Labor-Market Matching with Precautionary Savings

Bence Bardóczy*

October 25, 2017

1 Introduction

In this note, I discuss how to solve Krusell, Mukoyama, and Şahin (2010) (KMS) in a continuous-time setting using a finite-difference method as explained in Achdou, Han, Lasry, Lions, and Moll (2017). KMS depart from the textbook Diamond-Mortensen-Pissarides (DMP) model in that workers can insure themselves against job loss by accumulating assets.[1] In short, the employed will save and the unemployed dissave to smooth consumption. The main computational challenge compared to the standard models comes from wage setting. Heterogeneity in wealth creates heterogeneity in the value of unemployment. Therefore, bilateral bargaining between individual workers and firms leads to a wage schedule that is increasing in wealth. Thus, solving the model involves finding this endogenous wage schedule, another infinite-dimensional object.

In discrete time, the problem is indeed difficult. KMS solves for the wage function by explicit maximization on a coarse grid (50-125 nodes), which is then interpolated when fed into the Bellman equations of workers and firms. In continuous time however, the wage schedule can be obtained in closed form. As such, this exercise suggests that the continuous-time approach is especially powerful when it comes to models with bilateral bargaining.

The rest of the note is organized as follows. In section 2, I show how to set up the model in continuous time. It is here, where I explain why wage setting simplifies compared to discrete time. Section 3 is about the stationary equilibrium. Here, I offer a brief explanation of the quantitative differences compared to the original discrete-time solution of KMS. Finally, section 4 is about transition dynamics following an unanticipated shock.

2 Setup

Time is continuous. There is a measure 1 of workers in the economy, and a large mass of potential firms. Each worker is either employed or unemployed. When employed, each worker produces $zF(k)$, where z is aggregate productivity, k is capital used by that worker, and $F(\cdot)$ is an increasing and strictly concave production function. Output is used for consumption, investment, and vacancy creation.

2.1 Labor market

Vacant jobs and unemployed workers are matched randomly in every instant according to the the aggregate matching function, $M(u, v)$. As usual in the literature, I assume that the matching function is increasing in both arguments and is homogeneous of degree one. Thus, the job-finding and filling rates

$$f_t = \frac{M(u_t, v_t)}{u_t} = M(1, \theta_t) \qquad \text{and} \qquad q_t = \frac{M(u_t, v_t)}{v_t} = M(\theta_t^{-1}, 1), \tag{2.1}$$

*I wish to thank Gideon Bornstein, Egor Kozlov, Gustavo Mellior, and Ben Moll for helpful comments and discussions.

[1]Alternatively, one might think about it as an Aiyagari model with endogenous job-finding rate.

are functions of the labor market tightness, θ. Notice that $f_t \equiv \theta_t q_t$ holds as an identity.

Matched workers and jobs separate exogenously at Poison rate σ. From the above assumptions, it follows that unemployment evolves according to

$$\dot{u}_t = \sigma(1 - u_t) - f_t u_t. \tag{2.2}$$

2.2 Asset market

There are two assets in this economy: capital, which is used for production; and equity, which is a claim to firm's profit. Let r be the return on capital, p the price of equity, and d the dividend it pays in every instant. In the absence of aggregate shocks, these are both riskless assets, and no arbitrage dictates that they offer the same return:

$$r_t - \delta = \frac{d_t + \dot{p}_t}{p_t}. \tag{2.3}$$

Moreover, since capital and equity are equivalent from the worker's perspective, we do not have to keep track of the portfolio composition. That is, the relevant individual state variable is just assets a, defined as

$$a_t = s_t^K K_t + s_t^p p_t = s_t(K_t + p_t),$$

where s_t is the share of total wealth in the economy held by a particular worker. For convenience, I assume that the portfolio composition is the same for all workers. If we took the distinction between assets seriously, we could calibrate portfolio weights s^K, s^p conditional on wealth and employment from micro data. However, this choice would not affect the stationary equilibrium at all, and have a very limited effect on transition dynamics, because capital stock dwarfs equity in equilibrium.

2.3 Workers

Workers can be either employed or unemployed. Let $s_t \in \{e, u\}$ denote their labor market status. They choose how much to consume and save, subject to prices and the job-finding probability, which they take as given. Since they do not value leisure, they always search for a job. The unemployed engage in home production[2] resulting in a flow income of h, while the employed earn wage $\omega(a_t)$ to be determined via Nash bargaining. In its most general form, their problem can be written as

$$\begin{aligned}
W(a_0, s_0) = \max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty & e^{-\rho t} u(c_t)\,\mathrm{d}t \\
\dot{a}_t &= y(a_t, s_t) + (r_t - \delta)a_t - c_t \\
a_t &\geq \underline{a} \\
s_t &\in \{e, u\} \text{ is a Poisson process with intensities } f_t, \sigma \\
y_t(a_t, s_t) &= \begin{cases} \omega(a_t) & \text{for } s_t = e \\ h & \text{for } s_t = u \end{cases}
\end{aligned}$$

where y_t is just a generic notation for income.

In recursive form, the problem of an employed worker can be summarized by a pair of an HJB equation and state-constraint boundary condition:

$$\rho W(a, e, t) = \max_c \Big\{ u(c) + \partial_a W(a, e, t)\big[\omega(a, t) + (r_t - \delta)a - c\big] + \sigma\big[W(a, u, t) - W(a, e, t)\big] + \partial_t W(a, e, t) \Big\} \tag{2.4}$$

$$\partial_a W(\underline{a}, e, t) \geq u'\big(\omega(\underline{a}, t) + (r_t - \delta)\underline{a}\big) \tag{2.5}$$

[2] I call it home production because it is not financed from taxes. That would be easy to change.

Notice that workers internalize the effect of higher wealth on their wage. This gives them an extra incentive to save—on top of preparing for job loss. Analogously, the problem of an unemployed worker is

$$\rho W(a,u,t) = \max_c \Big\{ u(c) + \partial_a W(a,u,t)\big[h + (r_t - \delta)a - c\big] \\ + f_t\big[W(a,e,t) - W(a,u,t)\big] + \partial_t W(a,u,t) \Big\} \tag{2.6}$$

$$\partial_a W(\underline{a},u,t) \geq u'\big(h + (r_t - \delta)\underline{a}\big). \tag{2.7}$$

Taking first-order conditions yield the policy functions for consumption and saving. Because of the symmetry of the two HJB equations, it is convenient to express the policy functions using the more general notation:

$$c(a,s,t) = (u')^{-1}\left(\partial_a W(a,s,t)\right),$$
$$\dot{a}(a,s,t) = y(a,s,t) + (r_t - \delta)a - c(a,s,t).$$

The savings function is then used in the Kolmogorov forward equations

$$\partial_t g(a,e,t) = -\partial_a\big[\dot{a}(a,e,t)g(a,e,t)\big] + f_t g(a,u,t) - \sigma g(a,e,t), \tag{2.8}$$
$$\partial_t g(a,u,t) = -\partial_a\big[\dot{a}(a,u,t)g(a,u,t)\big] + \sigma g(a,e,t) - f_t g(a,u,t), \tag{2.9}$$

that characterize the dynamics of the distribution of workers over individual states, $g(a,s)$.

