Why Piketty Says $r - g$ Matters for Inequality

Supplementary Lecture Notes "Income and Wealth Distribution"

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June 1, 2014

These Notes

My version of a hybrid of

1. Section 5.4 of Piketty and Zucman (2014)
   http://gabriel-zucman.eu/files/PikettyZucman2014HID.pdf

2. Benhabib, Bisin and Zhu (2013)
   http://www.econ.nyu.edu/user/benhabib/lineartail31.pdf

Warning: high probability of algebra mistakes. If you find one, please email me
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These Notes

Summary:

- Standard explanation of high observed wealth concentration (e.g. top 1% own 30%): idiosyncratic capital income risk
- In theories with capital income risk, $\bar{r} - g$ is one main determinant of top wealth inequality
- Theory suggests slight modification: $\bar{r} - g - \bar{c}$ where $\bar{c}$ is the marginal propensity to consume out of wealth for rich people

What these notes are not about:

- the aggregate capital-output ratio K/Y: different story (inequality across groups)
- see Piketty and Zucman (2014)
- and critical reviews by Ray and Krusell-Smith:
  `http://www.econ.nyu.edu/user/debraj/Papers/Piketty.pdf`
  `http://aida.wss.yale.edu/smith/piketty1.pdf`

Outline

1. Simplest possible case
    - Brownian capital income risk
    - exogenous MPC
2. Generalizations
    - endogenous savings/MPC
    - labor income risk
    - more general capital income processes
3. Transition dynamics

Simplest Possible Case

- Continuum of individuals, heterogeneous in
  - wealth b
  - labor income w
- Wealth evolves as

$$db_t = [w_t + r_t b_t - c_t]dt$$

- Labor income w_t grows deterministically $w_t = we^{gt}$ (e.g. GDP grows and constant labor share: $w_t = (1-\alpha)Y_t$)
- Capital income r_t is stochastic

$$r_t = \bar{r} + \sigma dW_t$$

where W_t is a standard Brownian motion, that is $dW_t \equiv \lim_{\Delta t \to 0} \varepsilon_t \sqrt{\Delta t}$, with $\varepsilon_t \sim \mathcal{N}(0,1)$

Simplest Possible Case

- Combining

$$db_t = [w_t + \bar{r} b_t - c_t]dt + \sigma b_t dW_t$$

- b_t is non-stationary because w_t is growing
- $\Rightarrow$ define detrended wealth: $a_t = b_t e^{-gt}$
- Using $da_t / a_t = db_t / b_t - gdt$:

$$da_t = [w + (\bar{r} - g)a_t - c_t]dt + \sigma a_t dW_t$$

- For now: assume **exogenous MPC** out of wealth $c_t = \bar{c}a$. Assume $\bar{c} > r - g$
- Endogenize saving behavior later
    - reinterpret $\bar{c} = \lim_{a \to \infty} c(a)/a$ where c=consumption policy fn

Stationary Wealth Distribution

- De-trended wealth follows stationary stochastic process

$$da_t = [w + (\bar{r} - g - \bar{c})a_t]dt + \sigma a_t dW_t \qquad (*)$$

- Characterize stationary distribution?
- **Definition:** a has a Pareto tail if there exists $C > 0$ and $\zeta > 0$ such that

$$\lim_{a \to \infty} a^{\zeta} \Pr(\tilde{a} > a) = C.$$

- Note: $\zeta =$ "tail parameter." Top wealth inequality $= 1/\zeta$
- **Result:** The stationary wealth distribution has a Pareto tail with tail parameter (recall $\bar{c} > r - g$)

$$\zeta = 1 - \frac{\bar{r} - g - \bar{c}}{\sigma^2/2} > 1, \quad \frac{1}{\zeta} = \frac{\sigma^2/2}{\sigma^2/2 - (\bar{r} - g - \bar{c})}$$

