Hopenhayn Model in Continuous Time

This note translates the Hopenhayn (1992) model of firm dynamics, exit and entry in a stationary equilibrium to continuous time. For a brief summary of the original Hopenhayn model, see slides 17 to 20 of these lecture notes `https://web.stanford.edu/~jdlevin/Econ%20257/Industry%20Dynamics.pdf`. We also build on Shaker Akhtekhane (2017).

The model features incumbent and entrant firms. Before proceeding to writing down and solving the full model, we first consider a simplified version with mechanical entry. In this version, firms choose optimally when to exit and for each exiting firm there is a new entrant such that the mass of firms stays exactly constant. This is inspired by the "return process" in Luttmer (2007) and the model with "exit and reinjection" in Gabaix et al. (2016). We then endogenize entry: firms not only choose optimally when to exit but also choose optimally whether to enter the industry.

Mathematically, firms in the Hopenhayn model solve stopping time problems. These can be formulated as so-called Hamilton-Jacobi-Bellman Variational Inequalities (HJBVIs). See Bensoussan and Lions (1982, 1984) and Bardi and Capuzzo-Dolcetta (1997). More recently, Bertucci (2017) analyzes Mean Field Games with stopping, with the prototypical problem featuring a coupled system of an HJBVI for agents' stopping time problem and the corresponding variant of a Kolmogorov Forward equation for the evolution of the distribution of agents.

1 Mechanical Entry

The only interesting problem is that of incumbent firms. Their productivity z_t evolves according to an exogenous diffusion process on a bounded intervals $z_t \in [0, 1]$, with reflecting barriers at $z = 0$ and $z = 1$. Any other productivity process that can be written in terms of an infinitesimal generator, e.g. a jump-diffusion, is possible as well. Incumbent firms choose employment n_t each period and a stopping time τ at which to exit. In a stationary equilibrium, their problem is:

$$v(z) = \max_{\{n_t\}_{t\geq 0},\tau} \left\{ \mathbb{E}_0 \int_0^\tau e^{-\rho t}(pf(z_t, n_t) - wn_t - c_f)dt + e^{-\rho\tau} v^* \right\}$$
$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t, \quad z_0 = z.$$

Here $f(z, n)$ is the firm's production function, p is the price of final goods, w is the wage rate, c_f is a per-period operating cost and v^* is a scrap value. For each exiting firm, there is a new entrant that starts with some initial productivity z_0 drawn from a distribution $\psi(z)$. Hence the total mass of active firms is constant. We therefore normalize it to one. We also assume that the support of ψ is such that firms do not immediately exit again after entering, i.e. any draw from ψ is sufficiently high.

For future reference, it is useful to define firm profits as

$$\pi(z) = \max_n \{pf(z,n) - wn\} - c_f$$

so that an incumbent firm's problem can be written as

$$v(z) = \max_\tau \left\{ \mathbb{E}_0 \int_0^\tau e^{-\rho t}\pi(z_t)dt + e^{-\rho\tau}v^* \right\} \tag{1}$$
$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t, \quad z_0 = z.$$

Similarly, denote the optimal labor demand and output policy functions by $n(z)$ and $q(z)$. We denote by $g(z)$ the stationary density of firms with productivity z. g is the state of the economy.

We close the model by assuming – as in Hopenhayn's orginal model – that there is an exogenous product demand and labor supply to the industry. Hence p and w are determined by the inverse product demand and labor supply curves that are functions of the total production and employment in the industry

$$p = D(Q), \qquad w = W(N), \qquad Q = \int_0^1 q(z)g(z)dz, \qquad N = \int_0^1 n(z)g(z)dz.$$

1.1 Equilibrium System

The firm's problem (1) can be characterized in terms of an HJB variational inequality (HJBVI)

$$\min\left\{\rho v(z) - v'(z)\mu(z) - \frac{1}{2}v''(z)\sigma^2(z) - \pi(z), v(z) - v^*\right\} = 0, \quad \text{all } z \in (0,1).$$

See `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf` for background on HJBVI's. Denote by $\mathcal{Z}$ the set of productivities such that firms remain in the industry, i.e. the inaction region. For standard production functions, the exit decision is characterized by a simple threshold rule: remain active as long as $z_t \geq x$ and so $\mathcal{Z} = [x, 1]$. That being said, in general, the inaction region $\mathcal{Z}$ could be a more complicated set. At the upper boundary, v satisfies the boundary conditions for a reflecting barrier $v'(1) = 0$. The same condition is satisfied if there is no exit

