|_\infty \leqslant \frac{1}{2}$. *Then*

$$u_{\Delta x,\Delta t} \to u \quad \textit{uniformly in } \mathbb{R} \times [0,\ T] \textit{ as } |\Delta x| + |\Delta t| \to 0. \qquad \square$$

We leave the proof to the reader since, in fact, all the assumptions of Theorem 2.1 are easily checkable, (2.5) being a consequence of the results of [13–17].

We now present one more example which is motivated from the theory of stochastic differential games. (See also Kushner [18,19] where the analogous problem is treated for convex F's via purely probabilistic methods). For simplicity we consider the problem

$$u_t + \max_\alpha \min_\beta \left\{ -a_{ij}^{\alpha,\beta}(x)\, u_{x_i x_j} - b_i^{\alpha,\beta}(x)\, u_{x_i} \right\} = 0 \quad \text{in } \mathbb{R}^N \times (0,\ T), \tag{3.15a}$$

$$u = u_0 \quad \text{on } \mathbb{R}^n \times \{0\}, \tag{3.15b}$$

where for all α, β the matrix $((a_{ij}^{\alpha,\beta}))$ is uniformly elliptic and the functions $a_{ij}^{\alpha,\beta}$ and $b_i^{\alpha,\beta}$ are bounded and uniformly Lipschitz continuous. Following [18,19], we approximate $u_{x_i}(x)$ either by $(u(x + e_i h) - u(x))/h$ if $b_i^{\alpha,\beta}(x) \geqslant 0$ or by $(u(x) - u(x - e_i h))/h$ if $b_i^{\alpha,\beta}(x) < 0$. In either case we write $D_{i,x}^h u$ for the difference quotient. As far as the second derivatives go, we approximate $u_{x_i x_i}(x)$ by

$$\frac{u(x + e_i h) + u(x - e_i h) - 2u(x)}{h^2},$$

and $u_{x_i x_j}(x)$ by either

$$\frac{2u(x) + u(x + e_i h + e_j h) + u(x - e_i h - e_j h)}{2h^2}$$
$$- \frac{u(x + e_i h) + u(x - e_i h) + u(x + e_j h) + u(x - e_j h)}{2h^2}$$

if $a_{ij}(x) \geqslant 0$ or by

$$\frac{-2u(x)+u(x+e_ih-e_jh)+u(x-e_ih+e_jh)}{2h^2} + \frac{u(x+e_ih)+u(x-e_ih)+u(x+e_jh)+u(x-e_jh)}{2h^2}$$

if $a_{ij}(x) < 0$. In all cases, we write $D^k_{ij,x}u$. The approximating operator is defined by

$$S(\rho)u(x) = u(x) - \rho \max_{\alpha} \min_{\beta} \left\{ -\alpha^{\alpha\beta}_{ij}(x) D^{\rho}_{ij,x}u - b^{\alpha,\beta}_{i}(x) D^{\rho}_{i,x}u \right\}.$$

Theorem 3.4. *Assume that*

$$a^{\alpha,\beta}_{ii}(x) - \sum_{j \neq i} \left| a^{\alpha,\beta}_{ij}(x) \right| \geqslant 0 \quad \text{for all } i,\ x,\ \alpha,\ \beta.$$

The scheme defined by S converges locally uniformly to the unique solution of (3.15). □

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