2.4 Firm

The representative firm creates jobs, rents capital from consumers and produces. Due to matching frictions, the firm earns a positive profit—even with free entry into vacancy creation—which it pays out as dividend to the shareholders. The firm discounts future profits at rate $r - \delta$, which is not only the market rate, but also the marginal rate of substitution of anyone with positive holdings of the firm.

As in the standard DMP model, I write the value functions for a single job and single vacancy. The time-dependent HJB equations can be written as

$$(r_t - \delta)J(a,t) = \max_k \Big\{ z_t F(k) - r_t k - \omega(a,t) + \partial_a J(a,t)\dot{a}(a,e,t) + \sigma(V(t) - J(a,t)) + \partial_t J(a,t) \Big\} \tag{2.10}$$

$$(r_t - \delta)V(t) = -\xi + q_t \int_{\underline{a}}^{\infty} J(a,t)\frac{g(a,u,t)}{u_t}\,\mathrm{d}a + \partial_t V(t), \tag{2.11}$$

where ξ is the flow vacancy cost; and $g(a,u)/u$ is the marginal distribution of the unemployed over assets. Free entry implies that $V(t) \equiv 0$ and thus

$$(\sigma + r_t - \delta)J(a,t) = \max_k \Big\{ z_t F(k) - r_t k - \omega(a,t) + \partial_a J(a,t)\dot{a}(a,e,t) + \partial_t J(a,t) \Big\}.$$
$$\xi = q_t \int_{\underline{a}}^{\infty} J(a,t)\frac{g(a,u,t)}{u_t}\,\mathrm{d}a.$$

Taking the FOC in the former gives the usual factor demand equation, $z_t F'(k_t) = r_t$.

2.5 Wage setting

Once a match is formed, it produces a positive surplus that has to be shared between the firm and the worker. Traditionally, most authors working with DMP models—including KMS—have adopted Nash bargaining. To my knowledge, there is no deep a priori reason to do so. Indeed, there are other bargaining solutions that one could think of using. l'Haridon, Malherbet, and Pérez-Duarte (2013) compare three of them—Nash, Egalitarian, and Kalai-Smorodinsky—in the context of a standard DMP model without precautionary savings. They find that if workers are risk-neutral, all three lead to equivalent outcomes. If workers are risk averse, this equivalence breaks, but the differences are quantitatively very small. They conclude that the policy implications of the DMP model are not sensitive to the choice of the bargaining solution.

In this section, I'll describe how the Nash and Egalitarian bargaining solutions work in our setup[3]. This discussion is relevant, because the Egalitarian bargaining solutions turns out to be easy to solve in closed form, while Nash requires an additional step of approximation.[4]

In what follows, let β denote the bargaining power of the worker, and let me focus on the stationary case for notational simplicity. I'll use the notation $\tilde{W}(w,a,e), \tilde{J}(w,a)$ when I want to make explicit their dependence on the flow wage. U, V don't need this, because by assumption, bargaining is always over the flow wage without commitment to the future, and U and V only depend on future wages.

2.5.1 Nash bargaining

The wage $\omega(a)$ is implicitly characterized by

$$\omega(a) = \arg\max_{w} \left[\tilde{W}(w,a,e) - W(a,u)\right]^{\beta} \left[\tilde{J}(w,a)\right]^{1-\beta}. \tag{2.12}$$

The FOC can be written as

$$\frac{\beta \partial_w \tilde{W}}{W(a,e) - W(a,u)} + \frac{(1-\beta)\partial_w \tilde{J}}{J(a)} = 0.$$

Differentiating the HJB equations (2.4) and (2.10) with respect to the flow wage gives

$$\partial_w W = \frac{\partial_a W}{\rho + \sigma}, \qquad \text{and} \qquad \partial_w J = \frac{-1 + \partial_a J \cdot \partial_w \dot{a}}{\sigma + r - \delta},$$

where $\partial_w \dot{a}(a,e) = 1 - \partial_w c(a,e)$. The firm is *not choosing savings optimally*, hence the envelope theorem does not apply to this term and $\partial_w c(a,e)$ cannot be ignored. Substituting in the Nash FOC yields

$$\beta \frac{(\sigma + r - \delta) J(a) \partial_a W}{1 - \partial_a J (1 - \partial_w c_e)} = (1-\beta)(\rho + \sigma)\left[W(a,e) - W(a,u)\right]. \tag{2.13}$$

First, the advantage of continuous time is that W and J are both linear in $\omega(a)$, and therefore it can be expressed in closed form from (2.13). Indeed, a little algebra yields

$$\omega(a) = \beta \frac{zF(k) - rk + \partial_a J(a)\left[(r-\delta)a - c_e\right]}{1 - \partial_a J(a)(1 - \partial_w c_e)} - (1-\beta)\frac{u(c_e) + \partial_a W\left[(r-\delta)a - c_e\right] - \rho W(a,u)}{\partial_a W(a,e)} \tag{2.14}$$

Second, the problem with Nash bargaining is that knowing the value functions for a given wage does not give us $\partial_w c(a,e)$.[5] Now, the instantaneous MPC from a completely transitory increase in the wage is most probably very low in this model, so one way to proceed is to set $\partial_w c(a,e) = 0$.

The wage function obtained with this simplifying assumption looks exactly as the continuous-time analog of that reported by KMS can be expected to look like. Nevertheless, this dirty fix is admittedly unappealing, which leads us to Egalitarian bargaining.

[3] Each bargaining solution can be characterized with a simple surplus-sharing rule. These are (implicit) equations that characterize the wage. The equations can be derived from axiomatic foundations, but the derivations require the set of feasible outcomes to satisfy some regularity conditions (like convexity, compactness). I don't show these formally, so you might want to think of these as ad hoc surplus-sharing rules inspired these bargaining solutions, rather than the real thing. That's fine by me.

[4] Kalai-Smorodinsky bargaining also avoids the problem Nash has, but it requires solving a new pair of HJB equations.

[5] I have a conjecture that it is just $\partial_a c(a,e)$ appropriately scaled, but I could not prove this. The intuition is that how much you consume from an additional dollar does not depend on where it comes from, just the fact that it is a one-time transfer.

