Viscosity Solutions for Dummies
(including Economists)

Online Appendix to

"Income and Wealth Distribution in Macroeconomics:
A Continuous-Time Approach"

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Viscosity Solutions

For our purposes, useful for two reasons:

1. problems with kinks, e.g. coming from non-convexities
2. problems with "state constraints"
    - borrowing constraints
    - computations with bounded domain

Things to remember

1. viscosity solution $\Rightarrow$ no concave kinks (convex kinks are allowed)
2. uniqueness: HJB equations have unique viscosity solution
3. constrained viscosity solution: "boundary inequalities"

References

Relatively accessible ones:

- Lions (1983) "Hamilton-Jacobi-Bellman Equations and the Optimal Control of Stochastic Systems" – report of some results, no proofs

  `http://www.mathunion.org/ICM/ICM1983.2/Main/icm1983.2.1403.1418.ocr.pdf`
- Crandall (1995) "Viscosity Solutions: A Primer"

  `http://www.princeton.edu/~moll/crandall-primer.pdf`
- Bressan (2011) "Viscosity Solutions of Hamilton-Jacobi Equations and Optimal Control Problems"

  `https://www.math.psu.edu/bressan/PSPDF/hj.pdf`
- Yu (2011) `http://www.math.ualberta.ca/~xinweiyu/527.1.08f/lec11.pdf`

Less accessible but often cited:

- Crandall, Ishii and Lions (1992) "User's Guide to Viscosity Solutions"
- Bardi and Capuzzo-Dolcetta (2008)
  `https://www.dropbox.com/s/hsihfm8xwnnncvy/bardi-capuzzo-wholebook.pdf?dl=0`
- Fleming & Soner (2006)
  `https://www.dropbox.com/s/wbekmg2icp5u9i1/fleming-soner-wholebook.pdf?dl=0`

Outline

1. Definition of viscosity solution
2. No concave kinks
3. Constrained viscosity solutions
4. Uniqueness

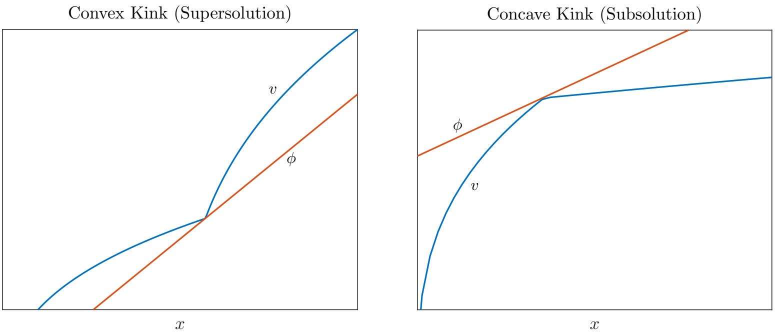
Viscosity Solutions: Definition

Consider HJB for generic optimal control problem

$$\rho v(x) = \max_{\alpha \in A} \left\{ r(x,\alpha) + v'(x) f(x,\alpha) \right\} \qquad \text{(HJB)}$$

- Next slide: definition of viscosity solution
- Basic idea: v may have kinks i.e. may not be differentiable
- replace $v'(x)$ at point where it does not exist (because of kink in v) with derivative of smooth function ϕ touching v
- two types of kinks: concave and convex $\Rightarrow$ two conditions
  - concave kink: ϕ touches v from above
  - convex kink: ϕ touches v from below
- Remark: definition allows for concave kinks. In a few slides: these never arise in maximization problems

Convex and Concave Kinks



Viscosity Solutions: Definition

Definition: A viscosity solution of (HJB) is a continuous function v such that the following hold:

1. (Subsolution) If ϕ is any smooth function and if $v - \phi$ has a local maximum at point x^* (v may have a concave kink), then

$$\rho v(x^*) \leq \max_{\alpha \in A} \left\{ r(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha) \right\}$$

2. (Supersolution) If ϕ is any smooth function and if $v - \phi$ has a local minimum at point x^* (v may have a convex kink), then

$$\rho v(x^*) \geq \max_{\alpha \in A} \left\{ r(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha) \right\}.$$

