

Consumption, Saving and Wealth Distribution with Robustness Concerns

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1 Model Description

1.1 Consumption-Saving Problem without Robustness Concerns

Recall the standard consumption-saving problem from Achdou, Han, Lasry, Lions, and Moll (2017):

$$\begin{aligned} v(a, z) &= \max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.} \\ da_t &= (z_t + ra_t - c_t) dt \\ dz_t &= \mu(z_t) dt + \sigma(z_t) dW_t \\ a_t &\geq \underline{a} \\ (a_0, z_0) &= (a, z). \end{aligned} \tag{1}$$

We will use this problem as a benchmark and introduce robustness concerns.

1.2 Consumption-Saving Problem with Robustness Concerns

Problem (1) assumes that every individual knows perfectly the dynamics of her consumption dynamics. We relax this assumption by allowing the individual to doubt the specification of the income process. Hansen and Sargent's framework allows us to capture the fear of model misspecification (Hansen, Sargent, Turmuhambetova, and Williams, 2006; Hansen and Sargent, 2008). The trick is to imagine that the individual is immersed in a zero-sum game with a fictitious player who systematically tries to harm her by distorting the probability distribution underlying the income process.

To be clear, let q^0 be a probability measure defined by the Brownian motion process in the reference model describing income dynamic and q an alternative model. Let $q \in Q$ a convex set. The distance between the two models is measured by the expected log likelihood ratio also called *relative entropy*:

$$\mathcal{R}(q) = \rho \int_0^\infty e^{-\rho t} \left[\int \log \left(\frac{dq_t}{dq_t^0} \right) dq_t \right] dt$$

¹The views expressed herein are those of the authors and not necessarily those of the Bank of Canada.

The robust control counterpart of the previous problem is then written as:

$$\begin{aligned}
v(a, z) &= \max_{\{c_t\}_{t \geq 0}} \min_q \int_0^\infty e^{-\rho t} \left[\int u(c_t) dq_t \right] dt + \theta \mathcal{R}(q) \quad \text{s.t.} \\
da_t &= (z_t + ra_t - c_t) dt \\
dz_t &= \mu(z_t) dt + \sigma(z_t) dW_t \\
a_t &\geq \underline{a} \\
(a_0, z_0) &= (a, z).
\end{aligned} \tag{2}$$

Using Girsanov's Theorem and letting h be a progressively measurable process, the relative entropy becomes:

$$\mathcal{R}(q) = \frac{1}{2} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} |h_t|^2 dt \right]$$

The same theorem allows us to view the alternative model q as induced by the following Brownian motion with drift:

$$\tilde{W}_t = W_t - \int_0^t h_s ds.$$

The individual's problem is then transformed as:

$$\begin{aligned}
v(a, z) &= \max_{\{c_t\}_{t \geq 0}} \min_{\{h_t\}_{t \geq 0}} \mathbb{E}_0 \left(\int_0^\infty e^{-\rho t} \left[u(c_t) + \frac{\theta}{2} |h_t|^2 \right] dt \right) \quad \text{s.t.} \\
da_t &= (z_t + ra_t - c_t) dt \\
dz_t &= [\mu(z_t) + \sigma(z_t)h] dt + \sigma(z_t) d\tilde{W}_t \\
a_t &\geq \underline{a} \\
(a_0, z_0) &= (a, z).
\end{aligned} \tag{3}$$

The key observation is that h appears as a control variable, just like consumption c . The Hamiltonian-Jacobi-Bellman equation for this problem is

$$\rho v(a, z) = \max_c \min_{h \in H} u(c) + \frac{\theta}{2} h^2 + (z + ra - c) \partial_a v(a, z) + (\mu(z) + \sigma(z)h) \partial_z v(a, z) + \frac{1}{2} \sigma^2(z) \partial_{zz} v(a, z) \tag{4}$$

Again, note that h appears as a control variable, just like consumption c . So we can treat them exactly symmetrically. The first-order conditions for c and h are

$$\begin{aligned}
u'(c) &= \partial_a v(a, z) \\
\theta h &= -\sigma(z) \partial_z v(a, z)
\end{aligned}$$

The solutions $c(a, z)$ and $h(a, z)$ can then be plugged into (4) so as to obtain an HJB equation without max and min operators (we do not state it here because it is not particularly instruc-

tive). An interesting feature of this problem is that, even though the true drift of the income process $\mu(z)$ is independent of wealth, the perceived drift $\mu(z) + \sigma(z)h(a, z)$ actually depends on wealth. So there is an interesting feedback from the individual's wealth to her (perceived) income dynamics.

Having solved, the HJB equation, we can also solve for the stationary joint distribution of income and wealth $g(a, z)$. This joint distribution satisfies the following Kolmogorov Forward equation:

$$0 = -\partial_a(s(a, z)g(a, z)) - \partial_z(\mu(z)g(a, z)) + \frac{1}{2}\partial_{zz}(\sigma^2(z)g(a, z)) \quad (5)$$

where $s(a, z) = z + ra - c(a, z)$ is the optimal saving policy function corresponding to the HJB equation (4). Note that the equation for the distribution features the true drift of the income process $\mu(z)$ rather than the perceived drift $\mu(z) + \sigma(z)h(a, z)$.