- **Observations:**
  1. inequality $1/\zeta$ increasing in $\bar{r} - g$
  2. but also depends on $\bar{c}, \sigma$ (decreasing in $\bar{c}$, increasing in σ)

Proof of Result

- Wealth distribution $f(a,t)$ satisfies Kolmogorov Forward/Fokker-Planck equation

$$\partial_t f(a,t) = -\partial_a((w + (\bar{r} - g - \bar{c})a)f(a,t)) + \frac{\sigma^2}{2}\partial_{aa}(a^2 f(a,t))$$

- Stationary wealth distribution $f(a)$ satisfies:

$$0 = -\partial_a((w + (\bar{r} - g - \rho)a)f(a)) + \frac{\sigma^2}{2}\partial_{aa}(a^2 f(a))$$

- Guess and verify $f(a) \propto a^{-\zeta-1}$

$$0 = w(\zeta+1)a^{-\zeta-2} + \zeta(\bar{r} - g - \bar{c})a^{-\zeta-1} + (\zeta-1)\zeta\frac{\sigma^2}{2}a^{-\zeta-1}$$

- We are interested in f as $a \to \infty$: **first term drops!**

$$0 = \zeta(\bar{r} - g - \bar{c}) + (\zeta-1)\zeta\frac{\sigma^2}{2}$$

- Collecting terms yields formula on previous slide.□
- Note: swept some technical issues under the rug e.g. existence of stationary distribution. Should follow from fact that (∗) is Kesten process (random growth process with intercept). See Benhabib-Bisin-Zhu.

The Effect of Taxes on Wealth Inequality

- Introduce taxes
  - labor income tax τ_w
  - capital income tax τ_r

$$db_t = [(1-\tau_w)w_t + (1-\tau_r)r_t b_t - c_t]dt$$

$$da_t = [(1-\tau_w)w + ((1-\tau_r)\bar{r} - g - \bar{c})a_t]dt + \sigma(1-\tau_r)a_t dW_t$$

- **Result:** Formula for tail parameter generalizes to

$$\zeta = 1 - \frac{(1-\tau_r)\bar{r} - g - \bar{c}}{\sigma^2(1-\tau_r)^2/2}$$

$$\frac{1}{\zeta} = \frac{(1-\tau_r)^2\sigma^2/2}{(1-\tau_r)^2\sigma^2/2 - (\bar{r} - g - \bar{c}) + \tau_r \bar{r}}$$

- **Observations:**
  1. inequality decreasing in τ_r for two reasons: capital income pays lower return $\bar{r}$, and is less volatile
  2. inequality does not depend on labor income tax

Discussion

- **Other sources of randomness in wealth growth**
  - Piketty-Zucman (Section 5.4) have stochastic savings/bequests rather than stochastic capital income
  - this is mathematically isomorphic: everything identical if set

$$r_t = \bar{r}, \quad c_t(a) = \bar{c}a + \sigma a dW_t$$

  - what matters is that a_t follows random growth process like $(*)$
  - randomness in bequests would work similarly (e.g. in more general model with OLG structure and Poisson death)
  - while mathematically isomorphic, economics obviously different

- **Partial vs. General Equilibrium**
  - obviously both $\bar{r}$ and g are endogenous, and so the above analysis is potentially misleading
  - GE extension interesting/desirable, especially for counterfactuals/policy
  - but PE with exogenous $\bar{r}, g$=useful starting point

Generalizations

1. Optimally chosen savings
2. stochastic labor income w_t
3. more general process for capital income r_t

Optimal Savings + Stochastic w_t

- Individuals solve

$$V(b,\tilde{w}) = \max_{\{c_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.}$$