2.5.2 Egalitarian bargaining

The wage $\omega(a)$ is implicitly characterized by

$$(1-\beta)\big[W(a,e)-W(a,u)\big]=\beta J(a). \tag{2.15}$$

Clearly, the egalitarian solution avoids the envelope-theorem issue. The resulting wage is

$$\begin{aligned}\omega(a)\left[(1-\beta)\frac{\partial_a W(a,e)}{\rho+\sigma}+\beta\frac{1-\partial_a J(a)}{r-\delta+\sigma}\right]&=\beta\frac{zF(k)-rk+\partial_a J(a)\big[(r-\delta)a-c_e\big]}{r-\delta+\sigma}\\ &\quad-(1-\beta)\frac{u(c_e)+\partial_a W\big[(r-\delta)a-c_e\big]-\rho W(a,u)}{\rho+\sigma}\end{aligned} \tag{2.16}$$

Note that flow wage is extracted from $W(a,e)$ and $J(a)$, but not $W(a,u)$. The same is true for the derivation of (2.14). This reflects the lack of commitment mentioned in the beginning of the section.

3 Stationary equilibrium

This section is dedicated to the stationary equilibrium. First, a formal definition. Second, the calibration. Third, some graphs illustrating qualitative properties of the equilibrium. For details of the algorithm, see appendix A.

3.1 Definition

The equations describing the stationary recursive equilibrium can be obtained by dropping the time arguments, indices and partials from all the equations above. The formal definition is then the following. The stationary recursive equilibrium consists of a set of value functions $\{W(a,s), J(a), V\}$; a set of policy functions $\{c(a,s), \dot{a}(a,s)\}$; a distribution over assets and employment $g(a,s)$; a set of prices $\{r, p, \omega(a)\}$; capital stock per worker k; and labor market tightness θ such that

- Worker optimization:

  Given prices and labor market tightness, the decision rules $\{c(a,s), \dot{a}(a,s)\}$ solve the optimization problem given by (2.4) (2.5) and (2.6) (2.7), with corresponding value function $W(a,s)$.

- Consistency of stationary distribution:

  Given the worker decision rules and tightness, the distribution $g(a,s)$ satisfies the Kolmogorov forward equations (2.8) (2.9).

- Firm optimization:

  Given prices, tightness, and the stationary distribution, k solves (2.10), with corresponding value functions $J(a), V$.

- Labor market:

  The job finding and filling probabilities are given by tightness according to (2.1). Unemployment can be calculated in two ways:

  $$u=\frac{\sigma}{\sigma+f}=\int_{\underline{a}}^{\infty} g(a,e)+g(a,u)\,\mathrm{d}a.$$

  The measure of vacancies is then given by $v=u\theta$.

- Free entry:

  Given prices, tightness, and the stationary distribution, V satisfies (2.11). A good way to think about this is that vacancies (a jump variable) adjust.

- Wage setting:

  The wage satisfies (2.14) or (2.16).

- Asset market:

  The asset market clears:

$$\int_{\underline{a}}^{\infty} a[g(a,e)+g(a,u)]\,\mathrm{d}a = K + p.$$

  The components of supply are determined as follows. Aggregate capital is $K = k(1-u)$. The market value of firms satisfies the no arbitrage condition (2.3) for $\dot{p} = 0$. Finally, dividends are the aggregate flow profit:

$$d = \int \pi(a)g(a,e)\,\mathrm{d}a - \xi v, \qquad \pi(a) = zk^{\alpha} - rk - \omega(a).$$

- Goods market:

  The final good is produced in firms and at home, and is spent on consumption and investment in physical capital and vacancy creation:

$$\underbrace{\int_{\underline{a}}^{\infty} \sum_{s=e,u} c(a,s)g(a,s)\,\mathrm{d}a}_{C_t} + \underbrace{\dot{K}_t - \delta K_t + \xi v_t}_{I_t} = \underbrace{(1-u_t)z_t F(k_t) + u_t h.}_{Y_t}$$

3.2 Calibration

The baseline calibration follows KMS closely. Some parameters are just set to standard values. In the stationary problem, it is natural to set $z = 1$. Let the relative risk aversion be $\gamma = 1$ and assume that agents can only save: $\underline{a} = 0$. Let the elasticity of matching with respect to unemployment be $\eta = 0.72$, and impose the Hosios condition, $\beta = \eta$. In the standard DMP with linear utility, this condition guarantees efficiency. In our setting it does not, because of the over-accumulation of capital induced by precautionary saving.

The rest of the parameters are calibrated to match various moments of the data. Table 1 makes clear how. The choice of bargaining solution only affects the vacancy cost.

Table 1: Calibration

moment	target	achieved	parameter	value	in model
Annual real rate of return on capital	0.04	0.039	ρ	0.01	$1-e^{-4(r-\delta)}$
Investment-output ratio	0.2	0.204	δ	0.021	$\delta k^{1-\alpha}$
Home production	0.4 $\bar{w}$	0.41 $\bar{w}$	h	0.75	h
Monthly separation rate	0.034	0.034	σ	0.1038	$1-e^{-\sigma/3}$
Monthly job-finding probability	0.45	0.45	χ	1.7935	$1-e^{-f/3}$
Labor market tightness (nash)	1	1	ξ	0.395	θ
Labor market tightness (egalitarian)	1	1	ξ	0.199	θ

3.3 Results

Figure 1 shows the policy functions, the wage schedule, and the stationary distribution for the simplified Nash case. Except for very poor workers, the consumption of the employed and unemployed is almost the same. This consumption smoothing is achieved by the unemployed tapping deeply into their savings. The precautionary savings motive is so strong that even the very rich employed would prefer to save, so the upper bound of the grid is a binding boundary constraint. This, however, is unlikely to influence the results, because the support of the stationary distribution is so far from this upper bound. We can also see that the wage schedule is almost flat, all the curvature is close to the borrowing constraint. Finally, the asset distribution is dispersed and right-skewed, but almost identical for employed and unemployed agents.

The egalitarian bargaining case depicted on figure 2 is similar. The wage curve is somewhat steeper. Relatedly, the firm's surplus decreases faster in assets.

3.4 Comparison to discrete time

Qualitatively, everything is the same. Quantitatively, two things change markedly: agents hold less capital and earn lower wages in the continuous-time environment. The reason is that the precautionary motive is weaker in continuous time, so agents accumulate less capital, which in turn lowers labor productivity. Why is that? In discrete time, agents who lose their job have to stay unemployed for at least one period (6 weeks in KMS), which is a long time. In the continuous-time model—and conditional on the same 6-week job-finding probability—workers have a good chance of finding a job within a period. I think that the latter is a better description of reality.