As long as firms remain active, i.e. when productivity $z_t \in \mathcal{Z}$, z_t evolves according to the diffusion process in (1). Therefore the evolution of g is characterized by a standard Kolmogorov Forward (KF) equation with a correction for firm entry. To this end, we denote by m the rate at which firms enter the industry. By assumption m is equal to the exit rate (so as to ensure a constant mass of firms). Considering for the moment the non-stationary distribution $g(z,t)$, the KF equation is:

$$\partial_t g(z,t) = -\partial_z(\mu(z)g(z,t)) + \frac{1}{2}\partial_{zz}\left(\sigma^2(z)g(z,t)\right) + m(t)\psi(z), \quad \text{all } z \in \mathcal{Z} \tag{2}$$

where $m(t)$ is the entry rate which is pinned down by the requirement that the total mass of firms is constant $\int_0^1 g(z,t)dz = 1$ for all z and t. The exact expression for $m(t)$ depends on the shape of the inaction region $\mathcal{Z}$, i.e. on the exit rule.

If there is a simple threshold rule for exit, i.e. exit whenever z falls below x, then

$$m(t) = \frac{1}{2}\partial_z\left(\sigma^2(x)g(x,t)\right) \tag{3}$$

To see this, integrate (2) from $z = x$ to $z = 1$ using that $\int_x^1 g(z,t)dz = 1$ and hence $\int_x^1 \partial_t g(z,t)dz = 0$ to get

$$\begin{aligned} 0 &= -\int_x^1 \partial_z(\mu(z)g(z,t))dz + \frac{1}{2}\int_x^1 \partial_{zz}\left(\sigma^2(z)g(z,t)\right)dz + m(t)\int_x^1 \psi(z)dz \\ &= \mu(x)g(x,t) - \frac{1}{2}\partial_z\left(\sigma^2(x)g(x,t)\right) + m(t) \end{aligned}$$

Next note that the density of firms exactly at the exit threshold has to be zero, $g(x,t) = 0$. Using this we have (3).

Formula (3) generalizes to more general inaction regions $\mathcal{Z}$ as follows. Denote by $\mathcal{A}$ the infinitesimal generator of the diffusion process $\mathcal{A}v := \mu(z)\partial_z v + \frac{1}{2}\sigma^2(z)\partial_{zz}v$ and by $\mathcal{A}^*$ its adjoint $\mathcal{A}^*g = -\partial_z(\mu(z)g) + \frac{1}{2}\partial_{zz}(\sigma^2(z)g)$ so that the KF equation (2) is $\partial_t g = \mathcal{A}^*g + m(t)\psi$. We then have

$$m(t) = -\int_{\mathcal{Z}}(\mathcal{A}^*g)(z,t)dz \tag{4}$$

Analogous to the derivation with a threshold rule, this expression is found by integrating the KF equation $\partial_t g = \mathcal{A}^*g + m(t)\psi$ over all z in the inaction region $\mathcal{Z}$ and rearranging.

In a stationary equilibrium, the stationary versions of (2) and (3) (or (4)) hold.

Summary of Equilibrium System. Summarizing, a stationary equilibrium are thus functions v and g (from $[0,1]$ to $\mathbb{R}$) and scalars (p, w, Q, N) such that

$$0 = \min\left\{\rho v(z) - v'(z)\mu(z) - \frac{1}{2}v''(z)\sigma^2(z) - \pi(z), v(z) - v^*\right\}, \quad \text{all } z \in (0,1), \tag{5}$$

$$0 = -(\mu(z)g(z))' + \frac{1}{2}\left(\sigma^2(z)g(z)\right)'' + m\psi(z), \quad \text{all } z \in \mathcal{Z}, \tag{6}$$

$$m = -\int_{\mathcal{Z}} (\mathcal{A}^* g)(z)dz, \tag{7}$$

$$p = D(Q), \quad w = W(N), \quad Q = \int_{\mathcal{Z}} q(z)g(z)dz, \quad N = \int_{\mathcal{Z}} n(z)g(z)dz. \tag{8}$$

where the inaction region in (6), (7) and (8) is determined from (5). The system (5) to (8) constitutes a Mean Field Game with optimal stopping. See Bertucci (2017) for a rigorous analysis of such systems using an equivalent variational formulation.

1.2 Numerical Solution

We numerically solve the model under the assumptions that

$$f(z,n) = zn^\alpha, \qquad D(Q) = Q^{-\varepsilon}, \quad w(N) = N^\phi, \quad v^* = 0, \quad \psi(z) = U([p,1])$$

with $\alpha \in (0,1), \varepsilon > 0, \phi > 0, p = 1/2$.

