Human Capital Model

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This note describes a deterministic human capital accumulation problem with one asset in continuous time. Agents can accumulate financial assets and human capital by dividing their time between labor and education. Human capital is accumulated via a DRS production function and depreciates over time. Labor income receives a wage per unit of time worked and per unit of human capital. The code is `human_capital.m`.

1 Model Setup

The household solves the following problem:

$$\max_{\{c_t,s_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt \quad s.t.$$

$$\dot{a}_t = r a_t + w h_t (1 - s_t) - c_t$$

$$\dot{h}_t = \theta (s_t h_t)^\alpha - \delta h_t$$

$$a_t \geq \underline{a}$$

Here a_t denotes wealth, h_t human capital, c_t consumption and s_t the time units spent on education. The interest rate is denoted by r and w denotes the wage. δ is the human capital depreciation rate and θ and α are the parameters of the human capital production function where $\theta > 0$ and $\alpha \in (0, 1)$. There is a lower bound on wealth denoted by $\underline{a}$. Utility is assumed to be CRRA with parameter σ

$$u(c) = \begin{cases} \log(c) & \text{if } \sigma = 1 \\ c^{1-\sigma}/(1-\sigma) & \text{otherwise} \end{cases}$$

2 Recursive formulation

2.1 HJB Equation

The HJB equation for the above problem is

$$\begin{aligned}\rho V(a,h) = \max_{c,s} \quad & u(c) + V_a(a,h)[ra + wh(1-s) - c] \\ & + V_h(a,h)[\theta(sh)^\alpha - \delta h]\end{aligned} \tag{1}$$

and the first order conditions follow:

$$u'(c) = V_a(a,h) \tag{2}$$

$$V_a(a,h)wh = V_h(a,h)[\theta\alpha s^{\alpha-1}h^\alpha] \tag{3}$$

2.2 State constraint

The boundary constraint $a_t \geq \underline{a}$ implies $r\underline{a}+wh(1-s(\underline{a},h))-c(\underline{a},h) \geq 0$ where $s(a,h)$ satisfies (3) evaluated at $\underline{a}$. Using the first order conditions yields $V_a(\underline{a},h) \geq u'(r\underline{a}+wh(1-s(\underline{a},h)))$. A similar condition is imposed on the upper end of the a space.

3 Numerical solution

3.1 Main loop

See `human_capital.m`. To solve the HJB equation (1). We are using an implicit upwind finite difference method. We discretize the state space denoting the grid points by $a_i, i = 1, \ldots, N_a$ and $h_j, j = 1, \ldots, N_h$, with $\Delta a_i^- = a_i - a_{i-1}$ and $\Delta a_i^+ = a_{i+} - a_i$ and similarly for the h-space. Finally

$$V_{i,j} \equiv V(a_i, h_j).$$

We approximate the derivatives of the value function with respect to a and h with either a forward or backward-difference approximation

$$\partial_a^F V_{i,j} = \frac{V_{i+1,j} - V_{i,j}}{\Delta a_i^+} \tag{4}$$

$$\partial_a^B V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{\Delta a_i^-} \tag{5}$$

$$\partial_h^F V_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{\Delta h_j^+} \tag{6}$$

$$\partial_h^B V_{i,j} = \frac{V_{i,j} - V_{i,j-1}}{\Delta h_i^-} \tag{7}$$

Now the discretized version of (1) is

$$\begin{aligned}\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} + \rho V_{i,j}^{n+1} = u(c_{i,j}^n) &+ \partial_a^F V_{i,j}^{n+1}\left[\mu_{a;i,j}^{FF}\mathbb{1}_{a;i,j}^{FF} + \mu_{a;i,j}^{FB}\mathbb{1}_{a;i,j}^{FB}\right] \\ &+ \partial_a^B V_{i,j}^{n+1}\left[\mu_{a;i,j}^{BF}\mathbb{1}_{a;i,j}^{BF} + \mu_{a;i,j}^{BB}\mathbb{1}_{a;i,j}^{BB}\right] \\ &+ \partial_h^F V_{i,j}^{n+1}\left[\mu_{h;i,j}^{FF}\mathbb{1}_{h;i,j}^{FF} + \mu_{h;i,j}^{BF}\mathbb{1}_{h;i,j}^{BF}\right] \\ &+ \partial_h^B V_{i,j}^{n+1}\left[\mu_{h;i,j}^{FB}\mathbb{1}_{h;i,j}^{FB} + \mu_{h;i,j}^{BB}\mathbb{1}_{h;i,j}^{BB}\right]\end{aligned} \tag{8}$$