4 Transition dynamics

Lastly, I will study the transition dynamics following an unanticipated and transitory shock to technology. For details of the algorithm, see appendix A. The experiment is as follows. The economy is in stationary equilibrium with $z = 1$, when suddenly it jumps to $z_0 = 1.02$. Following this initial shock, agents correctly anticipate that z_t will return to it's steady state value at exponential rate ν. Formally,

$$\begin{cases} \dot{z}_t &= -\nu(z_t - 1), \\ z_0 &= 1.02. \end{cases}$$

Figures 3 and 4 show the impulse responses of selected variables in the case of simplified Nash bargaining. The case of egalitarian bargaining is similar and is not shown. As firms become more productive, vacancy creation surges, which translates into an increase in job-finding rates. Since the inflow to unemployment (separation) is fixed, this implies that unemployment falls. However, these effects are quantitatively modest, because much of the productivity gain goes into higher wages. In other words, the Shimer puzzle is alive and well in the model. We can see that the shock moves the entire wage schedule almost uniformly.

Why is increase in the average wage more persistent than the shock, or the other labor market variables? The answer is on the next page. The productivity shock induced agents to save more temporarily. Since the precautionary motive is so strong that only the unemployed dissave, capital stays high for a very long time. This increases wages through two channels: (i) higher marginal product of labor; (ii) richer workers bargain for a higher wage. This extra capital pushes down the interest rate. So, firms pay less for capital and more for labor. On net, their profits return to their stationary levels much faster than wages, interest rates or capital. This can explain why the effect on job-finding rate and unemployment is so short-lived.

Finally, we can see how the two components of assets react differently. Capital is a stock so it cannot jump on impact. Equity price is forward looking. Total assets is the sum of the two.

References

Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2017): "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach," Tech. rep.

Krusell, P., T. Mukoyama, and A. Şahin (2010): "Labour-Market Matching with Precautionary Savings and Aggregate Fluctuations," *Review of Economic Studies*, 77, 1477–1507.

l'Haridon, O., F. Malherbet, and S. Pérez-Duarte (2013): "Does Bargaining Matter in the Small Firm's Matching Model?" *Labour Economics*, 21, 42–58.

A Algorithms

The HJB equations are solved via the finite difference method as described in Achdou et al. (2017). Finding the equilibrium requires a fixed-point algorithm over $\{\theta, k\}$.

A.1 Stationary equilibrium

1. Set up a grid for assets, $\{a_i\}_{i=1}^{I}$.

   KMS use 1000 equidistant points on $[0, 500]$. I found that a smaller interval suffices. There is an economic reason. In discrete time, workers who lose their job have to stay unemployed for one period at least. For the same monthly job-finding probability, workers of the continuous-time world have a non-negligible probability to find a job some time within the period. Ceteris paribus, this weakens the precautionary motive and leads agents accumulate less wealth. How to choose an interval? The smaller, the more efficient, but what is large enough? There are two considerations that can be checked ex post: (i) cover the stationary distribution, (ii) upper bound is irrelevant for agents' decisions. The second is usually much stricter. Figure 1 shows that all the mass is below 60, but employed workers would stop saving only around 700. I think in this case an upper bound of 100 is reasonable, because it will not affect decisions in the relevant region.

2. Guess tightness, θ.

   This guess will be updated based on the free entry condition. When calibrating the model, we can target $\theta = 1$ and adjust ξ accordingly. Start from $\xi = 0.3$. Given θ, get the labor market variables are

$$f = \chi\theta^{1-\eta}, \qquad q = \chi\theta^{-\eta}, \qquad u = \frac{\sigma}{\sigma + f}, \qquad v = \theta u.$$

3. Guess capital per worker, k.

   Let's start from $k = 1.1k_{cm}$, where

$$k_{cm} = \left(\frac{\alpha z}{\rho + \delta}\right)^{\frac{1}{1-\alpha}}$$

   is steady state capital with complete markets. In our setting, precautionary saving will lead to higher capital. This gives aggregate capital and interest rate:

$$K = (1-u)k, \qquad r = \alpha z k^{\alpha - 1}.$$

4. Guess wage schedule, $\omega(a)$.

   Let's start from $\omega(a) = \beta[zk^\alpha - rk]$, a fraction of the flow profit without vacancy cost. Alternatively, it could be pinned down by the initial guesses for the value functions W, J. I found that it converges very quickly anyway.

5. Solve the worker's problem.

   The discretized HJB equation is

$$\rho W_{i,s} = u(c_{i,s}) + \underbrace{\frac{W_{i+1,s} - W_{i,s}}{\Delta a}}_{\partial_a W_{i,s}^F} [\dot{a}_{i,s}]^+ + \underbrace{\frac{W_{i,s} - W_{i-1,s}}{\Delta a}}_{\partial_a W_{i,s}^B} [\dot{a}_{i,s}]^- + \lambda_{s,s'} \big[W(i,s') - W(i,s)\big].$$

   How is $\dot{a}_{i,s}$ defined? First, define consumption associated with increasing, decreasing, and not-touching assets, respectively:

$$c_{i,s}^F = (\partial_a W_{i,s}^F)^{-\frac{1}{\gamma}}, \qquad c_{i,s}^B = (\partial_a W_{i,s}^B)^{-\frac{1}{\gamma}}, \qquad c_{i,s}^0 = y_{i,s} + (r - \delta)a_i$$

These imply savings decisions

$$\dot{a}^F_{i,s} = y_{i,s} - (r-\delta)a_i - c^F_{i,s}, \qquad \dot{a}^B_{i,s} = y_{i,s} - (r-\delta)a_i - c^B_{i,s}.$$

By concavity of the value function,

$$\partial_a W^F_{i,s} < \partial_a W^B_{i,s} \implies \dot{a}^F_{i,s} < \dot{a}^B_{i,s}$$

and thus there are three cases to distinguish:

$$\mathcal{I}^B = \{(i,s) : \dot{a}^B_{i,s} < 0\}, \qquad \mathcal{I}^F = \{(i,s) : 0 < \dot{a}^F_{i,s}\}, \qquad \mathcal{I}^0 = \{(i,s) : \dot{a}^F_{i,s} < 0 < \dot{a}^B_{i,s}\}.$$

Intuitively, these correspond to decreasing assets for sure, increasing assets for sure, and ambiguous. The upwinding scheme is conservative in the sense that it chooses the final policies

$$c_{i,s} = \mathbb{1}_{\{I^F\}} c^F_{i,s} + \mathbb{1}_{\{I^B\}} c^B_{i,s} + \mathbb{1}_{\{I^0\}} c^0_{i,s}, \quad \dot{a}_{i,s} = y_{i,s} - (r-\delta)a_i - c_{i,s}.$$