We use a finite difference method on an equi-spaced grid with I points, $z_i, i = 1, ..., I$ and grid spacing Δz. We use the notation $v_i = v(z_i)$ and $g_i = g(z_i)$ and denote the vector of all v_i's and g_i's by $\mathbf{v}$ and $\mathbf{g}$.

The numerical solution of the HJBVI (5) is as explained in `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf` and implemented in `http://www.princeton.edu/~moll/HACTproject/option_simple.m`. The discretized HJBVI equation can be written as

$$0 = \min\left\{\rho\mathbf{v} - \mathbf{A}\mathbf{v} - \pi, \mathbf{v} - \mathbf{v}^*\right\} \tag{9}$$

and can then be solved as a linear complementarity problem (LCP). In particular (9) is equivalent to

$$\begin{aligned} (\mathbf{v} - \mathbf{v}^*)^{\mathrm{T}}(\rho\mathbf{v} - \mathbf{A}\mathbf{v} - \pi) &= 0 \\ \mathbf{v} - \mathbf{v}^* &\geq 0 \\ \rho\mathbf{v} - \mathbf{A}\mathbf{v} - \pi &\geq 0. \end{aligned} \tag{10}$$

Let's denote the "excess value" $\mathbf{x} = \mathbf{v} - \mathbf{v}^*$ and $\mathbf{B} = \rho\mathbf{I} - \mathbf{A}$. Then the second equation is $\mathbf{x} \geq 0$

and the third equation is

$$\mathbf{B}\mathbf{x} + \mathbf{q} \geq 0$$

where $\mathbf{q} = -\pi + \mathbf{B}\mathbf{v}^*$. Summarizing

$$\begin{aligned} \mathbf{x}^{\mathrm{T}}(\mathbf{B}\mathbf{x} + \mathbf{q}) &= 0 \\ \mathbf{x} &\geq 0 \\ \mathbf{B}\mathbf{x} + \mathbf{q} &\geq 0 \end{aligned}$$

This is the standard form for LCPs `https://en.wikipedia.org/wiki/Linear_complementarity_problem` so can solve it with an LCP solver.

The numerical solution of the KF equation in the presence of entry and exit is more involved. Denote by $\mathcal{I}$ the set of grid points in the inaction region $\mathcal{Z}$, i.e. grid points at which firms do not exit and by $A_{i,j}$ the entries of the transition matrix $\mathbf{A}$ from the discretized HJBVI equation (and note that because $\mathbf{A}$ is a transition matrix its sum to zero, $\sum_{j=1}^{I} A_{i,j} = 0$). Then the discretized KF equation (6) is

$$0 = \sum_{j=1}^{I} A_{j,i} g_j + m\psi_i, \quad \text{all } i \in \mathcal{I} \tag{11}$$

Summing over all $i \in \mathcal{I}$ and using that $\sum_{i\in\mathcal{I}} \psi_i \Delta z = 1$, we obtain the following discretized version of (7)

$$m = -\sum_{i\in\mathcal{I}} \sum_{j=1}^{I} A_{j,i} g_j \Delta z$$

Using the fact that $\sum_{j=1}^{I} A_{i,j} = 0$ and hence $\sum_{k\in\mathcal{I}} A_{j,k} = -\sum_{k\notin\mathcal{I}} A_{j,k}$, this can be usefully rewritten as

$$m = \sum_{i\notin\mathcal{I}} \sum_{j=1}^{I} A_{j,i} g_j \Delta z, \tag{12}$$

i.e. the exit/entry rate m is the mass of firms that the process sends into the exit region. Finally, for all grid points in the exit region the density is necessarily zero: $g_i = 0$ all $i \notin \mathcal{I}$.

Let us manipulate the discrete KF equation a bit more to generate some further insights. Substitute (12) into (11) to write

$$\begin{aligned} &0 = \sum_{j=1}^{I} B_{j,i} g_j, \quad \text{all } i \in \mathcal{I} \quad \text{with} \quad B_{i,j} := A_{i,j} + \psi_j \Delta z \sum_{k\notin\mathcal{I}} A_{i,k} \\ &g_i = 0, \quad \text{all } i \notin \mathcal{I} \end{aligned} \tag{13}$$

The first line of this equation says that the process induced by the transition matrix $\mathbf{A}$ with entry and exit equals the transition matrix $\mathbf{B}$ of another process with entries $B_{i,j} = A_{i,j} + \psi_j \Delta z \sum_{k \notin \mathcal{I}} A_{i,k}$. That is, whenever the $\mathbf{A}$-process leads into the exit region, we simply move the corresponding entries of the transition matrix and spread them across the inaction region in proportion to the entry distribution $\psi_i \Delta z$. In particular, note that the entries of $\mathbf{B}$ sum to zero when summed over the inaction region $\sum_{j \in \mathcal{I}} B_{i,j} = \sum_{j \in \mathcal{I}} A_{i,j} + \sum_{j \in \mathcal{I}} \psi_j \Delta z \sum_{k \notin \mathcal{I}} A_{i,k} = 0$, thereby guaranteeing mass preservation in the inaction region.

