Online Appendix D to "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach"

Viscosity Solutions for Dummies (including Economists)

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This Appendix presents a brief introduction to the theory of viscosity solutions of Hamilton-Jacobi-Bellman (HJB) equations (Crandall and Lions, 1983), focusing on dynamic maximization problems of the type that commonly arise in economics. These include problems with non-convexities that typically feature kinked value functions.

After outlining a generic optimal control problem and the corresponding HJB equation, we first define what is meant by a viscosity solution to this HJB equation. Second, we provide a heuristic derivation of this definition that also provides some intuition by making a connection to discrete-time dynamic programming. Third, we show that, for typical maximization problem, the definition of a viscosity solution rules out concave kinks, i.e. only convex kinks are admissible. This appendix is also available as a set of slides at https://benjaminmoll.com/viscosity_slides/. The slides also feature some additional results. In particular, on slides 21-29, we show how to define viscosity solutions for problems with state constraints (Soner, 1986a,b; Capuzzo-Dolcetta and Lions, 1990) and discuss how to interpret them, namely as imposing certain "boundary inequalities." And on slides 30-42 we provide some intuition for why there is a unique viscosity solution to an HJB equation.

As already stated, this appendix aims only to give an intuitive introduction to viscosity solutions. The mathematics literature contains various introductions and lecture notes, some of which are relatively accessible. The more accessible ones include Crandall (1997), Yu (2008) and Bressan (2011). There are also a number of standard references and books that are more encyclopedic but sometimes less accessible. These include Crandall, Ishii, and Lions (1992), Bardi and Capuzzo-Dolcetta (1997) and Fleming and Soner (2006).

A.1 A Generic Optimal Control Problem

Consider the following generic optimal control problem:

$$v(x) = \max_{\{\alpha(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} r(x(t), \alpha(t)) dt \quad \text{s.t.} \quad \dot{x}(t) = f(x(t), \alpha(t)), \quad x(0) = x. \tag{1}$$

Here x the state, α the control, $r(x, \alpha)$ a period return function, and $f(x, \alpha)$ the law of motion of the state. For simplicity, we focus on problems that are (i) one-dimensional, i.e. x is a scalar and (ii) deterministic, i.e. there is no uncertainty. The problem is to choose a function $\alpha : \mathbb{R}^+ \to A \subset \mathbb{R}$ to control the time path $x : \mathbb{R}^+ \to \mathbb{R}$. Throughout this appendix we confine ourselves to cases where the value function v in (1) is continuous.[1] However, we do not impose that it is differentiable.

By using standard arguments, one can show that the value function v has to satisfy the HJB equation:

$$\rho v(x) = \max_{\alpha \in A} \{r(x, \alpha) + v'(x) f(x, \alpha)\}. \tag{2}$$

Examples. A particularly simple example is the deterministic consumption-saving problem (41) in the main text which we restate here for convenience

$$\rho v(a) = \max_{c} \{u(c) + v'(a)(y + ra - c)\}. \tag{3}$$

Viscosity solutions are also designed to handle problems with non-convexities which arise in many important economic applications. One example is the problem with indivisible housing (48) in the main text. Another example is a growth model with a non-convexity à la Skiba (1978)

$$\rho v(k) = \max_{c} \{u(c) + v'(k)(F(k) - \delta k - c)\}, \tag{4}$$

where the production function is convex-concave, e.g.

$$F(k) = \max\{F_L(k), F_H(k)\}, \quad F_L(k) = A_L k^\alpha, \quad F_H(k) = A_H((k-\kappa)^+)^\alpha, \quad \kappa > 0, A_H > A_L.$$

In such applications, the value function typically features kinks, i.e. points of non-differentiability. This raises the question of what is meant by saying that "v satisfies the HJB equation" if the derivative v' that appears in it does not exist. The theory of viscosity solutions provides an answer.