Let ℓ index the iterations of the value function. In matrix form, the HJB equation can be written as

$$\frac{W^{\ell+1} - W^\ell}{\Delta} = u^\ell + (\mathbf{A}^\ell + \mathbf{\Lambda}^\ell) W^{\ell+1} - \rho W^{\ell+1} \implies W^{\ell+1} = \left[\left(\frac{1}{\Delta} + \rho\right)\boldsymbol{I} - \boldsymbol{A}^\ell - \mathbf{\Lambda}^\ell\right]^{-1}\left(u^\ell + \frac{1}{\Delta}W^\ell\right)$$

where $\boldsymbol{A}^\ell$, the transition matrix along the asset dimension, is defined as

$$\boldsymbol{A}^\ell = \begin{bmatrix} \boldsymbol{A}_e & 0 \\ 0 & \boldsymbol{A}_u \end{bmatrix}, \qquad \boldsymbol{A}_s = \begin{bmatrix} y_{1,s} & z_{1,s} & 0 & 0 & \dots & 0 \\ x_{2,s} & y_{2,s} & z_{2,s} & 0 & \dots & 0 \\ 0 & x_{3,s} & y_{3,s} & z_{3,s} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & x_{I,s} & y_{I,s} \end{bmatrix}, \qquad \begin{cases} x_{i,s} = -\frac{[\dot{a}_{i,s}]^-}{\Delta a}, \\ z_{i,s} = \frac{[\dot{a}_{i,s}]^+}{\Delta a}, \\ y_{i,s} = -x_{i,s} - z_{i,s} \end{cases}$$

and $\mathbf{\Lambda}^\ell$, the transition matrix along the employment dimension, is defined as

$$\mathbf{\Lambda}^\ell = \begin{bmatrix} -\sigma\boldsymbol{I} & \sigma\boldsymbol{I} \\ f\boldsymbol{I} & -f\boldsymbol{I} \end{bmatrix}.$$

Note that choosing a large Δ makes the iteration faster but less robust. Intuitively, it is the step size along a time dimension. For this problem, even huge numbers (≈ 2000) work well, making the iteration very fast. Δ is too large if it makes the matrix

$$\left(\frac{1}{\Delta} + \rho\right)\boldsymbol{I} - \boldsymbol{A}^\ell - \mathbf{\Lambda}^\ell$$

ill-conditioned.

6. Calculate the stationary distribution of workers.

The most convenient way of discretizing the Kolmogorov forward equation is

$$0 = -\frac{[\dot{a}_{i,s}]^+ g_{i,s} - [\dot{a}_{i-1,s}]^+ g_{i-1,s}}{\Delta a} - \frac{[\dot{a}_{i+1,s}]^- g_{i+1,s} - [\dot{a}_{i,s}]^- g_{i,s}}{\Delta a} + \lambda_{s',s} g_{i,s'} - \lambda_{s,s'} g_{i,s},$$

because it leads to

$$(\boldsymbol{A}^\ell + \mathbf{\Lambda}^\ell)' g = 0.$$

Notice that this equation has infinitely many solutions, since any solution can be scaled arbitrarily. The true solution is pinned down by the fact that g is a density and thus has to integrate to one. In practice, one can fix one coordinate of g, find the unique solution of the resulting linear system, and rescale it ex post. This is implemented by replacing the first row of $(\boldsymbol{A}^\ell + \mathbf{\Lambda}^\ell)'$ by $[1, 0, \dots, 0]$ and setting the first coordinate of the zero vector on right-hand side to an arbitrary positive number. There are probably better methods to find eigenvector for zero eigenvalue, but this is simple and fast enough for this application.

7. Solve the firm's problem.

   This is relatively easy since assets are exogenous for the firm and we already calculated their transition matrix. The discretized HJB equation can be written as

$$\frac{J^{\ell+1}-J^{\ell}}{\Delta} = \pi^{\ell} + \boldsymbol{A}_e^{\ell} J^{\ell+1} - (\sigma + r - \delta) J^{\ell+1} \implies J^{\ell+1} = \left[\left(\frac{1}{\Delta} + \sigma + r - \delta\right)\boldsymbol{I} - \boldsymbol{A}_e^{\ell}\right]^{-1}\left(\pi^{\ell} + \frac{1}{\Delta}J^{\ell}\right),$$

   where π is the flow profit from a filled job:

$$\pi_i = zk^{\alpha} - rk - \omega_i.$$

8. Evaluate the free entry condition:

$$FE = -\xi + qJ' \frac{g_u}{u} \Delta a.$$

   If FE is positive, firms are too profitable and the guess for θ will have to be increased. Update at the end of the loop according to

$$\theta^{\ell+1} = \theta^{\ell} + \Delta\theta \cdot FE^{\ell},$$

   where $\Delta\theta > 0$ is a small number.

9. Evaluate asset market clearing.

   Aggregate profit to be payed out as dividends are given by

$$d = \pi' g(e) \Delta a - \xi v.$$

   No arbitrage implies that the price of equity has to satisfy

$$p = \frac{d}{r-\delta}.$$

   Excess demand on the asset market is:

$$AD = a'\big[g(e) + g(u)\big]\Delta a - K - p.$$

   If it is positive, the guess for k was too low. Update according to

$$k^{\ell+1} = k^{\ell} + \Delta k \cdot AD^{\ell},$$

   where $\Delta k > 0$ is a small number.

10. Update the wage schedule according to (2.14).

11. Update k and θ as described above. Repeat until convergence from step 5.

A.2 Transition dynamics

The economy starts from and returns to the stationary equilibrium we know. The idea is to iterate forward-looking variables (prices, value functions) backwards, and backward-looking variables (states) forwards. Now, the a fixed-point algorithm is implemented over p_0 and the sequences $\{k_t, \theta_t\}_t$. Guessing the equity price on impact is necessary because the unexpected shock changes equity price and thus revalues the assets of workers in the economy.

1. Solve for corresponding stationary equilibrium. It will provide initial and terminal conditions.

2. Set up a grid for time steps: $\{\Delta t_n\}_{n=1}^{N-1}$.

   The $N-1$ time steps define N time points as follows:

$$t_n = \begin{cases} 0 & n = 1 \\ \sum_{m=1}^{n-1} \Delta t_m & n = 2, \dots, N \end{cases}$$

   To speed up the algorithm, it is useful to define a non-uniform grid that gets wider as n increases. This way, we can capture the action right after the shock has hit, but economize on later periods when the transition is smooth.

