Consumption and Saving with an Indivisible Durable[1]

1 Model Description

Individuals have flow utility over non-durable consumption c_t and durable consumption d_t

$$u(c_t) + \kappa d_t.$$

Durable consumption is indivisible: $d_t \in \{0, 1\}$. For instance, d_t could present car ownership: individuals either own a car, $d_t = 1$, or they do not, $d_t = 0$. For concreteness we will therefore refer to the durable as "car." Individuals who do not own a car can purchase it at price p_0. Individuals who already own a car can sell it at price p_1 with $p_1 < p_0$. At any time when they do not buy or sell a car, individuals' wealth a_t accumulates according to $\dot{a}_t = y + ra_t - c_t$ where y is their constant labor income and r is the interest rate. If they buy a car their wealth jumps down by p_0, and if they sell their car their wealth jumps up by p_1.

Denote by $v_d(a)$ the value of having wealth a and car ownership state $d \in \{0, 1\}$. Individuals in state $d = 0$, optimally choose consumption and the *stopping time* τ at which to purchase the car:

$$v_0(a) = \max_{\{c_t\}_{t\geq 0}, \tau} \int_0^\tau e^{-\rho t} u(c_t) dt + e^{-\rho \tau} v_0^*(a_\tau)$$
$$\dot{a}_t = y + ra_t - c_t, \quad a_t \geq \underline{a}, \quad a_0 = a.$$

where $v_0^*(a)$ is the value of buying a car given by:

$$v_0^*(a) = \begin{cases} v_1(a - p_0), & \text{if } a - p_0 \geq \underline{a} \\ -\infty, & \text{if } a - p_0 < \underline{a} \end{cases}$$

The second branch takes care of the borrowing constraint: individuals cannot buy a car if doing so would lead them to violate the borrowing constraint. The problem for individuals already owning a car is symmetric:

$$v_1(a) = \max_{\{c_t\}_{t\geq 0}, \tau} \int_0^\tau e^{-\rho t} (u(c_t) + \kappa) dt + e^{-\rho \tau} v_1^*(a_\tau)$$
$$\dot{a}_t = y + ra_t - c_t, \quad a_t \geq \underline{a}, \quad a_0 = a.$$

[1] We thank Victor Rios-Rull for suggesting this Problem

where $v_1^*(a)$ is the value of selling a car given by:

$$v_1^*(a) = v_0(a + p_1)$$

Because we will solve the problem on a bounded grid $\underline{a} \leq a \leq a_{\max}$, we will make the simplifying assumption that $v_1^*(a) = v_0(\max\{a+p_1, a_{\max}\})$, i.e. if selling the car would take the individual's wealth above $a_{\max}$ then she receives a smaller price.

The individual's problem boils down to a system of "HJB Variational Inequalities" (HJBVIs)

$$0 = \min\{\rho v_0(a) - \max_c \{u(c) + v_0'(a)(y + ra - c)\}, v_0(a) - v_0^*(a)\}, \tag{1}$$

$$0 = \min\{\rho v_1(a) - \max_c \{u(c) + \kappa + v_1'(a)(y + ra - c)\}, v_1(a) - v_1^*(a)\} \tag{2}$$

See `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf` for an explanation of HJBVIs.

2 Algorithm

The Matlab code at `http://www.princeton.edu/~moll/HACTproject/car.m` solves the system (3) and (4) under the assumption of CRRA utility $u'(c) = c^{-\gamma}$. It uses a similar algorithm as in `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf` and `http://www.princeton.edu/~moll/HACTproject/option_simple.m`.

A sketch is as follows: the discretized HJBVIs are basically:

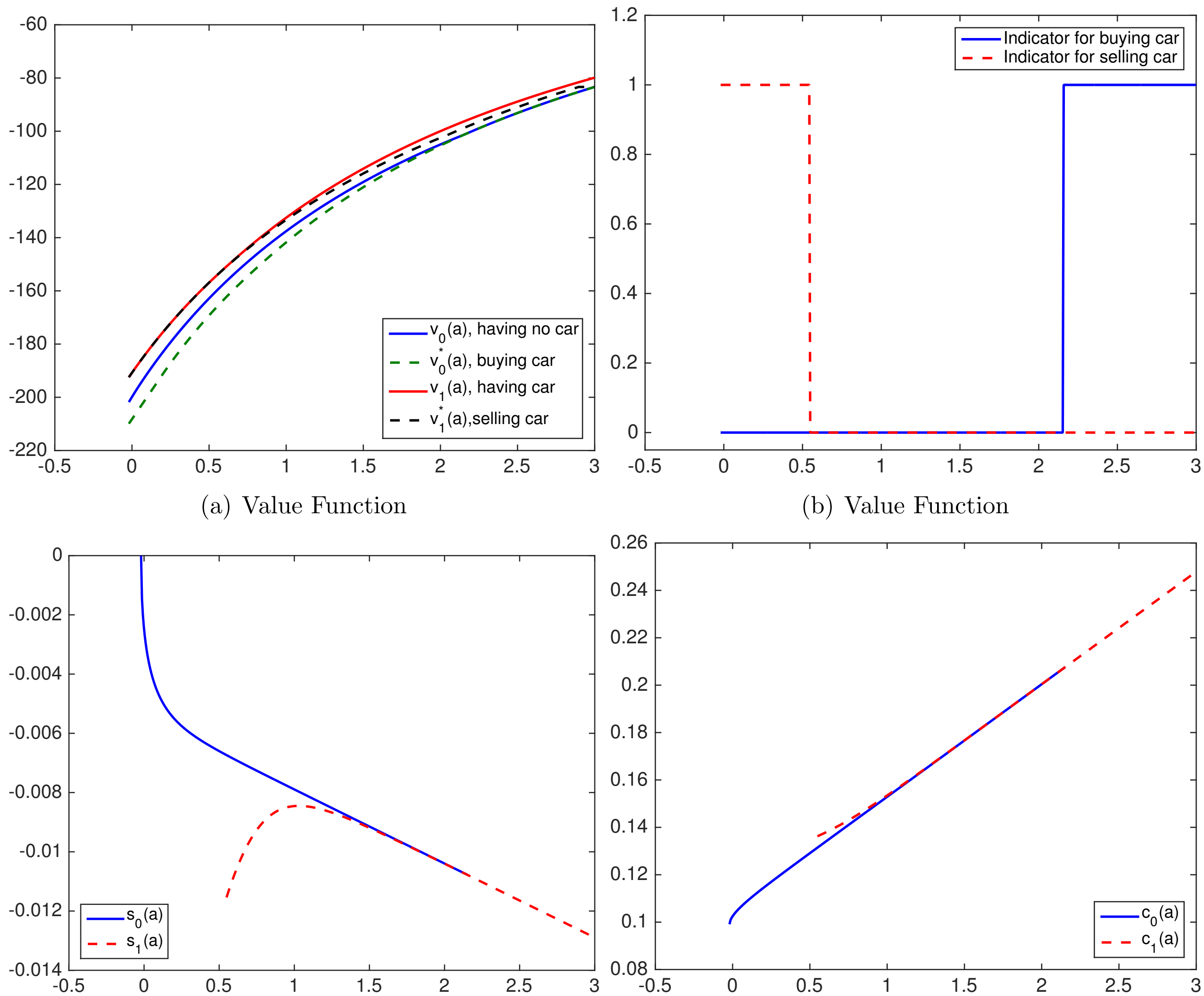
$$0 = \min\{\rho v_0 - u(v_0) - \mathbf{A}(v_0)v_0, v_0 - v_0^*(v_1)\}, \tag{3}$$

$$0 = \min\{\rho v_1 - u(v_1) - \mathbf{A}(v_1)v_1, v_1 - v_1^*(v_0)\} \tag{4}$$

These can then be converted into a Linear Complementarity Problem (LCP) that can be solved with readily available solvers.

3 Results

Figure 1 plots the value and policy functions. As is intuitive, poor individuals sell their cars and rich individuals buy a car. The policy functions in panels (c) and (d) are only plotted for wealth levels at which individuals neither buy or sell a car, i.e. for which the indicators in panel (c) equal zero. The policy functions at other wealth levels are irrelevant because individuals immediately jump from state $d = 0$ to $d = 1$ or vice versa.



(a) Value Function

(b) Value Function

(c) Saving Policy Function

(d) Consumption Policy Function

Figure 1: Value and Policy Functions