Remark 1: In the mathematics literature, many authors write (2) as

$$\rho v(x) = H(x, v'(x)) \tag{5}$$

where the function H is called the "Hamiltonian" and given by

$$H(x, p) = \max_{\alpha \in A} \{r(x, \alpha) + p f(x, \alpha)\} \tag{6}$$

[1] There is also a theory of discontinuous viscosity solutions. A good summary of known results is in Barles (2013). Also see Bardi and Capuzzo-Dolcetta (1997, Chapter V). In economic applications, problems with kinked value functions are far more natural than problems with discontinuous ones.

For example, the consumption-saving problem in (3) can be written as

$$\rho v(a) = H(a, v'(a)), \quad H(a,p) = \max_c \{u(c) + p(y + ra - c)\} \tag{7}$$

Remark 2: In maximization problems, the Hamiltonian defined by (6) is weakly convex in p, and in practice it is often strictly convex. To see that the Hamiltonian must be weakly convex, note that for fixed (x, α), the function $r(x, \alpha) + pf(x, \alpha)$ is linear in p. The Hamiltonian $H(x, p)$ defined in (6) is the upper envelope of a family of such linear functions. It is therefore a convex function of p. The intuition is the same as that for a familiar result in economics: that the profit function of a profit-maximizing firm is convex in the price faced by the firm. Intuitively, the firm always has the option not to reoptimize as the price changes and secure a linear profit function; but by reoptimizing it can usually do better and hence the profit function is convex.

As an example, one can verify that the Hamiltonian in the consumption-saving problem (7) is indeed convex (and in fact strictly so):

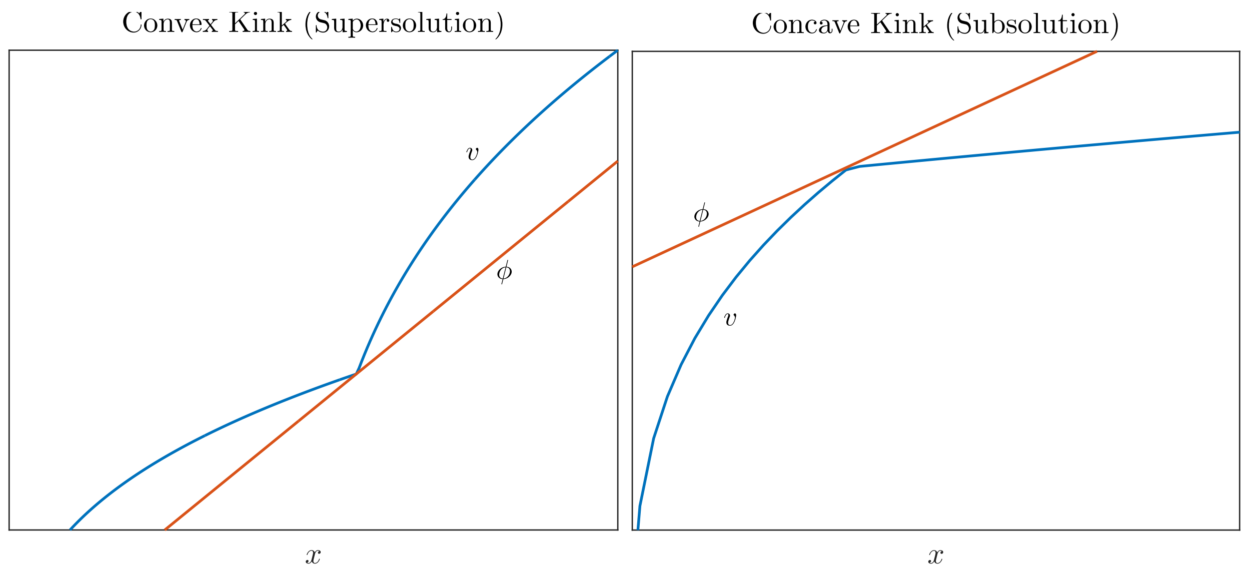
$$H_{pp} = -\frac{1}{u''((u')^{-1}(p))} > 0.$$

Finally, note that a symmetric argument establishes that the Hamiltonian corresponding to a *minimization* problem is weakly concave in p.