and the first order conditions

$$u'\left(c_{i,j}^n\right) = \partial_a V_{i,j}^n \tag{9}$$

$$\partial_a V_{i,j}^n wh = \partial_h V_{i,j}^n [\theta \alpha s_{i,j}^{\alpha-1} h^\alpha] \tag{10}$$

where the μ_a and μ_b terms represent the drift of the endogenous state variables, with the convention that $\mu_a^{FB} = ra + wh(1 - s(a,h)) - c(a,h)$ is computed using the forward difference in the a dimension and the backward difference in the h dimension and so on. The indicator functions pick the combination of backward/forward differences consistent with the upwind scheme, e.g. $\mathbb{1}_{a;i,j}^{FB}$ equals to 1 if at the point (i,j) we have $\mu_a > 0$ and $\mu_h \leq 0$ when using the FB approximation.

Finally, given a guess for the value function and a choice of which finite difference approximation to choose, the first order conditions in (9) and (10) implicitly define the choice for $c_{i,j}^n$ and $s_{i,j}^n$.

Similarly to Achdou et al. (2017), given equation (4), the discretized value function can be put on a vector V^n of size $(N_a \times N_h) \times (1)$ and the equation can be stacked into

$$\frac{V^{n+1} - V^n}{\Delta} + \rho V^{n+1} = u^n + \mathbf{A}^n V^{n+1}$$

where the matrix $\mathbf{A}$ of size $(N_a \times N_h) \times (N_a \times N_h)$ picks up the terms in brackets in the first four lines of equation (4).

3.2 Imposing boundary conditions

Boundary conditions are imposed at $a_{min}, a_{max}, h_{min}$ and h_{max}, where those values are the upper and lower bounds of the state-space. For illustration we describe here the boundary conditions imposed at $a_{min} = \underline{a}$. We showed in section 2.2 that the derivative of the value function at $\underline{a}$ must satisfy $V_a(\underline{a}, h) \geq u'(r\underline{a} + wh(1 - s(\underline{a}, h)))$. When this expression holds with equality and using CRRA preferences, we have $r\underline{a} + wh(1 - s(\underline{a}, h)) = V_a(\underline{a}, h)^{-\frac{1}{\sigma}}$. Now, using equation (3) to get an expression for $s(a, h)$ we get the boundary condition

$$r\underline{a} + wh \left[1 - \left(\frac{\theta\alpha}{w} \frac{V_h}{V_h} h^{\alpha-1} \right)^{\frac{1}{1-\alpha}} \right] = V_a(\underline{a}, h)^{-\frac{1}{\sigma}}$$

Now, for each $j = 1, \ldots, N_h$, we solve for $\partial_a^B V_{1,j}$ in the nonlinear equation

$$ra_1 + wh_j \left[1 - \left(\frac{\theta\alpha}{w} \frac{\partial_h^F V_{1,j}}{\partial_a^B V_{1,j}} h_j^{\alpha-1} \right)^{\frac{1}{1-\alpha}} \right] = \partial_a^B V_{1,j}^{-\frac{1}{\sigma}} \tag{11}$$

Having obtained $\partial_a^B V_{1,j}$ we compute $\mu_{h;1,j}^{BF}$. If $\mu_{h;1,j}^{BF} \geq 0$ we update $\mu_{a;1,j}^{BF}$ and $\mu_{h;1,j}^{BF}$. Otherwise, we solve equation (11) using $\partial_h^B V_{1,j}$ instead of $\partial_h^F V_{1,j}$ and update $\mu_{a;1,j}^{BB}$ and $\mu_{h;1,j}^{BB}$ if the resulting $\mu_{h;1,j}^{BB}$ is non-positive. If neither $\mu_{h;i,j}^{BF} \geq 0$ nor $\mu_{h;i,j}^{BB} \leq 0$, then the drift values are set to zero.

