Productivity Losses from Financial Frictions: Can Self-financing Undo Capital Misallocation?

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G Online Appendix: The Model in Discrete Time and with iid Shocks

This Appendix presents a version of the model that is set up in discrete time and features productivity shocks that are iid over time. Its purpose is to make the paper more accessible to readers who are perhaps unfamiliar with some of the mathematical tools used in the continuous time model in the main text (in particular stochastic calculus). While the setup there is more complicated in terms of the mathematics, it also allows me to derive more general results, particularly with regard to the persistence of shocks which is the central theme in the paper. In this Appendix, shocks are iid over time so persistence is ruled out by assumption - see the last paragraph of this Appendix for more discussion. The model presented here is the exact discrete time analogue to the model in the main text. I therefore confine myself to outlining the model in the briefest possible fashion, only highlighting where the two setups differ; motivations for modeling choices can be found in the main text.

Preferences and Technology. Time is discrete. There is a continuum of entrepreneurs that are indexed by their productivity z and their wealth a. At each point in time t, the state of the economy is some joint distribution $g_t(a, z)$. The marginal distribution of productivity is denoted by $\psi(z)$. Each period, entrepreneurs draw a productivity shock from this distribution. Importantly, this productivity shock is not only iid across entrepreneurs but also iid across time. Entrepreneurs have preferences

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t. \tag{1}$$

Each entrepreneur owns a private firm which uses k units of capital and l units of labor to produce

$$y = f(z, k, l) = (zk)^{\alpha} l^{1-\alpha}$$

units of output, where $\alpha \in (0, 1)$. Capital depreciates at the rate δ. There is also a measure one of workers. Each worker is endowed with one efficiency unit of labor which he supplies inelastically.

Budgets. Denote by a_t an entrepreneur's wealth and by r_t and w_t the (endogenous) interest and wage rates. Entrepreneurs can rent capital k_t in a rental market at a rental rate $R_t = r_t + \delta$. Then their wealth evolves according to

$$a_{t+1} = f(z_t, k_t, l_t) - w_t l_t - (r_t + \delta)k_t + (1 + r_t)a_t - c_t \tag{2}$$

As in the main text, the setup with a rental market is chosen solely for simplicity. As in Appendix B of the paper, one can show that it is equivalent to a setup in which entrepreneurs own and accumulate capital.[1] Entrepreneurs face collateral constraints:

$$k_t \leq \lambda a_t, \quad \lambda \geq 1. \tag{3}$$

I assume that workers cannot save so that they are in effect hand-to-mouth workers who immediately consume their earnings. Workers can therefore be omitted from the remainder of the analysis.[2]

Individual Behavior. Entrepreneurs maximize the present discounted value of utility from consumption (1) subject to their budget constraints (2). Their production and savings/consumption decisions separate in a convenient way. Define the profit function

$$\Pi(a, z) = \max_{k,l} \{ f(z, k, l) - wl - (r + \delta)k \quad \text{s.t.} \quad k \leq \lambda a \},$$

and rewrite the budget constraint (2) as

$$a_{t+1} = \Pi(a_t, z_t) + (1 + r_t)a_t - c_t.$$

Lemma 1 in the main text shows that factor demands and profits are linear in wealth and there is a productivity cutoff $\underline{z}$ for being active. The profit function is

$$\Pi(a, z) = \max\{z\pi - r - \delta, 0\}\lambda a,$$

implying a law of motion for wealth that is linear in wealth

$$a_{t+1} = [\lambda \max\{z_t \pi_t - r_t - \delta, 0\} + 1 + r_t]\, a_t - c_t.$$

[1] All derivations in this Appendix apply after setting the length of time periods, $\Delta = 1$.

[2] Footnote 20 in the main text still applies. That is, even if I allowed workers to save, in the long-run they would endogenously choose to be hand-to-mouth workers because the interest rate is smaller than the rate of time preference.

This linearity allows me to derive a closed form solution for the optimal savings policy function.

Lemma 3 *Entrepreneurs save a constant fraction of their profits so that savings are linear in wealth.*

$$a_{t+1} = \beta \left[\lambda \max\{z_t \pi_t - r_t - \delta, 0\} + 1 + r_t\right] a_t. \tag{4}$$

Equilibrium and Aggregate Dynamics. An *equilibrium* are sequences of prices $\{(r_t, w_t)\}_{t=0}^{\infty}$ and corresponding quantities, such that (i) entrepreneurs maximize (1) subject to (2) taking as given equilibrium prices, and (ii) the capital and labor markets clear at each point in time

$$\int k_t(a,z) dG_t(a,z) = \int a dG_t(a,z), \tag{5}$$

$$\int l_t(a,z) dG_t(a,z) = L. \tag{6}$$

The linearity of individual savings policy functions (Lemma 1) implies that the economy aggregates nicely as in the main text. In fact, the assumption of iid productivity shocks makes the analysis simpler than there. This assumption implies that, at each point in time, wealth a_t and productivity z_t are independent. This follows because an entrepreneur chooses his wealth a_t one period before (at $t-1$), when he *does not know* his productivity draw z_t (see equation (4)). By construction, therefore, a_t is correlated only with z_{t-1} but not with z_t. The main simplification implied by this independence is that it immediately delivers an expression for the wealth shares that were my main tool for aggregation in the main text. Like there, these wealth shares are defined as

$$\omega_t(z) \equiv \frac{1}{K_t} \int_0^{\infty} a g_t(a,z) da.$$

Denoting the marginal distribution of wealth by $\varphi_t(a)$, we have that $g_t(a,z) = \psi(z)\varphi_t(a)$, which immediately implies that $\omega_t(z) = \psi(z)$ for all t. That wealth shares simply equal the marginal distribution of productivity is the main simplification obtained from the assumption of iid shocks.[3]

The following Proposition is the analogue to Proposition 2 in the main text.