$$db_t = [\tilde{w}_t + r_t b_t - c_t]dt$$

$$d\tilde{w}_t = (g + \mu_w(\tilde{w}_t))dt + \sigma_w(\tilde{w}_t) dW_t$$

$$r_t = \bar{r} + \sigma dW_t$$

$$b_t \geq 0, \quad (b_0, \tilde{w}_0) = (b, \tilde{w})$$

- Assume CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

Generalizations

- Detrended problem: $w_t = \tilde{w}_t e^{-gt}, a_t = b_t e^{-gt}$

$$v(a, w) = \max_{\{c_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.}$$

$$da_t = [w_t + (\bar{r} - g)a_t - c_t]dt + \sigma a_t dW_t$$

$$dw_t = \mu_w(w_t)dt + \sigma_w(w_t)dW_t$$

$$a_t \geq 0, \quad (a_0, w_0) = (a, w)$$

- HJB equation:

$$\rho v(a, w) = \max_c u(c) + \partial_a v(a, w)(w + (\bar{r} - g)a - c) + \partial_{aa} v(a, w)\frac{\sigma^2 a^2}{2} + \partial_w v(a, w)\mu_w(w) + \partial_{ww} v(a, w)\frac{\sigma_w^2(w)}{2}$$

with a state constraint boundary condition to enforce the borrowing constraint.

Tail Saving Behavior & Implied Inequality

Proposition (Asymptotic Linearity)

Consumption policy functions are asymptotically linear, i.e. MPCs out of wealth are asymptotically constant:

$$\lim_{a\to\infty} \frac{c(a,w)}{a} = \bar{c} = \frac{\rho - (1-\gamma)(\bar{r}-g)}{\gamma} + (1-\gamma)\frac{\sigma^2}{2}$$

Corollary

Formula for tail parameter becomes

$$\zeta = 1 - \frac{(\bar{r}-g-\rho)/\gamma - (1-\gamma)\frac{\sigma^2}{2}}{\sigma^2/2}, \quad \frac{1}{\zeta} = \frac{\sigma^2/2}{(2-\gamma)\frac{\sigma^2}{2} - (\bar{r}-g-\rho)/\gamma}$$

Observations:

1. inequality still depends on $\bar{r}-g$
2. but quantitative mapping different, e.g. depends on γ

Proof of Linearity Prop.: Homogeneity

auxiliary result from Achdou, Lasry, Lions and Moll (2014)

Proposition (Homogeneity)

For any $\xi > 0$,

$$v(\xi a, w) = \xi^{1-\gamma} v_\xi(a, w)$$

where v_ξ solves

$$\rho v_\xi(a, w) = \max_c \; u(c) + \partial_a v_\xi(a, w)(w/\xi + (\bar{r} - g)a - c) + \partial_{aa} v_\xi(a, w)\frac{\sigma^2}{}$$

$$+ \partial_w v_\xi(a, w)\mu_w(w) + \partial_{ww} v_\xi(a, w)\frac{\sigma_w^2(w)}{2}$$

Corollary

For large a, individuals behave as if they had no labor income:

$$\lim_{a \to \infty} \frac{v(a, w)}{\tilde{v}(a)} = 1 \quad \textit{where } \tilde{v}(a) \textit{ solves}$$

$$\rho \tilde{v}(a) = \max_c \; u(c) + \tilde{v}'(a)((\bar{r} - g)a - c) + \tilde{v}''(a)\frac{\sigma^2 a^2}{2} \qquad (**)$$

Proof of Linearity Proposition

- Next step: find explicit solution for policy function of $(**)$

$$\rho\tilde{v}(a) = H(\tilde{v}'(a)) + \tilde{v}'(a)(\bar{r}-g)a + \tilde{v}''(a)\frac{\sigma^2 a^2}{2}$$
$$H(p) = \max_c u(c) - pc = \frac{\gamma}{1-\gamma}p^{\frac{\gamma-1}{\gamma}}$$

- Guess and verify $\tilde{v}(a) = Ba^{1-\gamma}, \tilde{v}'(a) = (1-\gamma)Ba^{-\gamma}$,