4 Results

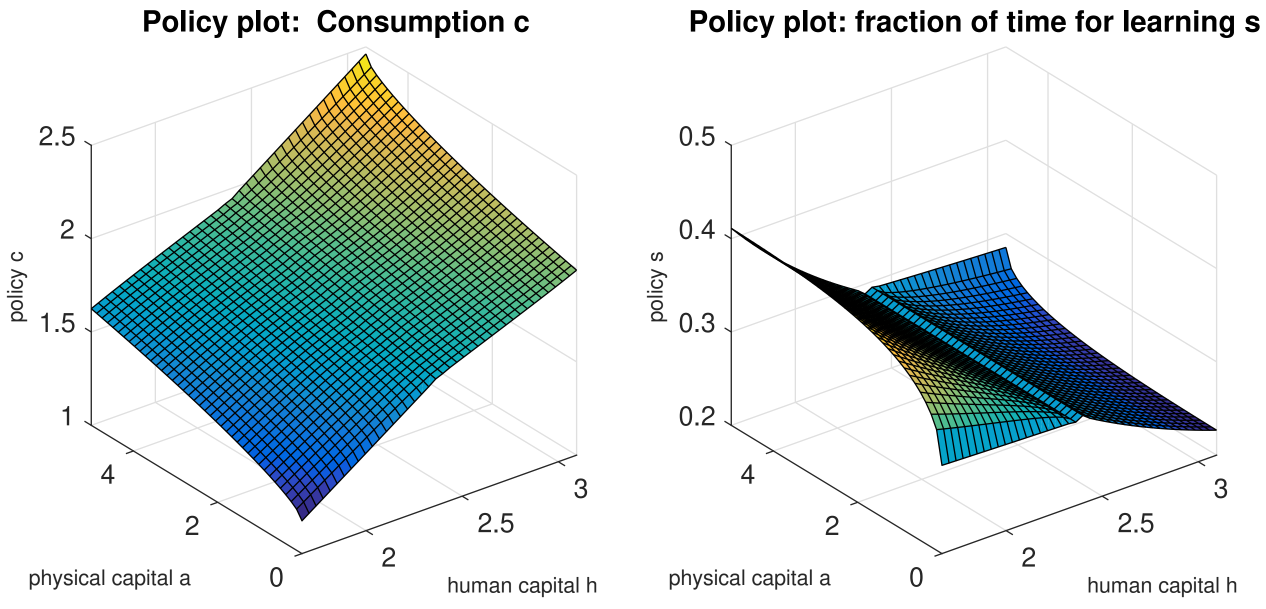

Figure 1: Policy Functions

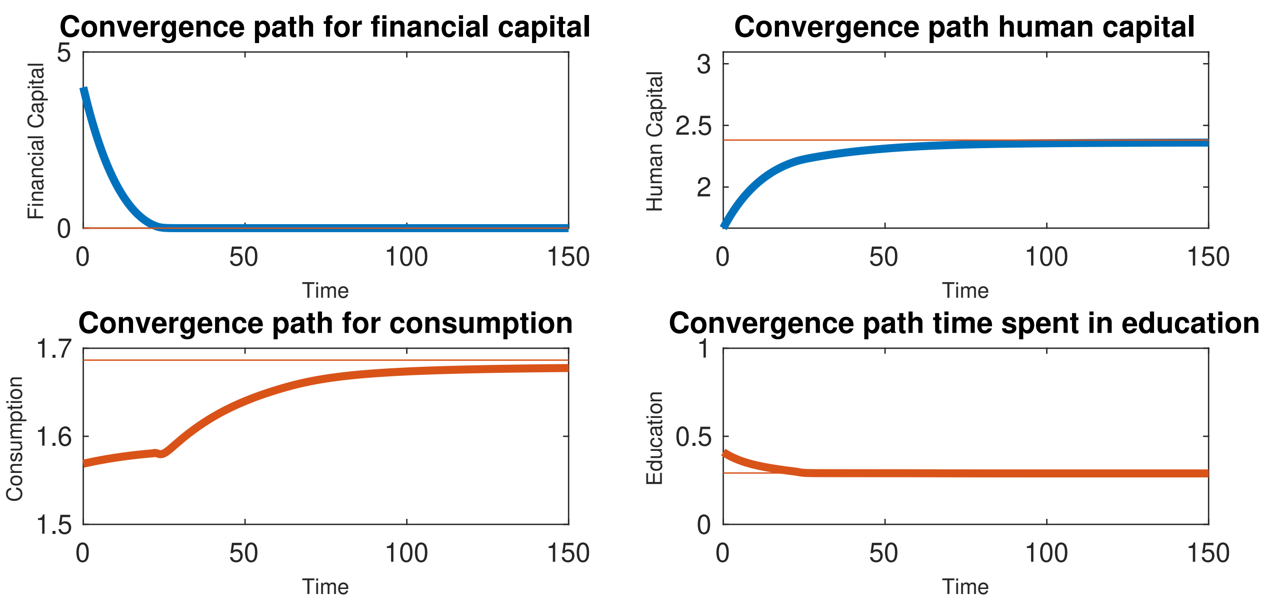

Figure 2: Transition Dynamics