A.2 Definition of Viscosity Solution

Before formally defining the notion of viscosity solution, we first explain the basic idea. As already noted, in general we do not expect to find a classical solution to the HJB equation (2). In particular, the value function v may have kinks, i.e. points of non-differentiability. The basic idea is to replace the derivative v' at a point where it does not exist (because of a kink in v) with the derivative ϕ' of a smooth function ϕ (a "test function") that "touches v", and to define a viscosity solution as a function v that satisfies an alternative equation that features ϕ' instead of v'.

In principle, two types of kinks can occur: convex kinks and concave kinks. The situation is depicted in Figure 1. The definition of a viscosity condition features two conditions to deal with each of the situations: in the case of a convex kink, the smooth function ϕ can only touch v from below (see Figure 1(a)); in the case of a concave kink, it can only touch v from above (see Figure 1(b)). We are now ready to spell out the definition of a viscosity solution. Section A.3 provides a heuristic derivation of the two conditions that also provides some intuition by making a connection to discrete-time dynamic programming.



(a) Convex kink, covered by supersolution condition (b) Concave kink, covered by subsolution condition

Figure 1: Convex and concave kinks and definition of viscosity solution

Definition 1 A *viscosity solution* of (2) is a continuous function v such that the following hold:

1. (Supersolution) If ϕ is any smooth function and if $v - \phi$ has a local minimum at point x^* (v may have a convex kink), then

$$\rho v(x^*) \geq \max_{\alpha \in A} \left\{ r(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha) \right\}. \tag{8}$$

2. (Subsolution) If ϕ is any smooth function and if $v - \phi$ has a local maximum at point x^* (v may have a concave kink), then

$$\rho v(x^*) \leq \max_{\alpha \in A} \left\{ r(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha) \right\}. \tag{9}$$

The key observation is, of course, that this definition does not feature the derivative of v at potential kink points x^* where this derivative would not exist. Instead, it only uses the derivative of the smooth function ϕ which exists everywhere by assumption.

Remark 3: If a continuous function v satisfies (8) (but not necessarily (9)), we say that it is a "viscosity supersolution." Conversely, if v satisfies (9) (but not necessarily (8)) we say that it is a "viscosity subsolution." That is, we say that a continuous function v is a "viscosity solution" of (2) if it is both a "viscosity supersolution" and a "viscosity subsolution." We also sometimes say that "v satisfies (2) in the viscosity sense." The name "supersolution" comes from the ≥ 0 in its definition (8) and, similarly, the name "subsolution" from the ≤ 0 in (9).

Remark 4: If v is differentiable at a point x^*, then a local maximum or minimum of $v - \phi$ implies $v'(x^*) = \phi'(x^*)$. The sub- and supersolution conditions together then imply that a viscosity solution of (2) is just a classical solution.

Remark 5: While the definition of a viscosity solution above allows for both convex and concave kinks, we will show below in Section A.4 that, for typical maximization problems, a viscosity solution cannot have concave kinks, i.e. only convex kinks are admissible. For typical minimization problems, the situation is reversed, i.e. only concave kinks are admissible.

Remark 6: The name "viscosity" is in honor of the "method of vanishing viscosity" which was the method by which viscosity solutions were first analyzed. The method of vanishing viscosity essentially adds some stochasticity ϵ to the law of motion of the state but then considers the limit as this stochasticity vanishes, $\epsilon \to 0$. This selects the viscosity solution in the sense defined above.[2] The idea of adding vanishing viscosity was inspired by the work of various Russian mathematicians (particularly Stanislav Kruzhkov) on nonlinear conservation laws in fluid mechanics.