$\tilde{v}''(a) = -\gamma(1-\gamma)Ba^{-\gamma-1}, \; H(\tilde{v}'(a)) = \frac{\gamma}{1-\gamma}((1-\gamma)B)^{\frac{\gamma-1}{\gamma}}a^{1-\gamma}$

$$\rho = \gamma((1-\gamma)B)^{-\frac{1}{\gamma}} + (1-\gamma)(\bar{r}-g) - \gamma(1-\gamma)\frac{\sigma^2}{2}$$

- from FOC, $\tilde{c}(a) = \bar{c}a, \bar{c} = ((1-\gamma)B)^{-1/\gamma}$ and hence

$$\bar{c} = \frac{\rho - (1-\gamma)(\bar{r}-g)}{\gamma} + (1-\gamma)\frac{\sigma^2}{2}$$

- Asymptotic Linearity Proposition follows directly from Homogeneity Proposition and above.□

Further Generalization: General r Process

- Individuals solve

$$V(b, \tilde{w}, r) = \max_{\{c_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.}$$

$$db_t = [\tilde{w}_t + r_t b_t - c_t] dt$$

$$d\tilde{w}_t = (g + \mu_w(\tilde{w}_t)) dt + \sigma_w(\tilde{w}_t) dW_t$$

$$dr_t = \mu_r(r_t)) dt + \sigma_r(r_t) dB_t$$

$$(b_0, \tilde{w}_0, r_0) = (b, \tilde{w}, r)$$

- Assume CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

Further Generalization: General r Process

- Detrended problem: $w_t = \tilde{w}_t e^{-gt}, a_t = b_t e^{-gt}$

$$v(a, w, r) = \max_{\{c_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.}$$

$$da_t = [w_t + (r_t - g)a_t - c_t]dt$$

$$dw_t = \mu_w(w_t)dt + \sigma_w(w_t)dW_t$$

$$dr_t = \mu_r(r_t)dt + \sigma_r(r_t)dB_t$$

$$(a_0, w_0, r_0) = (a, w, r)$$

Tail Saving Behavior

Following similar steps as above, one can show:

Corollary

Consumption policy functions are asymptotically linear, i.e. MPCs out of wealth are asymptotically constant:

$$\lim_{a\to\infty} \frac{c(a,w,r)}{a} = \bar{c}(r)$$

The task is therefore to characterize the stationary distribution $f(a,w,r)$ of the following Kesten-type process:

$$da_t = [w_t + (r_t - g - \bar{c}(r_t))a_t]dt$$

$$dw_t = \mu_w(w_t)dt + \sigma_w(w_t)dW_t$$

$$dr_t = \mu_r(r_t)dt + \sigma_r(r_t)dB_t$$

Stationary Wealth Distribution

- Here's how to do it, based on Gabaix (2010) "On Random Growth Processes with Autocorrelated Shocks"

Proposition (Gabaix)

Assume w and r are stationary processes. Then the process for a has a stationary distribution with a Pareto tail $f(a,w,r) \sim \phi(w,r)a^{-\zeta-1}$ where the tail parameter ζ satisfies an eigenvalue problem

$$0 = \zeta(r - g - \bar{c}(r))e(w,r) + \mu_w(w)\partial_w e(w,r)] + \frac{1}{2}\sigma_w^2(w)\partial_{ww}e(w,r) + \mu_r(r)\partial_r e(w,r) + \frac{1}{2}\sigma_r^2(r)\partial_{rr}e(w,r) \qquad \text{(E)}$$

for some eigenfunction $e \geq 0$.

- Need to solve (E) numerically
- But can handle very general class of r-processes