A few remarks, terminology

- If v is differentiable at x^*, then
  - local max or min of $v - \phi$ implies $v'(x^*) = \phi'(x^*)$
  - sub- and supersolution conditions $\Rightarrow$ viscosity solution of (HJB) is just classical solution
- If a continuous function v satisfies condition 1 (but not necessarily 2) we say that it is a "viscosity subsolution"
- Conversely, if v satisfies condition 2 (but not necessarily 1), we say that it is a "viscosity supersolution"
- $\Leftrightarrow$ a continuous function v is a "viscosity solution" if it is both a "viscosity subsolution" and a "viscosity supersolution."
- "subsolution" and "supersolution" come from ≤ 0 and ≥ 0
- "viscosity" is in honor of the "method of vanishing viscosity": add Brownian noise and $\rightarrow 0$ (movements in viscuous fluid)

Viscosity Solutions: Intuition

- Consider discrete time Bellman:

$$v(x) = \max_{\alpha} \ r(x, \alpha) + \beta v(x'), \quad x' = f(x, \alpha)$$

- Think about it as an operator T on function v

$$(Tv)(x) = \max_{\alpha} \ r(x, \alpha) + \beta v(f(x, \alpha))$$

- Solution = fixed point: $Tv = v$
- Intuitive property of T ("monotonicity")

$$\phi(x) \leq v(x) \quad \forall x \quad \Rightarrow \quad (T\phi)(x) \leq (Tv)(x) \quad \forall x \qquad (*)$$

- Intuition: if my continuation value is higher, I'm better off
- Viscosity solution is exactly same idea
- Key idea: sidestep non-differentiability of v by using "monotonicity"

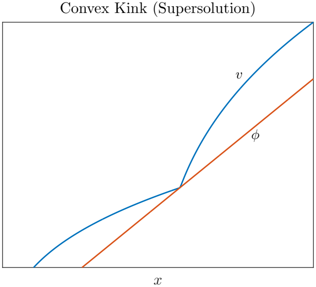
Viscosity Solutions: Heuristic Derivation

- Time periods of length Δ. Consider HJB equation

$$v(x_t) = \max_{\alpha} \Delta r(x_t, \alpha) + (1 - \rho\Delta) v(x_{t+\Delta}) \quad \text{s.t.}$$

$$x_{t+\Delta} = \Delta f(x_t, \alpha) + x_t$$

- Suppose v is not differentiable at x^* and has a convex kink
  - problem: when taking $\Delta \to 0$, pick up derivative v'
  - solution: replace continuation value with smooth function ϕ



Viscosity Solutions: Heuristic Derivation

- For now: consider ϕ such that $\phi(x^*) = v(x^*)$
  - local min of $v - \phi$ and $\phi(x^*) = v(x^*) \Rightarrow v(x) > \phi(x), x \neq x^*$
- Then for $x_t = x^*$, we have

$$v(x_t) \geq \max_{\alpha} \; \Delta r(x_t, \alpha) + (1 - \rho\Delta)\phi(x_{t+\Delta})$$

- Subtract $(1 - \rho\Delta)\phi(x_t)$ from both sides and use $\phi(x_t) = v(x_t)$

$$\Delta\rho v(x_t) \geq \max_{\alpha} \Delta r(x_t, \alpha) + (1 - \rho\Delta)(\phi(x_{t+\Delta}) - \phi(x_t))$$

- Dividing by Δ and letting $\Delta \to 0$ yields the supersolution condition

$$\rho v(x_t) \geq \max_{\alpha} r(x_t, \alpha) + \phi'(x_t) f(x_t, \alpha)$$

Viscosity Solutions: Heuristic Derivation

- Turns out this works for any ϕ such that $v - \phi$ has local min at x^*
  - define $\kappa = v(x^*) - \phi(x^*)$. Then $v(x^*) = \phi(x^*) + \kappa$ and
  $$v(x_t) \geq \max_{\alpha} \Delta r(x_t, \alpha_t) + (1 - \rho\Delta)(\phi(x_{t+\Delta}) + \kappa)$$
  - subtract $(1 - \rho\Delta)(\phi(x_t) + \kappa)$ from both sides
  $$\Delta\rho v(x_t) \geq \max_{\alpha} \Delta r(x_t, \alpha) + (1 - \rho\Delta)(\phi(x_{t+\Delta}) - \phi(x_t))$$
  - rest is the same...
- Derivation of subsolution condition exactly symmetric