Remark 7: As already mentioned, in the mathematics literature, the HJB equation is typically written as in (5) in terms of the Hamiltonian (6). The definition of the viscosity solution is then also written using the Hamiltonian notation: A *viscosity solution* of (5) is a continuous function v such that the following hold:

1. (Supersolution) If ϕ is any smooth function and if $v - \phi$ has a local minimum at point x^* (v may have a convex kink), then

$$\rho v(x^*) \geq H(x^*, \phi'(x^*)). \tag{10}$$

2. (Subsolution) If ϕ is any smooth function and if $v - \phi$ has a local maximum at point x^* (v may have a concave kink), then

$$\rho v(x^*) \leq H(x^*, \phi'(x^*)). \tag{11}$$

[2] Roughly, in (1) replace the law of motion of the state by $dx_t = f(x_t, \alpha_t)dt + 2\sqrt{\epsilon} dW_t$ where W_t is a standard Brownian motion so that the HJB equation (2) becomes $\rho v_\epsilon(x) = \max_{\alpha \in A} \{r(x,\alpha) + v_\epsilon'(x) f(x,\alpha)\} + \epsilon v_\epsilon''(x)$. The claim is that the viscosity solution can be obtained as ϵ vanishes: as $\epsilon \to 0$, $v_\epsilon(x) \to v(x)$ for all x.

A.3 Heuristic Derivation of Viscosity Solution and Intuition

We now provide a heuristic derivation of the definition of a viscosity solution from a dynamic programming problem with small time steps of length Δ. This derivation also provides some intuition for the sub- and supersolution conditions in this definition.

A.3.1 Intuition in terms of Discrete-Time Bellman Equation.

Before proceeding with the heuristic derivation, we first draw a parallel to standard discrete-time dynamic programming. To this end, consider the discrete-time dynamic optimization problem analogous to (1) and the corresponding Bellman equation

$$v(x) = \max_{\alpha} \{r(x,\alpha) + \beta v(x') \quad \text{s.t.} \quad x' = f(x,\alpha)\} \tag{12}$$

This Bellman equation is a functional equation. A useful way of interpreting the right-hand side is as an operator T on a function v. This operator is defined as

$$(Tv)(x) = \max_{\alpha} \{r(x,\alpha) + \beta v(f(x,\alpha))\}. \tag{13}$$

A solution to the Bellman equation (12) is then a solution to the fixed point problem $v = Tv$.

The "Bellman operator" T has a number of properties. One very intuitive property is "monotonicity", defined as follows (see e.g. Stokey, Lucas, and Prescott, 1989): for any two functions v and ϕ

$$v(x) \geq \phi(x) \quad \forall x \quad \Rightarrow \quad (Tv)(x) \geq (T\phi)(x) \quad \forall x \tag{14}$$

Intuitively, the monotonicity property states that "if my continuation value is higher, I am necessarily at least as well off today."

For future reference, note that we could have defined the solution to a Bellman equation in the following (unnecessarily complicated) fashion: A solution of (12) is a continuous function v such that the following hold:

1. if ϕ is any smooth function and $v \geq \phi$ for all x, then

$$v(x) \geq \max_{\alpha} \{r(x,\alpha) + \beta\phi(f(x,\alpha))\}$$

2. if ϕ is any smooth function and $v \leq \phi$ for all x, then

$$v(x) \leq \max_{\alpha} \{r(x,\alpha) + \beta\phi(f(x,\alpha))\}$$

In what follows we show that the definition of a viscosity solution makes use of essentially the same "monotonicity" property. In particular, the definition of a viscosity solution sidesteps non-differentiabilities by using a version of "monotonicity."

A.3.2 Heuristic Derivation of Viscosity Solution.

We now turn to this section's main argument, namely the derivation of the sub- and super-solution conditions. We start from a dynamic optimization problem with time periods of length Δ.

$$v(x_t) = \max_{\alpha} \left\{ \Delta r(x_t, \alpha) + (1-\rho\Delta) v(x_{t+\Delta}) \quad \text{s.t.} \quad x_{t+\Delta} = \Delta f(x_t, \alpha) + x_t \right\} \tag{15}$$

First assume that v is differentiable everywhere. As is standard, the HJB equation (2) can then be obtained by means of the following heuristic derivation: subtract $(1-\rho\Delta)v(x_t)$ from both sides, divide by Δ, and take $\Delta \to 0$. In particular, it is easy to see that taking this limit picks up a derivative v'.