Finally, for numerical convenience, we can write (13) in matrix notation as follows

$$0 = \widetilde{\mathbf{B}}^{\mathrm{T}} \mathbf{g}$$

where $\widetilde{B}_{i,j} = B_{i,j}$ for all columns in the inaction region $j \in \mathcal{I}$ and the columns in the exit region $j \notin \mathcal{I}$ are replaced by a column of zeros everywhere except for 1 on the diagonal. The latter case means that rows of $\widetilde{\mathbf{B}}^{\mathrm{T}}$ corresponding to the exit region $i \notin \mathcal{I}$ are rows of zeros except for 1 on the diagonal and hence $0 = \widetilde{\mathbf{B}}^{\mathrm{T}} \mathbf{g}$ implies that $g_i = 0$ for all $i \notin \mathcal{I}$.

2 Optimal Entry

We now endogenize entry as in the original paper by Hopenhayn (1992). Rather than the entry rate m being determined mechanically such that the total mass of firms is constant, it is now determined from a free-entry condition:

$$\int_0^1 v(z)\psi(z)dz = c_e \tag{14}$$

where c_e denotes the entry cost. To make the problem better behaved numerically, we change this equation slightly to have an elastic supply of entry:

$$m = \bar{m}\exp\left(\eta\left(\int_0^1 v(z)\psi(z)dz - c_e\right)\right) \tag{15}$$

where m is the entry rate and η is the elasticity of the supply of entrant and $\bar{m}$ is a parameter. The free-entry condition (14) is the special case $\eta \to \infty$, i.e. the Hopenhayn model has an infinitely elastic supply of entrants.[1]

The fact that entry is now chosen optimally also means that the condition (7) now needs to be dropped from the set of equilibrium conditions. Finally, note that the total mass of active firms will, in general, not be constant over time or with respect to parameter changes. We therefore no longer normalize $\int g(z,t)dz$ to one anymore.

2.1 Equilibrium System

A stationary equilibrium is therefore (v, g, w, p, m) and an exit region $\mathcal{Z}$ that solves

$$0 = \min\left\{\rho v(z) - v'(z)\mu(z) - \frac{1}{2}v''(z)\sigma^2(z) - \pi(z), v(z) - v^*\right\}, \quad \text{all } z \in (0,1), \tag{16}$$

$$0 = -(\mu(z)g(z))' + \frac{1}{2}\left(\sigma^2(z)g(z)\right)'' + m\psi(z), \quad \text{all } z \in \mathcal{Z}, \tag{17}$$

$$m = \bar{m}\exp\left(\eta\left(\int_0^1 v(z)\psi(z)dz - c_e\right)\right), \tag{18}$$

$$p = D(Q), \quad w = W(N), \quad Q = \int_{\mathcal{Z}} q(z)g(z)dz, \quad N = \int_{\mathcal{Z}} n(z)g(z)dz. \tag{19}$$

Note that the system is identical to before with the exception that we have replaced (7) by (14). The logic is that the entry rate m is pinned down by this condition. To see this note that from (17) a higher entry rate m means a larger mass of active firms $\int_0^1 g(z)dz$. This in turn

[1] We have $\frac{\log(m/\bar{m})}{\eta} = \int_0^1 v(z)\psi(z)dz - c_e$. And hence taking the limit $\eta \to \infty$ yields (15).

means larger (Q, N) which in turn means lower p and higher w. This depresses the value of a firm v. The equilibrium entry rate m is such that these forces balance out and the free-entry condition (18) holds.

2.2 Numerical Solution

See `hopenhayn.m` which was written by Riccardo Cioffi. Analogously to above, the discretized version of (17) is

$$
\begin{aligned}
0 &= \sum_{j=1}^{I} A_{j,i} g_j + m\psi_i, \quad \text{all } i \in \mathcal{I} \\
g_i &= 0, \quad \text{all } i \notin \mathcal{I}
\end{aligned}
$$