Viscosity Solution: Nonstandard but Intuitive Definition

Monotonicity logic: if my continuation value is higher, I'm better off

Suggests alternative definition that is less standard but more intuitive and in particular let's you remember which way $\leq$ and $\geq$ go

Definition: A viscosity solution of (HJB) is a continuous function v such that the following hold:

1. (Subsolution) If ϕ is any smooth function that touches v from above at point x^* (v may have a concave kink & "ϕ is better than v"), then

$$\rho v(x^*) \leq \max_{\alpha \in A} \left\{ r(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha) \right\}$$

Intuition: analogue of $\phi \geq v \Rightarrow T\phi \geq Tv$ for discrete-time Bellman

2. (Supersolution) If ϕ is any smooth function that touches v from below at point x^* (v may have a convex kink & "ϕ is worse than v"), then

$$\rho v(x^*) \geq \max_{\alpha \in A} \left\{ r(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha) \right\}$$

Intuition: analogue of $\phi \leq v \Rightarrow T\phi \leq Tv$ for discrete-time Bellman

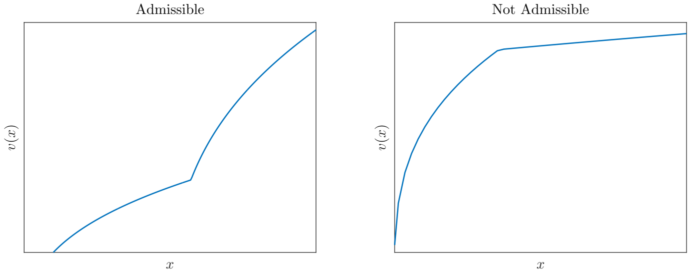
Viscosity + maximization ⇒ no concave kinks

Proposition

The viscosity solution of the maximization problem

$$\rho v(x) - \max_{\alpha \in A} \left\{ r(x,\alpha) + v'(x) f(x,\alpha) \right\} = 0$$

only admits convex (downward) kinks, but not concave (upward) kinks.



- opposite would be true for minimization problem

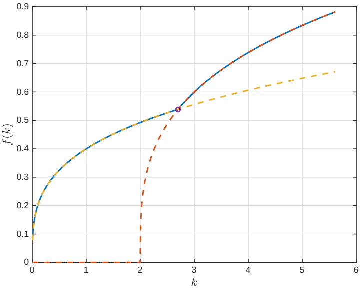
Why is this useful? Problems with non-covexities

- Consider growth model

$$\rho v(k) = \max_c \; u(c) + v'(k)(F(k) - \delta k - c).$$

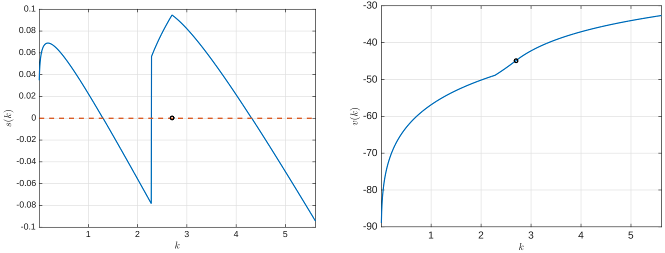
- But drop assumption that F is strictly concave. Instead: "butterfly"

$$F(k) = \max\{F_L(k), F_H(k)\},$$
$$F_L(k) = A_L k^\alpha,$$
$$F_H(k) = A_H((k-\kappa)^+)^\alpha, \quad \kappa > 0, \; A_H > A_L$$



Why is this useful? Problems with non-covexities

- Suppose restrict attention to solutions v with at most one kink
    - there is solution we found in Lecture 3 with convex kink