3. Specify the aggregate shock process:

$$z_n = 1 + (z_0 - 1)e^{-\nu t_n}.$$

4. Guess a path for tightness $\{\theta_n\}$.

   This pins down the job-finding and filling rates for all n as

$$f_n = \chi \theta_n^{1-\eta}, \qquad q_n = \chi \theta_n^{-\eta}.$$

5. Guess a path for capital per worker, $\{k_n\}$.

   Then the interest rate is given by

$$r_n = \alpha z_n k_n^{\alpha-1}.$$

6. Iterate the HJB equations backward from terminal conditions $W^N = W$ and $J^N = J$.

   Note that we have to solve the worker's and firm's problems simultaneously in order to keep track of the wage. The solution method is analogous to the stationary case and boils down to solving a linear system for workers:

$$\left[\left(\frac{1}{\Delta t_n} + \rho\right)\mathbf{I} - \mathbf{A}^{n+1} - \mathbf{\Lambda}^{n+1}\right] W^n = u^{n+1} + \frac{1}{\Delta t_n} W^{n+1},$$

   and for firms:

$$\left[\left(\frac{1}{\Delta t_n} + \sigma + r_{n+1} - \delta\right)\mathbf{I} - \mathbf{A}_e^{n+1}\right] J^n = \pi^{n+1} + \frac{1}{\Delta t_n} J^{n+1}.$$

7. Revalue assets in the economy.

   Let p denote equity price in the stationary equilibrium, and p_0 on impact. An agent that owned $a_i = s_i(K+p)$ prior to the shock, will have

$$a_i' = s_i(K + p_0) = a_i \left[1 + \frac{p_0 - p}{K + p}\right].$$

   Furthermore, we know that the new distribution can be defined as $g_0(a_i', s) = g(a_i, s)$, since the agents cannot do anything about the revaluation. They just wake up and realize the price change. Then, g_0 at the old grid points $\{a_i\}_i^I$ can be found via interpolation. The trick is that we need to ensure that the resulting function is still a density (positive, integrates to 1). In principle, integral-preserving interpolation can be done by integrating first, interpolating the integral, and finally taking derivatives again. (There are built-in functions to take derivatives of common interpolating functions such as splines.) In practice, I found that interpolating g directly using MATLAB's `pchip` method, and then simply rescaling works well.

8. Iterate the Kolmogorov forward equations, well, forward from the *revalued stationary distribution.*

   Using the implicit method, the discretized version can be written as (subscripts improve readability)

$$\frac{g_{n+1} - g_n}{\Delta t_n} = (\mathbf{A}^n + \mathbf{\Lambda}^n)' g_{n+1} \implies g_{n+1} = \left[\mathbf{I} - \Delta t_n (\mathbf{A}^n + \mathbf{\Gamma}^n)'\right]^{-1} g_n.$$

   I found this to be numerically more stable than the explicit method. Also, $\{\Delta t_n\}_n$ have to be small enough for matrices $\{\mathbf{I} - \Delta t_n(\mathbf{A}^n + \mathbf{\Gamma}^n)'\}_n$ to be well-conditioned.

9. Calculate unemployment and vacancies as

$$u_n = \sum_i g_{i,u,n} \Delta a, \qquad \text{and} \qquad v_n = \theta_n u_n.$$

   For large $\{\Delta t_n\}_n$, this is more stable than iterating the law of motion.

10. Evaluate free entry for all n :

$$FE_n = -\xi + q_n \sum_i J_{i,n} \frac{g_{i,u,n}}{u_n} \Delta a.$$

11. Check asset market clearing for all n.

    Flow profit from a job filled by a worker with assets a is

$$\pi_{i,n} = z_n k_n^\alpha - r_n k_n - \omega_{i,n}$$

    hence aggregate profits are

$$d_n = \sum_i \pi_{i,n} g_{i,e,n} \Delta a - \xi v_n.$$

    No arbitrage condition can be used to solve price of equity backward from terminal condition $p_N = p$:

$$\frac{p_{n+1} - p_n}{\Delta t_n} = p_n(r_n - \delta) - d_n \implies p_n = \left(\frac{1}{\Delta t_n} + r_n - \delta\right)^{-1} \left[d_n + \frac{p_{n+1}}{\Delta t}\right]$$

    Aggregate capital can be calculated as

$$K_n = (1 - u_n) k_n.$$

    Excess demand on the asset market is

$$AD_n = \sum_i a_i \left[g_{i,e,n} + g_{i,u,n}\right] \Delta a - K_n - p_n.$$

12. Adjust p_0 based on the backward iteration of the no-arbitrage condition. Adjust $\{k_n, \theta_n\}_n^{N-1}$ based on the asset market clearing and free entry as before.

    Formally, let ℓ index the current guess, $\tilde{p}_0$ the price obtained from the backward iteration above, and $\Delta_p, \{\Delta_{kn}\}_n, \{\Delta_{\theta n}\}_n$ positive constants. Then formally

$$\begin{aligned}
p_0^{\ell+1} &= \Delta_p \tilde{p}_0 + (1 - \Delta_p) p_0^\ell \\
k_n^{\ell+1} &= k_n^\ell + \Delta_{kn} \cdot AD_n \qquad \forall n \\
\theta_n^{\ell+1} &= \theta_n^\ell + \Delta_{\theta n} \cdot FE_n \qquad \forall n
\end{aligned}$$

    In practice, choosing decreasing sequences of $\{\Delta_{kn}, \Delta_{\theta n}\}_n$ helps convergence. The reason is that we know that all variables eventually return to steady state (some very fast), but failure of market clearing in early periods usually implies further failures down the road.

