Supplement to Lectures 5 and 6

Spectral Approach to Distributional Dynamics

(based on EF&G discussion of Alvarez-Lippi in March 2019)

Distributional Macroeconomics

Part II of ECON 2149

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Purpose of these Notes

- Explain "eigenvalue-eigenfunction approach" or "spectral approach" to distributional dynamics

- Used in two recent papers

  1. Gabaix-Lasry-Lions-Moll "The Dynamics of Inequality" (Ecma, 2016)

  2. Alvarez & Lippi (2019) "The Analytic Theory of a Monetary Shock"

  These notes are based on a discussion of Alvarez-Lippi at the NBER EF&G Meeting in San Francisco on March 1, 2019

Overview

Key message: even though GLLM, Alvarez-Lippi study environment w

- continuous time t
- continuous state x

conceptually everything is the same as with discrete time & states

Approach has two steps:

1. express transition dynamics in terms of eigenvalues & eigenvectors/functions
2. analytic solution for these

Plan for Explanation

1. Discrete time, discrete states
2. Continuous time, discrete states
3. Continuous time, continuous states

Again, point is: it's all the same!

1. Discrete time, discrete states

- $x_{it} \in \{x_1, ..., x_N\} \Rightarrow$ distribution = vector $\mathbf{p}_t \in \mathbb{R}^N$ (histogram)

- Dynamics of distribution

$$\mathbf{p}_{t+1} = \mathbf{A}^\top \mathbf{p}_t,$$

  where $\mathbf{A}$ = $N \times N$ transition matrix

- Example: symmetric two-state process, $\mathbf{A} = \begin{bmatrix} 1-\phi & \phi \\ \phi & 1-\phi \end{bmatrix}$

- Stationary distribution solves

$$\mathbf{p}_\infty = \mathbf{A}^\top \mathbf{p}_\infty$$

  i.e. eigenvector corresponding to unit eigenvalue: $\lambda \mathbf{v} = \mathbf{A}^\top \mathbf{v}, \lambda = 1$

- But what about transition dynamics? Spectral approach

1. Discrete time, discrete states

- Spectral approach to distributional dynamics
  - assume $\mathbf{A}^\top$ is diagonalizable
  - denote eigenvalues by $\lambda_1 > \lambda_2 > \ldots > \lambda_N$
  - corresponding eigenvectors by $\mathbf{v}_1, \ldots, \mathbf{v}_N$

Result: can write $\mathbf{p}_0 = \sum_{j=1}^{N} b_j \mathbf{v}_j$ and hence $\mathbf{p}_t = \sum_{j=1}^{N} \lambda_j^t b_j \mathbf{v}_j$

- Two-state example from previous slide: $\lambda_1 = 1$ and $\lambda_2 = 1 - 2\phi$

$$\Rightarrow \quad \mathbf{p}_t = b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1-2\phi)^t b_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Similarly, IRFs for moments of distribution

$$H_t := \sum_{i=1}^{N} f(x_i) p_{it} = \mathbf{f}^\top \mathbf{p}_t = \sum_{j=1}^{N} \lambda_j^t b_j \left(\mathbf{f}^\top \mathbf{v}_j\right)$$

1. Discrete time, discrete states

- Spectral approach to distributional dynamics
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- Similarly, IRFs for moments of distribution

$$H_t = \sum_{j=1}^{N} \lambda_j^t b_j[\mathbf{p}_0] b_j[\mathbf{f}], \quad b_j[\mathbf{f}] := \mathbf{f}^\top \mathbf{v}_j$$

Discrete analogue of Alvarez-Lippi's main formula

Quick summary so far

Approach has two steps:

1. express transition dynamics in terms of eigenvalues & eigenvectors
    - very general

2. analytic solution for these
    - only works in particular cases, e.g. two-state example

Will come back this...

2. Continuous time, discrete states

Assume process for x_{it} = finite-state Poisson process

Everything the same except

$$\dot{\mathbf{p}}(t) = \mathbf{A}^\top \mathbf{p}(t)$$

$$\lambda_j \leq 0$$

$$\mathbf{p}(t) = \sum_{j=1}^{N} e^{\lambda_j t} b_j \mathbf{v}_j$$

$$H(t) = \sum_{j=1}^{N} e^{\lambda_j t} b_j[\mathbf{p}_0] b_j[\mathbf{f}]$$

3. Continuous time, continuous states

Now suppose x is continuous rather than discrete

Everything still the same but need a bit of new vocabulary:

- vector $\mathbf{p} \Leftrightarrow$ function p
- matrix $\mathbf{A} \Leftrightarrow$ (linear) operator $\mathcal{A}$
- transpose $\mathbf{A}^\top \Leftrightarrow$ adjoint $\mathcal{A}^*$

For example: distribution is now a function

$$p(x, t)$$

rather than a vector $\mathbf{p}(t)$

3. Continuous time, continuous states

- Particular example: Brownian motion $dx_{it} = \mu dt + \sigma dW_{it}$
- Question: how characterize $p(x, t)$?
- Useful fact: p satisfies Kolmogorov Forward equation

$$\frac{\partial p(x,t)}{\partial t} = -\mu \frac{\partial p(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2}$$

- **Now comes the key:** write this in terms of differential operator

$$\frac{\partial p}{\partial t} = \mathcal{A}^* p, \qquad \mathcal{A}^* := -\mu \frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} \qquad (*)$$

which is exact analogue of

$$\dot{\mathbf{p}}(t) = \mathbf{A}^\top \mathbf{p}(t)$$

3. Continuous time, continuous states

- This goes further: just like $\mathbf{A}^{\top}$, $\mathcal{A}^*$ has eigenvalues & eigenvectors
- The eigenvalues λ_j and eigenfunctions $\varphi_j(x)$ of $\mathcal{A}^*$ solve

$$\lambda\varphi = \mathcal{A}^*\varphi \quad \Leftrightarrow \quad \lambda\mathbf{v} = \mathbf{A}^{\top}\mathbf{v}$$

- Also everything else is "the same"

$$\frac{\partial p}{\partial t} = \mathcal{A}^* p \quad \Leftrightarrow \quad \dot{\mathbf{p}}(t) = \mathbf{A}^{\top}\mathbf{p}(t)$$

$$p(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t} b_j \varphi_j(x) \quad \Leftrightarrow \quad \mathbf{p}(t) = \sum_{j=1}^{N} e^{\lambda_j t} b_j \mathbf{v}_j$$

  and similarly for IRFs ...