(a) Saving Policy Function (b) Value Function

-
    - but there is another solution with a concave kink!
- Proposition above rules out second solution

Proof that no concave kinks

- Step 1: maximization problem $\Rightarrow$ convex Hamiltonian
- Step 2: convex Hamiltonian $\Rightarrow$ no concave kinks
- Proof of Step 1: Hamiltonian is

$$H(x,p) := \max_{\alpha \in A} \{r(x,\alpha) + pf(x,\alpha)\}$$

- first- and second-order conditions:

$$r_\alpha + pf_\alpha = 0 \qquad \text{(FOC)}$$

$$r_{\alpha\alpha} + pf_{\alpha\alpha} \leq 0 \qquad \text{(SOC)}$$

- From (FOC)

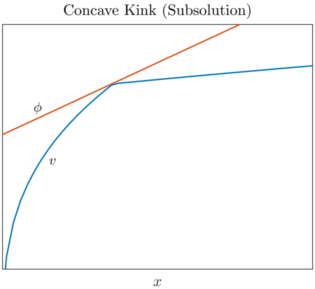
$$\alpha_p = -\frac{f_\alpha}{(r_{\alpha\alpha} + pf_{\alpha\alpha})}$$

- Differentiating Hamiltonian, we have $H_p(x,p) = f(x,\alpha(x,p))$ and

$$H_{pp} = f_\alpha \alpha_p = -\frac{(f_\alpha)^2}{(r_{\alpha\alpha} + pf_{\alpha\alpha})} > 0$$

Proof that no concave kinks (step 2)

- Proof by contradiction: suppose concave kink at x^*
- Then v, ϕ in subsolution condition look like in figure



- Note: left & right derivatives satisfy

$$\partial_- v(x^*) < \phi'(x^*) < \partial_+ v(x^*)$$

- Hence there exists $t \in (0,1)$ such that

$$\phi'(x^*) = t\partial_+ v(x^*) + (1-t)\partial_- v(x^*)$$

Proof that no concave kinks (step 2)

- Because H is convex, for t defined on previous slide

$$H(x^*, \phi'(x^*)) < tH(x^*, \partial_+ v'(x^*)) + (1-t)H(x^*, \partial_- v'(x^*)) \quad (*)$$

- By continuity of v

$$\rho v(x^*) = H(x^*, \partial_+ v(x^*)),$$
$$\rho v(x^*) = H(x^*, \partial_- v(x^*))$$

- Therefore $(*)$ implies

$$H(x^*, \phi'(x^*)) < \rho v(x^*)$$

- But this contradicts the subsolution condition

$$\rho v(x^*) \leq H(x^*, \phi'(x^*)). \quad \square$$

Something to Keep in Mind

- Viscosity solution can handle problems where v has kinks...
- ... but not discontinuities
- Theory can be extended in special cases, but no general theory of discontinuous viscosity solutions
- In economics, kinks seem more common than discontinuities
- Good reference on HJB equations with discontinuities: Barles and Chasseigne (2015) "(Almost) Everything You Always Wanted to Know About Deterministic Control Problems in Stratified Domains"
  `http://arxiv.org/abs/1412.7556`

Viscosity Solutions with State Constraints

Constrained Viscosity Soln: Boundary Inequalities

- How handle "state constraints"?
  - borrowing constraints
  - computations with bounded domain
- Example: growth model with state constraint

$$v(k_0) = \max_{\{c(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c(t)) dt \quad \text{s.t.}$$

$$\dot{k}(t) = F(k(t)) - \delta k(t) - c(t)$$

$$k(t) \geq k_{\min} \text{ all } t \geq 0$$

- purely pedagogical: constraint will never bind if $k_{\min} <$ st.st.
- HJB equation

$$\rho v(k) = \max_c \left\{ u(c) + v'(k)(F(k) - \delta k - c) \right\} \tag{HJB}$$

- Key question: how impose $k \geq k_{\min}$?