Now assume that v in (15) has a kink for some x^*. The problem is now that the derivative v' no longer exists. The idea is then to replace v with a smooth function ϕ for which the derivative does exist. As discussed above there are two situations to consider: convex and concave kinks. We consider each in turn.

Convex kink: heuristic derivation of supersolution condition (8). Consider first the case of a convex kink at some point x^*. See Figure 1(a). The idea is then to consider other smooth functions ϕ that do not have a kink. In particular we consider functions ϕ such that $v - \phi$ has a local minimum at point x^*.

We consider only functions ϕ for which $\phi(x^*) = v(x^*)$. As we show at the end of this proof, this is not restrictive because one can always shift ϕ by a constant. Together with the local minimum requirement, $\phi(x^*) = v(x^*)$ means that these functions ϕ touch v from below as in Figure 1(a). Clearly, from the figure, $v(x) \geq \phi(x)$ for all $x \neq x^*$ in a neighborhood of x^*. Hence, if we evaluate (15) at $x_t = x^*$ and replace the continuation value function v by ϕ we have

$$v(x_t) \geq \max_{\alpha} \left\{ \Delta r(x_t, \alpha_t) + (1-\rho\Delta)\phi(x_{t+\Delta}) \right\}. \tag{16}$$

This is the key step of the proof and it is here that we have used the monotonicity condition (14). Note that by doing so we have replaced the non-differentiable continuation value v with the differentiable one ϕ. Now follow the usual steps for deriving an HJB equation as the limiting equation as $\Delta \to 0$: Subtract $(1-\rho\Delta)\phi(x_t)$ from both sides and use $\phi(x_t) = v(x_t)$

$$\Delta \rho v(x_t) \geq \max_{\alpha} \left\{ \Delta r(x_t, \alpha) + (1-\rho\Delta)(\phi(x_{t+\Delta}) - \phi(x_t)) \right\} \tag{17}$$

Dividing by Δ and letting $\Delta \to 0$ yields

$$\rho v(x_t) \geq \max_{\alpha} \{r(x_t, \alpha) + \phi'(x_t) f(x_t, \alpha)\}$$

This is the supersolution condition (8).

Finally, we briefly discuss why it was not restrictive to only consider functions ϕ for which $\phi(x^*) = v(x^*)$, i.e. this restriction is not necessary and one can also consider other smooth functions ϕ for which $\phi(x^*) \neq v(x^*)$. To this end, assume that $v(x^*) = \phi(x^*) + \kappa$ for some constant κ which may be positive or negative but that ϕ satisfies the same other properties as before, in particular $v - \phi$ has a local minimum at x^*. Then

$$v(x_t) \geq \max_{\alpha} \{\Delta r(x_t, \alpha_t) + (1 - \rho\Delta)(\phi(x_{t+\Delta}) + \kappa)\}$$

Subtracting $(1 - \rho\Delta)(\phi(x_t) + \kappa)$ from each side again yields (17). The rest of the proof is as above.

Concave kink: heuristic derivation of subsolution condition. The argument is exactly symmetric.