B Figures

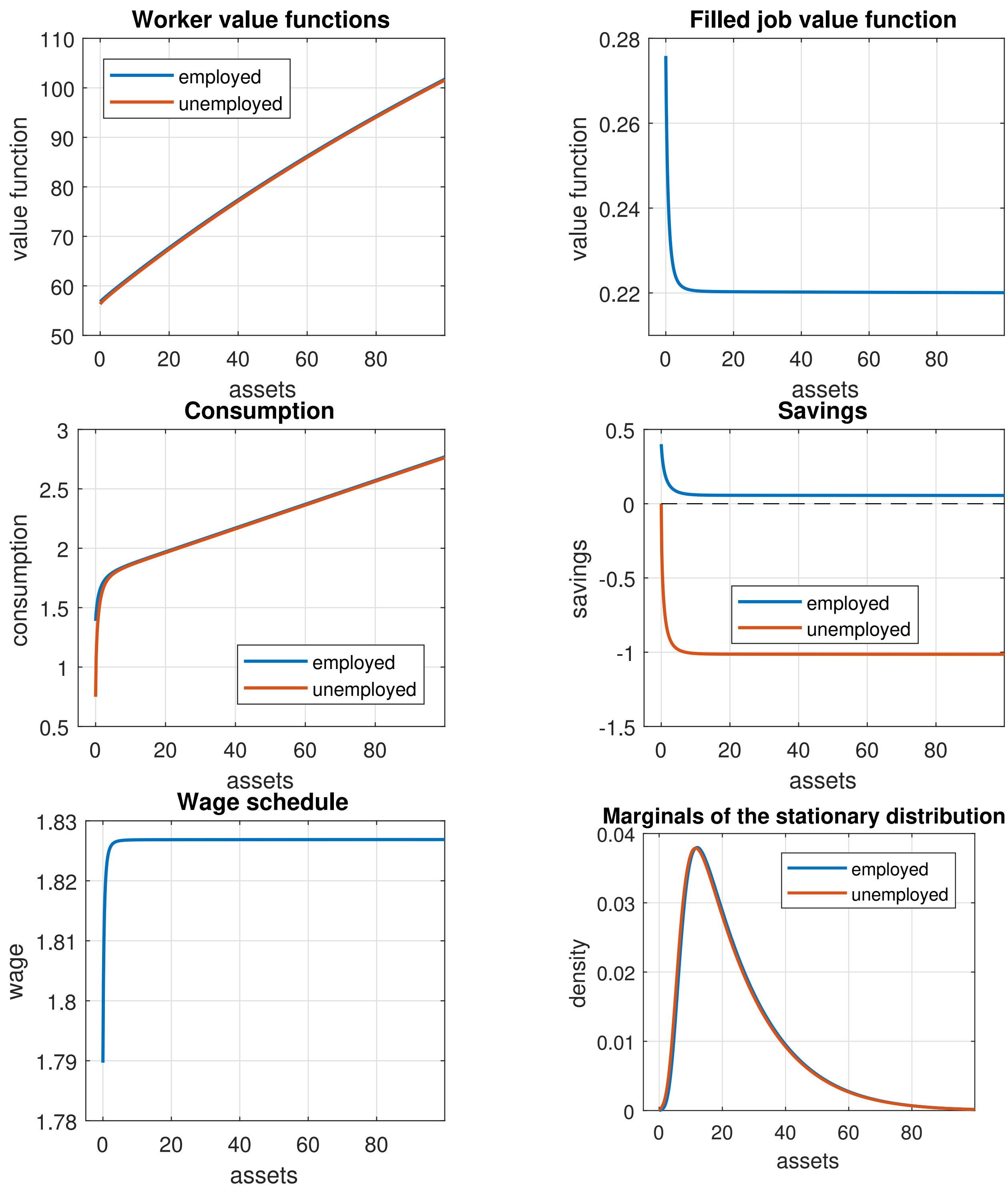

Figure 1: Stationary equilibrium with simplified Nash bargaining

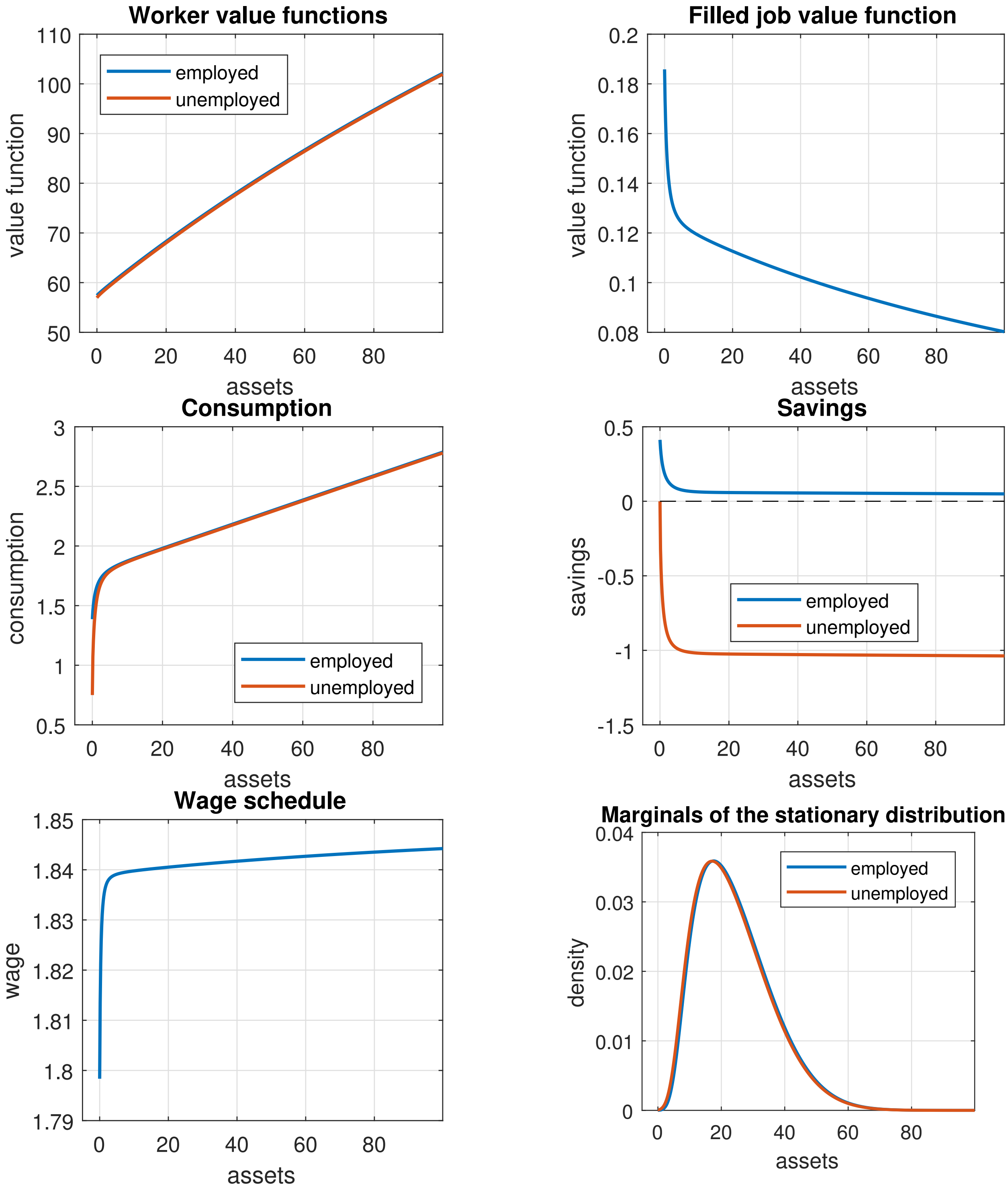

Figure 2: Stationary equilibrium with Egalitarian bargaining