Proof

- Stationary distribution satisfies

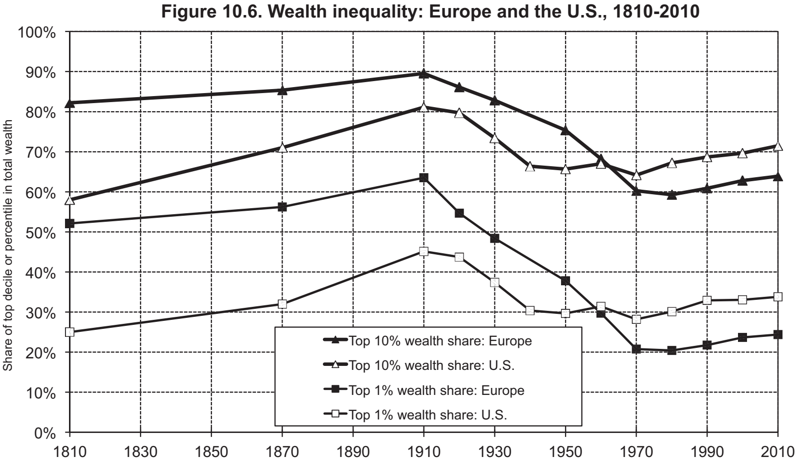
$$\begin{aligned}0 =& -\partial_a[(w+(r-g-\bar{c}(r))a)f(a,w,r)]\\&-\partial_w[\mu_w(w)f(a,w,r)]+\frac{1}{2}\partial_{ww}[\sigma_w^2(w)f(a,w,r)]\\&-\partial_r[\mu_r(r)f(a,w,r)]+\frac{1}{2}\partial_{rr}[\sigma_r^2(r)f(a,w,r)]\end{aligned}$$

- Guess $f(a,w,r)=\phi(w,r)a^{-\zeta-1}$ and substitute in.
- Divide by $a^{-\zeta-1}$ and use that we're interested in the tail as $a\to\infty$ and hence w/a drops:

$$\begin{aligned}0 =&\zeta(r-g-\bar{c}(r))\phi(w,r)\\&-\partial_y[\mu_w(w)\phi(w,r)]+\frac{1}{2}\partial_{ww}[\sigma_w^2(w)\phi(w,r)]\\&-\partial_r[\mu_r(r)\phi(w,r)]+\frac{1}{2}\partial_{rr}[\sigma_r^2(r)\phi(w,r)]\end{aligned}$$

- Using KF equation for (w,r), obtain (E).□

Transition Dynamics



Until the mid 20th century, wealth inequality was higher in Europe than in the United States.
Sources and series: see piketty.pse.ens.fr/capital21c.

Transition Dynamics

- So far: only focussed on stationary distributions
- But Piketty's whole point: world is **not stationary** (see Figure on previous slide)
  - wants to argue: that's because $\bar{r}_t - g_t$ varies over time
- Most interesting questions require extension to transition dynamics
  - simplest case: characterize $f(a,t)$ satisfying

$$\partial_t f(a,t) = -\partial_a((w + (\bar{r}_t - g_t - \bar{c}_t)a)f(a,t)) + \frac{\sigma^2}{2}\partial_{aa}(a^2 f(a,t))$$

  - probably need to go numerical
- Economics should be similar to comparing steady states
  - inequality depends on $\bar{r}_t - g_t - \bar{c}_t$:
  - how much does $\bar{c}_t$ vary over time relative to $\bar{r}_t - g_t$?
- **Open question:** how fast (or slow) are transitions?
  - e.g. if $\bar{r}_t - g_t - \bar{c}_t \uparrow$, how long until inequality $\uparrow$?

Richer Models

- Why would capital income be stochastic?
- One answer: entrepreneurship
    - Quadrini (1999, 2000)
    - Cagetti and DeNardi (2006)

Summary

- Standard explanation of high observed wealth concentration (e.g. top 1% own 30%): idiosyncratic capital income risk

- In theories with capital income risk, $\bar{r} - g$ is one main determinant of top wealth inequality
    - does not rely on weird assumptions about saving behavior (instead optimization w/ CRRA utility)
    - but theory suggests slight modification: $\bar{r} - g - \bar{c}$
    - other factors also potentially important, e.g. σ

- Changes in inequality over time? Open questions:
    - speed of transitions?
    - relative (quantitative) importance of different factors
    - e.g. how much does $\bar{c}_t$ vary over time relative to $\bar{r}_t - g_t$?