Example: Growth Model with Constraint

- Result: if v is (left-)differentiable at k_{min}, it needs to satisfy

$$v'(k_{\text{min}}) \geq u'(F(k_{\text{min}}) - \delta k_{\text{min}}) \qquad \text{(BI)}$$

- Intuition:
  - $v'(k_{\text{min}})$ is such that if $k(t) = k_{\text{min}}$ then $\dot{k}(t) \geq 0$
  - if v is differentiable, the FOC still holds at the constraint

$$u'(c(k_{\text{min}})) = v'(k_{\text{min}}) \qquad \text{(FOC)}$$

  - for constraint not to be violated, need

$$F(k_{\text{min}}) - \delta k_{\text{min}} - c(k_{\text{min}}) \geq 0 \qquad (*)$$

  - (FOC) and $(*)$ $\Rightarrow$ (BI).

- Next: state constraints in generic optimal control problem

Generic Control Problem with State Constraint

- Consider variant of generic maximization problem

$$v(x) = \max_{\{\alpha(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} r(x(t), \alpha(t)) dt \quad \text{s.t.}$$

$$\dot{x}(t) = f(x(t), \alpha(t)), \quad x(0) = x$$

$$x(t) \geq x_{\min} \quad \text{all } t \geq 0$$

- HJB equation

$$\rho v(x) = \max_{\alpha \in A} \{ r(x, \alpha) + v'(x) f(x, \alpha) \} \tag{HJB}$$

- Question: how impose $x \geq x_{\min}$? Two cases:

1. v is (left-)differentiable at $x_{\min}$: boundary inequality
2. v not differentiable at $x_{\min}$: "constrained viscosity solution"

State constraints if v is differentiable

- Use Hamiltonian formulation

$$\rho v(x) = H(x, v'(x)) \tag{HJB}$$

$$H(x,p) := \max_{\alpha \in A} \{r(x,\alpha) + pf(x,\alpha)\} \tag{H}$$

- From envelope condition

$$H_p(x, v'(x)) = f(x, \alpha^*(x)) = \text{optimal drift at } x$$

- If v' and H_p exist, (HJB) for generic control problem satisfies

$$H_p(x_{\min}, v'(x_{\min})) \geq 0 \tag{BI}$$

- see Soner (1986, p.553) and Fleming and Soner (2006, p.108)
  - write state constraint as $f(x, \alpha^*(x)) \cdot \nu(x) \leq 0$ for x at boundary where $\nu(x) =$ "outward normal vector" at boundary
  - show this implies $H_p(x, \nabla v(x)) \cdot \nu(x) \geq 0$

If v not differentiable: constrained viscosity solution

- Setting $H_p(x_{\min}, v'(x_{\min})) \geq 0$ obviously requires v to be (left-)differentiable $\Rightarrow$ what if not differentiable at $x_{\min}$?
- Definition: a constrained viscosity solution of (HJB) is a continuous function v such that
  1. v is a viscosity solution (i.e. both sub- and supersolution) for all $x > x_{\min}$
  2. v is a subsolution at $x = x_{\min}$: if ϕ is any smooth function and if $v - \phi$ has a local maximum at point $x_{\min}$, then

$$\rho v(x_{\min}) \leq \max_{\alpha \in A} \left\{ r(x_{\min}, \alpha) + \phi'(x_{\min}) f(x_{\min}, \alpha) \right\} \qquad (*)$$

- $(*)$ functions as boundary condition, or rather "boundary inequality"
- Note: minimization $\Rightarrow$ opposite, i.e. supersolution at boundary
  - for minimization: Fleming and Soner (2006), Section II.12
  - for maximization: e.g. Definition 3.1 in Zariphopoulou (1994)

Intuition why subsolution on boundary

- Follow Fleming and Soner (2006, p.108) with signs switched
- Lemma: If v is diff'ble at $x_{\min}$, then $f(x_{\min}, \alpha^*(x_{\min})) \geq 0$ implies

$$H(x_{\min}, p) \geq H(x_{\min}, v'(x_{\min})), \quad \text{for all} \quad p \geq v'(x_{\min}) \qquad \text{(BI')}$$