- Aside "for the interested nerd": comes from quantum mechanics
  - John von Neumann (1932) "Mathematical Foundations of Quantum Mechanics"

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3. Continuous time, continuous states

- Finally: these eigenvalue problems are differential equations, can be solved analytically in special cases
- For example: eigenvalue problem from previous slide

$$\lambda\varphi = \mathcal{A}^*\varphi, \qquad \mathcal{A}^* := -\mu\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} \quad \text{\& boundary conditions}$$

is simply an ODE

$$\lambda\varphi(x) = -\mu\varphi'(x) + \frac{\sigma^2}{2}\varphi''(x) \quad \text{\& boundary conditions}$$

- Analytic solutions with $\sigma^2/2 = 1$, reflected on $[0, 1]$

$$\lambda_0 = 0, \quad \lambda_j = \frac{\mu^2}{2} + \frac{\pi^2 j^2}{2}, \quad j = 1, 2, \ldots$$

$$\varphi_j(x) = \pm\frac{e^{-\mu x}}{\sqrt{1 + \mu^2/(\pi^2 j^2)}}\left\{\cos(x\pi j) + \frac{\mu}{\pi j}\sin(x\pi j)\right\}$$

which is similar to Alvarez Lippi's formulas – see Linetsky (2005) "On the Transition Densities for Reflected Diffusions"

Summary of Approach

Conceptually, everything is the same as with discrete time & states!

Two steps:

1. express transition dynamics in terms of eigenvalues & eigenvectors/functions

2. analytic solution for these

Analogies:

- vector $\mathbf{p} \Leftrightarrow$ function p

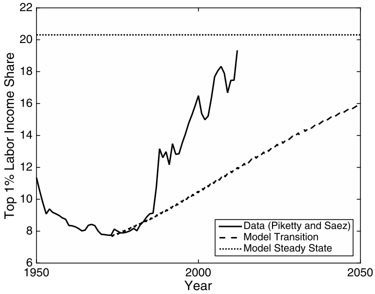
- matrix $\mathbf{A} \Leftrightarrow$ operator $\mathcal{A}$

- transpose $\mathbf{A}^\top \Leftrightarrow$ adjoint $\mathcal{A}^*$

Applications

GLLM (2016) "The Dynamics of Inequality"

Main message: standard theories of top inequality ⇒ very slow transition dynamics, too slow relative to data



GLLM main theorem in a nutshell:

1. spectral approach: $\mathbf{p}(t) = \sum_{j=1}^{N} e^{-|\lambda_j| t} b_j \mathbf{v}_j \approx b_1 \mathbf{v}_1 + e^{-|\lambda_2| t} b_2 \mathbf{v}_2$
2. analytic formula for $|\lambda_2| = \frac{1}{2}\frac{\mu^2}{\sigma^2} + \zeta$ (but not higher eigenvalues)
3. $|\lambda_2|$ very small for any reasonable parameterization

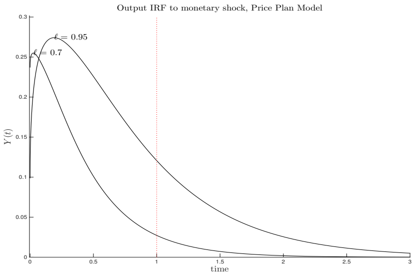
Alvarez-Lippi (2019) "The Analytic Theory of a Monetary Shock"

Main results:

1. "selection effect" decouples price adj frequency & output response (active in menu cost models but not in Calvo)
2. "price plans" may yield hump-shaped IRFs
3. monetary policy less effective when volatility is high (but only if monetary expansion lags volatility shock sufficiently)

Relative to GLLM: all eigenvalues rather than just spectral gap!

Analytic characterization of whole profile of IRF. Example:



Alvarez-Lippi's Main Theorem

Impulse response after t periods:

$$H(t; f, \hat{p}) = \sum_{j=1}^{\infty} e^{\lambda_j t} b_j[f] b_j[\hat{p}]$$

and analytic solutions for λ_j, $b_j[f]$, $b_j[\hat{p}]$, e.g.

$$\lambda_j = -\left[\zeta + \frac{\sigma^2}{8\bar{x}}(j\pi)^2\right], \quad j = 1, 2, \ldots$$

Exact analogue of

$$H_t = \sum_{j=1}^{N} \lambda_j^t b_j[\mathbf{p}_0] b_j[\mathbf{f}], \quad b_j[\mathbf{f}] := \mathbf{f}^\top \mathbf{v}_j$$

Other References

- Stefan Krieger (2002), “The General Equilibrium Dynamics of Investment and Scrapping in an Economy with Firm Level Uncertainty”

- Atkeson and Uhlig (2000), “Neoclassical growth with idiosyncratic and aggregate shocks: a linearization approach”

- Neither of these papers ever saw the light of the day, but you can find a draft of the former via google