This can be written in matrix form as

$$
0 = \widetilde{\mathbf{A}}^{\mathrm{T}} \mathbf{g} + m\psi \tag{20}
$$

where $\widetilde{A}_{i,j} = A_{i,j}$ for all columns in the inaction region $j \in \mathcal{I}$ and the columns in the exit region $j \notin \mathcal{I}$ are replaced by a column of zeros everywhere except for 1 on the diagonal. The latter case means that rows of $\widetilde{\mathbf{A}}^{\mathrm{T}}$ corresponding to the exit region $i \notin \mathcal{I}$ are rows of zeros except for 1 on the diagonal and hence $0 = \widetilde{\mathbf{A}}^{\mathrm{T}} \mathbf{g} + m\psi$ implies that $g_i = 0$ for all $i \notin \mathcal{I}$. Since $\widetilde{\mathbf{A}}^{\mathrm{T}}$ constructed in this fashion is non-singular, we can simply solve $\mathbf{g} = -(\widetilde{\mathbf{A}}^{\mathrm{T}})^{-1} m\psi$.

Algorithm:

i. Guess w^0

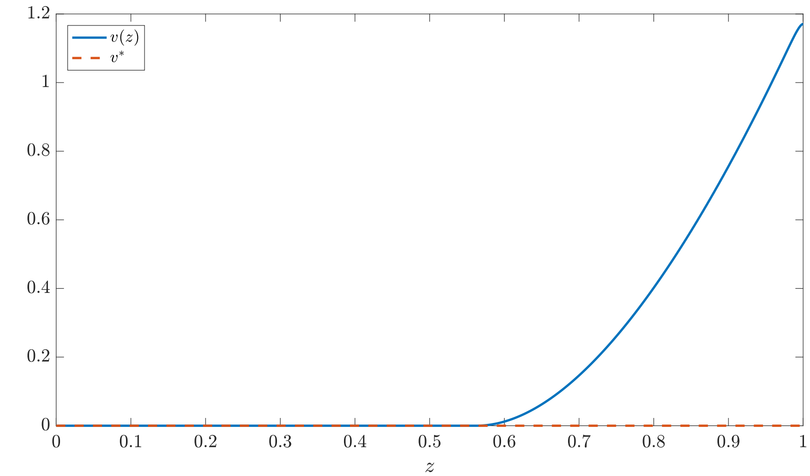
ii.
1. Guess p^0
2. Given (p^j, w^k) solve the HJBVI equation (16). This yields v and an exit region $\mathcal{Z}$.
3. Given v, compute m from the supply of entrants (18). To approximate the case of a perfectly elastic supply of entrants (14), we choose a large value of η.
4. Given the exit region $\mathcal{Z}$, and the entry rate m, solve the KF equation (17) to get g (numerically, solve (20)). Note that g will, in general, not integrate to one.
5. Given g, compute Q and update p: $p^{j+1} = (1 - \lambda_p)p^j + \lambda_p Q^{-\varepsilon}$
6. If p^{j+1} and $Q^{-\varepsilon}$ are close enough, go to *iii.*, otherwise back to 2.

iii. Given g, compute N and update w: $w^{k+1} = (1 - \lambda_w)w^{k+1} + \lambda_w N^{\phi}$

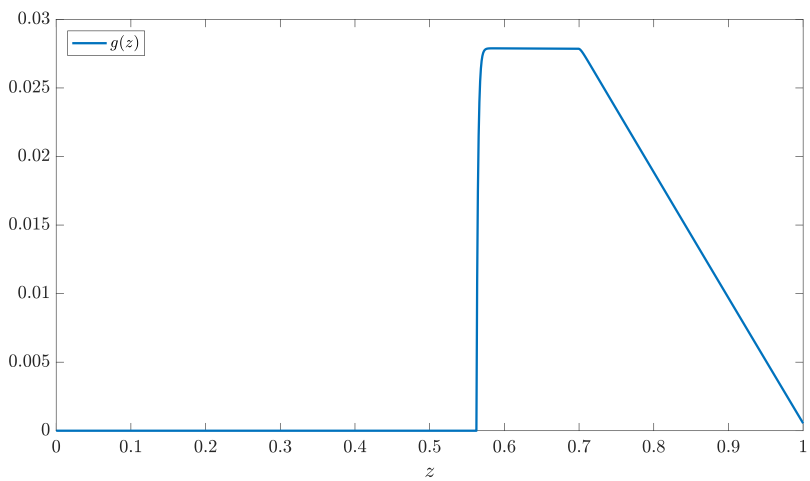
iv. If w^{k+1} and N^{ϕ} are close enough, exit, otherwise back to *ii.*

2.3 Results

Figure 1 plots the value function $v(z)$ and the size distribution of active firms $g(z)$. The mass of active firms equals 0.28.



(a) Value function $v(z)$



(b) Size distribution of active firms $g(z)$

Figure 1: Value Function and Size Distribution in Hopenhayn Model

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