- Proof: for any $p \leq v'(x_{\min})$

$$\begin{aligned} H(x_{\min}, p) &= \max_{\alpha \in A} \{h(x_{\min}, \alpha) + pf(x_{\min}, \alpha)\} \\ &\geq r(x_{\min}, \alpha^*(x_{\min})) + pf(x_{\min}, \alpha^*(x_{\min})) \\ &\geq r(x_{\min}, \alpha^*(x_{\min})) + v'(x_{\min})f(x_{\min}, \alpha^*(x_{\min})) \\ &= H(x_{\min}, v'(x_{\min})). \square \end{aligned}$$

- Remark: (BI') implies (BI), i.e. $H_p(x_{\min}, v'(x_{\min})) \geq 0$

Intuition why subsolution on boundary

- By continuity of v and v':

$$\rho v(x_{\min}) = H(x_{\min}, v'(x_{\min}))$$

- Combining with (BI'):

$$\rho v(x_{\min}) \leq H(x_{\min}, p) \quad \text{for all} \quad p \geq v'(x_{\min})$$

- Subsolution condition same statement without differentiability
  - if v (left-)differentiable: local max of $v - \phi$ at $x_{\min}$ $\Leftrightarrow v'(x_{\min}) \leq \phi'(x_{\min})$ (not $=$ bc/ corner)
  - but also applies if v not differentiable

Example: Growth Model with Bounded Domain

- Consider again HJB equation for growth model

$$\rho v(k) = \max_c \left\{ u(c) + v'(k)(F(k) - \delta k - c) \right\}$$

- For numerical solution, want to impose

$$k_{\min} \leq k(t) \leq k_{\max} \quad \text{all } t$$

- How can we ensure this?

- Answer: impose two boundary inequalities

$$v'(k_{\min}) \geq u'(F(k_{\min}) - \delta k_{\min})$$

$$v'(k_{\max}) \leq u'(F(k_{\max}) - \delta k_{\max})$$

Uniqueness of Viscosity Solution

Uniqueness of Viscosity Solution

- Theorem: Under some conditions, HJB equation has unique viscosity solution
- due to Crandall and Lions (1983), "Viscosity Solutions of Hamilton-Jacobi Equations"
- My intuition for uniqueness theorem with state constraints in one dimension
  - ODE has unique solution given one boundary condition
  - two boundary inequalities = one boundary condition ...
  - ... sufficient to pin down unique solution
- but much more powerful: generalizes to N dimensions, kinks etc

Intuition for Uniqueness: Boundary Inequalities

- Consider toy problem with explicit solution

$$v(x) = \max_{\{\alpha(t)\}_{t\geq 0}} \int_0^\infty e^{-t} \left(-3x(t)^2 - \frac{1}{2}\alpha(t)^2 \right) dt,$$

$$\dot{x}(t) = \alpha(t) \quad x(0) = x,$$

  and with state constraints $x(t) \in [x_{\min}, x_{\max}]$

- HJB equation

$$v(x) = \max_{\alpha} \left\{ -3x^2 - \frac{1}{2}\alpha^2 + v'(x)\alpha \right\}$$

- or maximizing out α using $\alpha = v'(x)$

$$v(x) = -3x^2 + \frac{1}{2}(v'(x))^2 \qquad \text{(HJB)}$$

- "Correct" solution is (verify: $-x^2 = -3x^2 + \frac{1}{2}(-2x)^2$)

$$v(x) = -x^2$$

- Respects state constraints, e.g. $\dot{x} = v'(x) = -2x < 0$ if $x > 0$

Intuition for Uniqueness: Boundary Inequalities

- $\dot{x}(t) = v'(x(t)) \Rightarrow$ natural state constraint boundary conditions

$$v'(x_{\min}) \geq 0, \qquad v'(x_{\max}) \leq 0 \tag{BI}$$

- ensure that $\dot{x} \geq 0$ at $x = x_{\min}$ and $\dot{x} \leq 0$ at $x = x_{\max}$
- Result: $v(x) = -x^2$ is the only continuously differentiable solution of (HJB) which satisfies the boundary inequalities (BI)
  - proof on next slide
- Result is striking because
  - inequalities (BI) strong enough to pin down unique solution of (HJB) even though, at correct solution $v(x) = -x^2$, neither holds with equality
  - boundary inequalities pin down unique solution even though this solution "does not see the boundaries"

