

Exercising an Option: Numerical Solution

1 Problem

$$v(x) = \max_{\tau} \mathbb{E}_0 \int_0^{\tau} e^{-\rho t} u(x_t) dt + e^{-\rho \tau} S(x_{\tau}), \quad dx_t = \mu(x_t) dt + \sigma(x_t) dW_t$$

This is the simplest problem from Stokey's book. One interpretation is that v is the value of a plant. u are profits and a is demand which evolves exogenously. One can close the value of the plant and sell it at a scrap value $S(x)$. The problem is to choose an optimal shut-down time (stopping time) τ . To make sure that the plant shuts down eventually assume $\mu(x) < 0$ for all x .

1.1 HJB Equation

For x such that the plant does not exit, the following HJB equation holds

$$\rho v(x) = u(x) + \mu(x)v'(x) + \frac{\sigma^2(x)}{2}v''(x) \quad (1)$$

In one-dimensional problems like the one described here, there will typically be an exit threshold \underline{x} such that the firm exits for $x \leq \underline{x}$. Hence for x above the exit threshold we have $v(x) \geq S(x)$ and for x below the exit threshold, we have $v(x) = S(x)$. Traditional methods impose “value matching” and “smooth pasting” conditions: $v(\underline{x}) = S(\underline{x})$ and $v'(\underline{x}) = S'(\underline{x})$. We here pursue a different strategy.

To this end let's denote the set of x for which there is no exit by X . Then

$$\begin{aligned} x \in X : \quad & v(x) \geq S(x), \quad \rho v(x) = u(x) + \mu(x)v'(x) + \frac{\sigma^2(x)}{2}v''(x) \\ x \notin X : \quad & v(x) = S(x), \quad \rho v(x) \geq u(x) + \mu(x)v'(x) + \frac{\sigma^2(x)}{2}v''(x) \end{aligned}$$

This can also be written compactly as follows¹

$$\min \left\{ \rho v(x) - u(x) - \mu(x)v'(x) - \frac{\sigma^2(x)}{2}v''(x), v(x) - S(x) \right\} = 0 \quad (2)$$

In mathematics, (2) is called a “HJB variational inequality.” See e.g. Barles, Daher and

¹To see this take any two functions $f(x)$ and $g(x)$ and consider the following statement: for all x either $f(x) \geq 0, g(x) = 0$ or $f(x) = 0, g(x) \geq 0$. This statement can be written compactly as $\min\{f(x), g(x)\} = 0$ for all x .

Romano (1995) and Tourin (2013).

An alternative, and perhaps economically more intuitive way of writing this equation is

$$\rho v(x) = \max \left\{ u(x) + \mu(x)v'(x) + \frac{\sigma^2(x)}{2}v''(x), \rho S(x) \right\} \quad (3)$$

However, the HJBVI formulation (2) will be useful in the numerical solution below.

Note that rather than *imposing* the smooth pasting condition as is usually done in economics, this is now a result. That is, one can prove that the HJBVI (2) implies the smooth pasting condition $v'(\underline{x}) = S'(\underline{x})$. See e.g. Oksendal's book <http://th.if.uj.edu.pl/~gudowska/dydaktyka/Oksendal.pdf> who calls “smooth pasting” the “high contact (or smooth fit) principle.”

2 Finite Difference Method

2.1 Solving as Linear Complementarity Problem (LCP)

This is inspired by Huang and Pang (1998) who realize the connection between the HJB for American options and LCP.

Without the exit option, we would have (1) which is discretized as

$$\rho v_i^n = u_i + \mu_i(v_i^n)' + \frac{\sigma_i^2}{2}(v_i^n)''$$

or in matrix form

$$\rho v = u + \mathbf{A}v$$

Instead, now v solves the variational inequality (2). The discretized analogue is:

$$\min\{\rho v - u - \mathbf{A}v, v - S\} = 0$$

Equivalently

$$\begin{aligned} (v - S)'(\rho v - u - \mathbf{A}v) &= 0 \\ v &\geq S \\ \rho v - u - \mathbf{A}v &\geq 0 \end{aligned}$$

Note that the second and third equations imply that the first equation actually has to hold *element-wise*.

Let's denote the "excess value" $z = v - S$ and $\mathbf{B} = \rho\mathbf{I} - \mathbf{A}$. Then the second equation is $z \geq 0$ and the third equation is

$$\mathbf{B}z + q \geq 0$$

where $q = -u + \mathbf{B}S$. Summarizing

$$z'(\mathbf{B}z + q) = 0$$

$$z \geq 0$$

$$\mathbf{B}z + q \geq 0$$

This is the standard form for LCPs https://en.wikipedia.org/wiki/Linear_complementarity_problem so can solve it with an LCP solver.

References

Huang, Jacqueline, and Jong-Shi Pang. 1998. "Option Pricing and Linear Complementarity." *Journal of Computational Finance*, 2: 31–60.