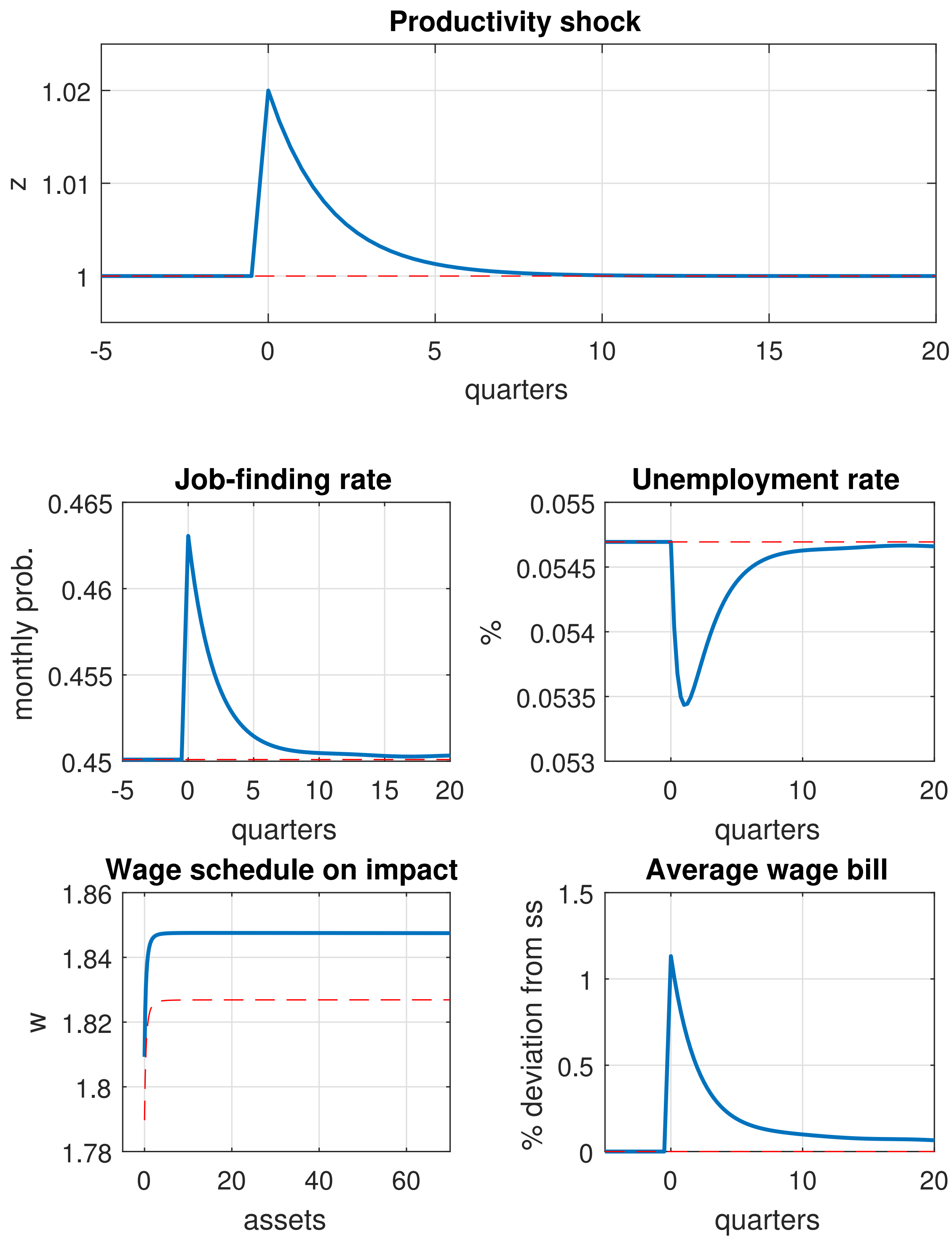

Figure 3: Impulse responses to a positive technology shock 1: labor market

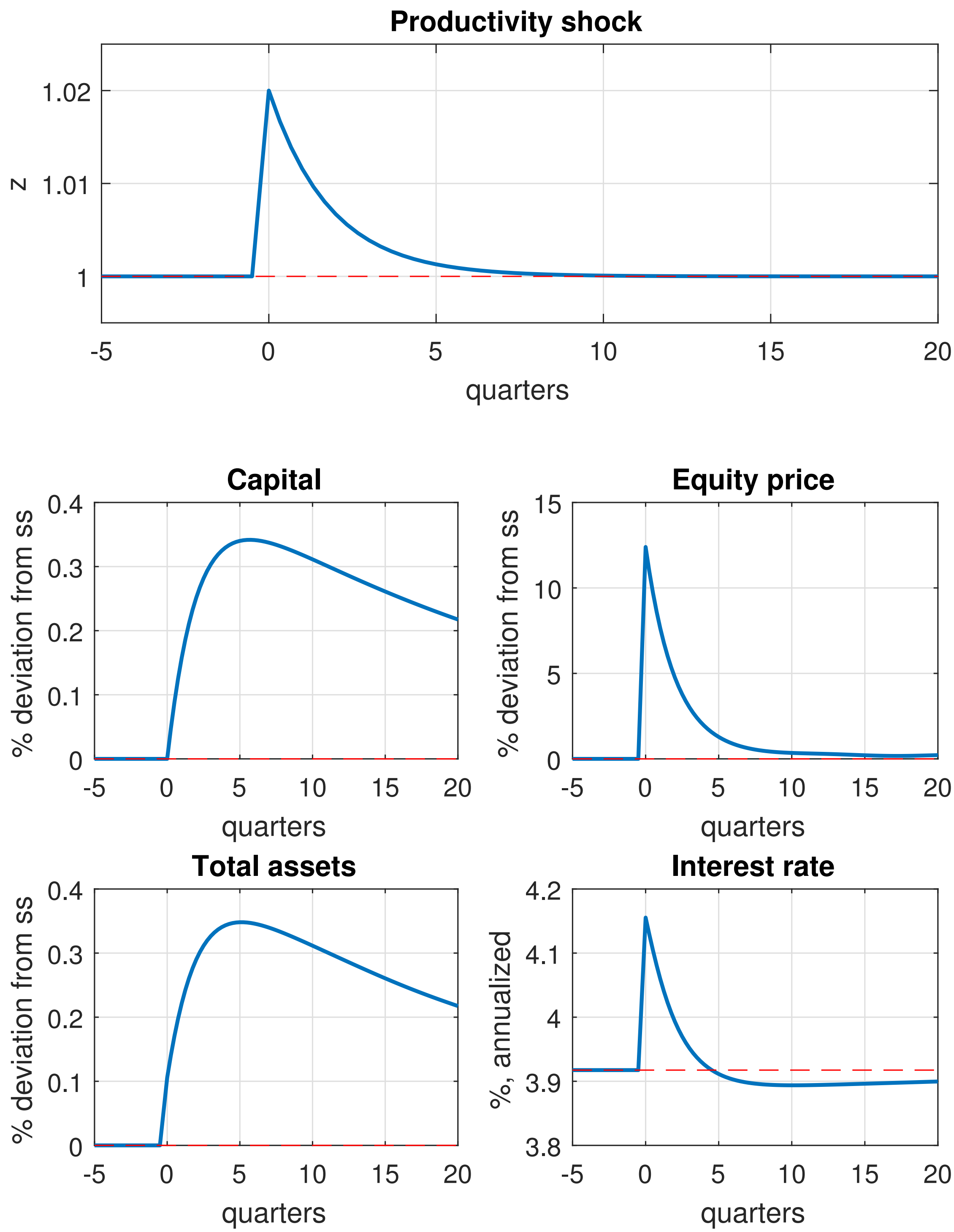

Figure 4: Impulse responses to a positive productivity shock 2: asset market