Proof of Result

- Let v be smooth solution of (HJB). Let $v(0) = c$. Rule out $c \neq 0$
- Easy to rule out $c < 0$: (HJB) at $x = 0$ is $c = \frac{1}{2}(v'(0))^2 \Rightarrow$ no solution for $v'(0)$ when $c < 0$
- Next consider $c > 0$. Inverting (HJB) for v' yields two branches

$$\text{Branch 1:} \quad v'(x) = +\sqrt{6x^2 + 2v(x)}$$
$$\text{Branch 2:} \quad v'(x) = -\sqrt{6x^2 + 2v(x)}$$

- Importantly, continuous $v' \Rightarrow v$ satisfies either branch 1 or branch 2 for all $x \in (x_{min}, x_{max})$, i.e. cannot switch branches
  - in particular $c > 0 \Rightarrow$ switching branches not allowed at $x = 0$
- Suppose v satisfies branch 1 for all x. Then
  - $v'(0) = \sqrt{2c} > 0$ and $v(x) > c$ for $x > 0$ near $x = 0$
  - Similarly $v(x) > c$, $v'(x) > 0$ all $x > 0 \Rightarrow$ violate $v'(x_{max}) \leq 0$
- Suppose v satisfies branch 2 for all x. Then $v'(0) = -\sqrt{2c} < 0$ ...
  - ... $v(x) > c$, $v'(x) < 0$ all $x < 0 \Rightarrow$ violate $v'(x_{min}) \geq 0$ $\square$

Remark: strengthening the result

- Have restricted attention to solutions such that v' is continuous
- In fact, result can be strengthened further to say
- Result 2: $v(x) = -x^2$ is the only solution of (HJB) without concave kinks ("viscosity solution") which satisfies boundary inequalities (BI)

Uniqueness Theorem: Logic

- Here outline case with state constraints: $x \in [x_{\min}, x_{\max}]$
  - due to Soner (1986), Capuzzo-Dolcetta & Lions (1990)
  - use notation $X := (x_{\min}, x_{\max})$ and $\overline{X} := [x_{\min}, x_{\max}]$
  - can extend to unbounded domain $x \in \mathbb{R}$ (references later)

- Key step in uniqueness proof: "comparison theorem"
- Consider HJB equation

$$\rho v(x) = \max_{\alpha \in A} \left\{ r(x, \alpha) + v'(x) f(x, \alpha) \right\} \qquad \text{(HJB)}$$

  with state constraints $x \in \overline{X}$

- **Comparison theorem**: (Under certain assumptions...) if v_1 is subsolution of (HJB) on $\overline{X}$ & v_2 is supersolution of (HJB) on X, then

$$v_1(x) \leq v_2(x) \quad \text{for all } x \in \overline{X}.$$

Remark: another comparison theorem you may know

- Here's another comparison theorem you may know that has same structure: Grönwall's inequality
  - If $v_1'(t) \leq \beta(t) v_1(t)$ then $v_1(t) \leq v_1(0) \exp\left(\int_0^t \beta(s) ds\right)$
  - This is really: If $v_1'(t) \leq \beta(t) v_1(t)$ and $v_2'(t) = \beta(t) v_2(t)$, then

$$v_1(t) \leq v_2(t) \quad \text{for all } t$$

Comparison Theorem immediately $\Rightarrow$ Uniqueness

- Corollary (uniqueness): there exists a unique constrained viscosity solution of (HJB) on $\overline{X}$, i.e. if v_1 and v_2 are both constrained viscosity solutions of (HJB) on $\overline{X}$, then $v_1(x) = v_2(x)$ for all $x \in \overline{X}$
- Proof: let v_1, v_2 be constrained viscosity solutions of (HJB) on $\overline{X}$

  1. Since v_1 and v_2 are constrained viscosity solutions, v_1 is also a subsolution on $\overline{X}$ and v_2 is a supersolution on X. By the comparison theorem, therefore $v_1(x) \leq v_2(x)$ for all $x \in \overline{X}$
  2. Reversing roles of v_1, v_2 in (1) $\Rightarrow$ $v_2(x) \leq v_1(x)$ for all $x \in \overline{X}$
  3. (1) and (2) imply $v_1(x) = v_2(x)$ for all $x \in \overline{X}$, i.e. uniqueness.$\square$