2 Algorithm

The algorithm is essentially the same as explained in Section XXX of http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf and implemented in http://www.princeton.edu/~moll/HACTproject/HJB_diffusion_implicit.m. The only difference is that there is an additional control variable h . But conceptually this generates no additional complications and h is treated exactly symmetrically to consumption c .

3 Results

Figure 1 plots the consumption and saving policy functions under the assumption that log income follows an Ornstein-Uhlenbeck process

$$d \log z_t = -\nu \log z_t + \bar{\sigma} dW_t$$

which implies, from Ito's Lemma, that the drift and diffusion coefficient of the income process are

$$\mu(z) = \left(-\nu \log(z) + \frac{\sigma^2}{2} \right) z, \quad \sigma(z) = \bar{\sigma} z$$

Panels (a) and (c) of Figure 1 plot the consumption and saving policy in the benchmark model *without* robustness concerns. Panels (b) and (d) plot the same policy functions in the model *with* robustness concerns. It can be seen that robustness concerns lead to higher precautionary saving (in the face of Knightian uncertainty).

Panels (e) and (f) plot the resulting stationary wealth distributions for the two models. As expected, robustness concerns lead to a distribution that places more mass at high wealth levels (i.e. there are more rich people).

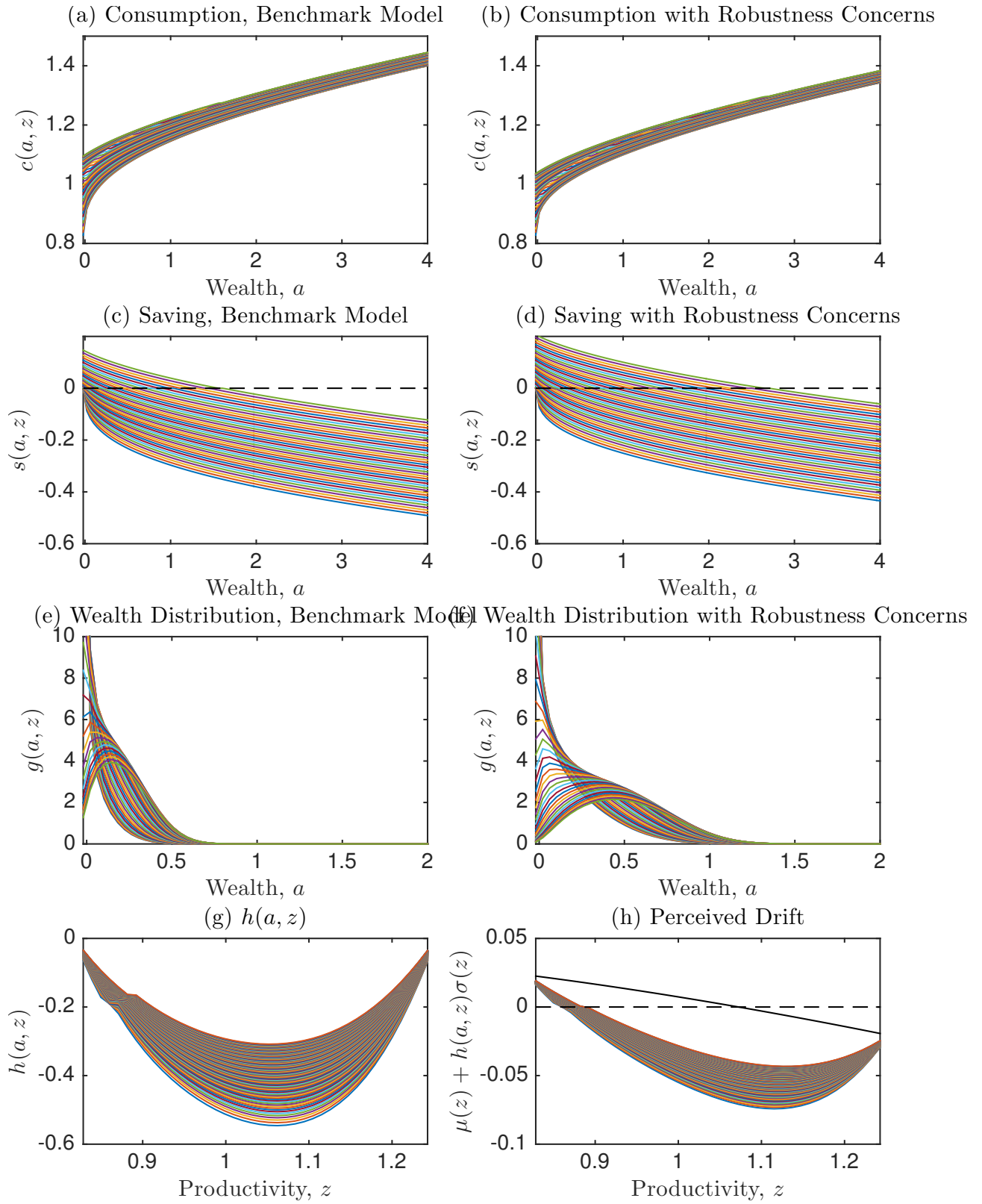


Figure 1: Policy Functions in Model with Robustness Concerns

Panels (g) and (h) examine the sources of this difference in saving behavior. Panel (e) plots the $h(a, z)$ that minimizes the HJB equation. Panel (f) plots the resulting *perceived* drift of income $\mu(z) + \sigma(z)h(a, z)$. Individuals are concerned that the drift of the income process is considerably more negative than it actually is.

References

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