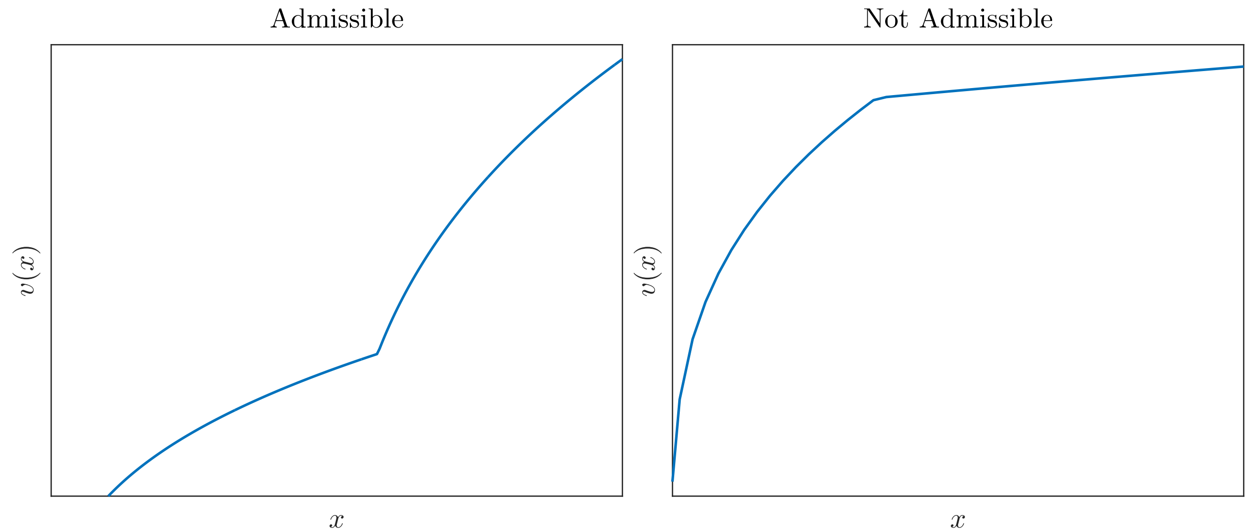
A.4 Viscosity solution rules out concave kinks for maximization

This section establishes that, for typical maximization problem, the definition of a viscosity solution rules out concave kinks, i.e. only convex kinks are admissible. For clarity, the situation is depicted in Figure 2. The argument has two steps. The first step is that maximization problems have convex Hamiltonians. We already proved this as part of Remark 2 in Section A.1. The second step is to show that the viscosity solution of an HJB equations with a convex Hamiltonian cannot have a concave kink. We here prove the second step.

Proposition 1 *Consider the HJB equation* (5) *with the Hamiltonian defined in* (6) *and assume that this Hamiltonian is strictly convex. Then the viscosity solution of this HJB equation does not admit a concave kink.*

Proof: to reach a contradiction, suppose that there is some point x^* at which v has a concave kink. Then we can construct a smooth function ϕ such that $v - \phi$ has a local maximum at point x^*. In particular, we can construct ϕ that "touches v from above" as in Figure 1(b). At this point x^* v and ϕ have to satisfy the subsolution condition (11).

While v is not differentiable at x^*, it still possesses a left derivative $v'_-(x^*) = \lim_{h \to 0^-} \frac{v(x^*+h) - v(x^*)}{h}$ and a right derivative $v'_+(x^*) = \lim_{h \to 0^+} \frac{v(x^*+h) - v(x^*)}{h}$. The function v is also continuous. Using continuity of v, one can show that the HJB equation (5) holds just to the left and just



(a) Convex kinks are admissible

(b) Concave kinks are not admissible

Figure 2: Viscosity solution rules out concave kinks for maximization

to the right of the kink:

$$\rho v(x^*) = H(x^*, v'_+(x^*)), \quad \rho v(x^*) = H(x^*, v'_-(x^*)). \tag{18}$$

Next, as can be seen clearly from Figure 1(b), the left and right derivatives satisfy

$$v'_+(x^*) < \phi'(x^*) < v'_-(x^*).$$

Hence there exists $t \in (0, 1)$ such that

$$\phi'(x^*) = tv'_+(x^*) + (1 - t)v'_-(x^*).$$

Because H is convex, for this t, we have

$$H(x^*, \phi'(x^*)) < tH(x^*, v'_+(x^*)) + (1 - t)H(x^*, v'_-(x^*)) \tag{19}$$

Substituting in $H(x^*, v'_+(x^*))$ and $H(x^*, v'_-(x^*))$ from (18) we have

$$H(x^*, \phi'(x^*)) < \rho v(x^*)$$

But this contradicts the subsolution condition (11).□

Remark 8: A symmetric argument shows that the viscosity solution of a minimization problem cannot have a convex kink. There are again two steps to the argument. The first step is that minimization problems have concave Hamiltonians (which we already noted as part of Remark 2 in Section A.1). And the second step is to show that viscosity solutions with concave Hamiltonians cannot have a convex kink.

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