Proof Sketch of Comparison Thm with smooth v_1, v_2

- **Thm**: v_1 = subsolution, v_2 = supersolution $\Rightarrow v_1(x) \leq v_2(x)$ all x
- **Proof** by contradiction: suppose instead $v_1(x) > v_2(x)$ for some x
- ... equivalently, $v_1 - v_2$ attains local maximum at some point $x^* \in \overline{X}$ with $v_1(x^*) > v_2(x^*)$
- Two cases:
  1. x^* in interior: $x^* \in X$
  2. x^* on boundary: $x^* = x_{\min}$ or $x^* = x_{\max}$
- Here: ignore case 2, i.e. possibility of x^* on boundary
  - requires more work, just like non-smooth v_1, v_2
  - for complete proof, see references

Proof Sketch of Comparison Thm with smooth v_1, v_2

- If $v_1 - v_2$ attains local maximum at interior x^*, then since v_1, v_2 are smooth also $v_1'(x^*) = v_2'(x^*)$
- v_1 is subsolution means: for any smooth ϕ and x^* such that $v_1 - \phi$ attains local max: $\rho v_1(x^*) \leq \max_{\alpha \in A}\{r(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha)\}$
- In particular, use $\phi = v_2$:

$$\rho v_1(x^*) \leq \max_{\alpha \in A}\left\{r(x^*, \alpha) + v_2'(x^*) f(x^*, \alpha)\right\} \tag{1}$$

- Repeat symmetric steps with supersolution condition for v_2, setting $\phi = v_1$ and noting that $v_2 - v_1$ attains local min at x^*

$$\rho v_2(x^*) \geq \max_{\alpha \in A}\left\{r(x^*, \alpha) + v_1'(x^*) f(x^*, \alpha)\right\} \tag{2}$$

- Subtracting (2) from (1) and recalling $v_1'(x^*) = v_2'(x^*)$ we have $v_1(x^*) \leq v_2(x^*)$. Contradiction.$\square$

Proof Sketch of Comparison Thm, non-smooth v_1, v_2

- Challenge: can no longer use $\phi = v_2$ in subsolution condition because not differentiable (and similarly for $\phi = v_1$)
- Use clever trick to overcome this: use two-dimensional extension $v_2(y) + \frac{1}{2\varepsilon}|x-y|^2$
  - key: $v_2(y) + \frac{1}{2\varepsilon}|x-y|^2$ is smooth as function of x
  - name of trick: "doubling of variables"
- For complete proof in state-constrained case, see
  - Soner (1986), Theorem 2.2
  - Capuzzo-Dolcetta and Lions (1990), Theorem III.1
  - Bardi and Capuzzo-Dolcetta (2008), Theorem IV.5.8

Variants on Comparison Theorem (still $\Rightarrow$ Uniqueness)

1. Known values of v at $x_{\min}, x_{\max}$ (“Dirichlet boundary conditions”)
    - simplest case and the one covered in most textbooks, notes
    - kind of uninteresting, mostly useful for understanding proof strategy
    - Yu (2011), Section 3
    - Bressan (2011), Theorem 5.1

2. Unbounded domain $x \in \mathbb{R}$
    - just like in discrete time, need certain boundedness assumptions on v_1, v_2 (e.g. at most linear growth)
    - Sections 5 and 6 of Crandall (1995)
    - Theorem III.2.12 in Bardi & Capuzzo-Dolcetta (2008)

Additional Results on HJB Equation in Economics

- Strulovici and Szydlowski (2015) "On the smoothness of value functions and the existence of optimal strategies in diffusion models"
  - very nice analysis of one-dimensional case (note: does not apply to $N > 1$)
  - provide conditions under which v is twice differentiable...
  - ... in which case whole viscosity apparatus not needed
  - no uniqueness result...
  - ... but they note that uniqueness implied because v=solution to "sequence problem" which has unique solution