Proposition 6 *Aggregate quantities satisfy*

$$Y_t = Z K_t^{\alpha} L^{1-\alpha}, \tag{7}$$

$$K_{t+1} = \beta \left[\alpha Z_t K_t^{\alpha} L^{1-\alpha} + (1-\delta) K_t\right], \tag{8}$$

[3] See also section 2.7 in the main text where this was obtained as a limit result, more precisely when letting the speed of mean reversion of the continuous time stochastic process (33 in main text) grow large, $\nu \to \infty$.

where K and L are aggregate capital and labor and

$$Z = \left(\frac{\int_{\underline{z}}^{\infty} z\psi(z)dz}{1 - \Psi(\underline{z})} \right)^{\alpha} = \mathbb{E}[z|z \geq \underline{z}]^{\alpha} \tag{9}$$

is measured TFP. The productivity cutoff $\underline{z}$ is defined by $\lambda(1 - \Psi(\underline{z})) = 1$. Factor prices are $w_t = (1-\alpha)Z_t K_t^{\alpha} L^{-\alpha}$ and $r_t = \alpha\zeta Z_t K_t^{\alpha-1} L^{1-\alpha} - \delta$, where $\zeta \equiv \underline{z}/\mathbb{E}[z|z \geq \underline{z}] \in [0,1]$.

The interpretation of this result is as in Proposition 2 in the main text. As already noted there is one main difference: there, the statement took as given the evolution of wealth shares $\omega(z,t), t \geq 0$. In contrast, here we have already solved for these wealth shares which are simply given by $\omega(z,t) = \psi(z)$, all t. This implies that TFP (9) is a simple *unweighted* truncated average of productivities. Because wealth shares are constant over time, also TFP is constant over time.

Steady State Equilibrium. The following Corollary is the analogue to Corollary 1 in the main text. The interpretation of most expressions there applies.

Corollary 2 *Aggregate steady state quantities solve*

$$Y = ZK^{\alpha}L^{1-\alpha} \tag{10}$$

$$\alpha Z K^{\alpha-1} L^{1-\alpha} = \rho + \delta, \tag{11}$$

where ρ is defined by $\beta = (1+\rho)^{-1}$, K and L are aggregate capital and labor and

$$Z = \left(\frac{\int_{\underline{z}}^{\infty} z\psi(z)dz}{1 - \Psi(\underline{z})} \right)^{\alpha} = \mathbb{E}[z|z \geq \underline{z}]^{\alpha}$$

is measured TFP. The productivity cutoff $\underline{z}$ is defined by $\lambda(1 - \Omega(\underline{z})) = 1$. Factor prices are $w = (1-\alpha)ZK^{\alpha}L^{-\alpha}$ and $r = \alpha\zeta ZK^{\alpha-1}L^{1-\alpha} - \delta = \zeta(\rho+\delta) - \delta$, where $\zeta \equiv \underline{z}/\mathbb{E}[z|z \geq \underline{z}] \in [0,1]$.

Discussion of iid Assumption Solving for wealth shares by assuming shocks to be iid is convenient and is the approach typically taken in the literature. See for example the papers by Angeletos (2007) and Kiyotaki and Moore (2008). In the present application, it is however only a useful benchmark; the assumption's biggest benefit in terms of tractability is also its biggest drawback in terms of the economics: iid productivity shocks leave no scope for self-financing which is the main theme of the paper. Because current productivity does not predict future productivity, more productive types cannot align their wealth to their productivity even though they save more. It is precisely this fact that is reflected in the independence of wealth and productivity in the cross section. For other applications the iid case may, however, be an extremely useful simplification.

A Pareto Example. The following simple example illustrates some important features of the model. Consider the expression for aggregate TFP (9). Since this expression does not impose any restrictions on the productivity distribution $\psi(z)$, one can pick the distribution of one's choice and compute TFP. Hence, let productivity be distributed Pareto on $[1, \infty)$, that is $\Psi(z) = 1 - z^{-\eta}, \eta > 1$. The parameter η is an inverse measure of the thickness of the tail of the distribution (a measure of the variance). Under this assumption, the productivity cutoff is simply $\underline{z} = \lambda^{1/\eta}$, and TFP is therefore

$$Z = \left(\frac{\eta}{\eta - 1} \lambda^{1/\eta} \right)^{\alpha}. \tag{12}$$

As already argued, TFP is strictly increasing in λ. More interesting is how TFP depends on the the productivity distribution $\psi(z)$, particularly the tail parameter η. Note that the elasticity of TFP with respect to the quality of credit markets is

$$\frac{\partial \log Z}{\partial \log \lambda} = \frac{\alpha}{\eta}.$$

Figure 1 plots TFP against the parameter measuring the development of credit markets λ for different values of the tail parameter η. TFP for $\lambda = 10$ is normalized to unity for sake

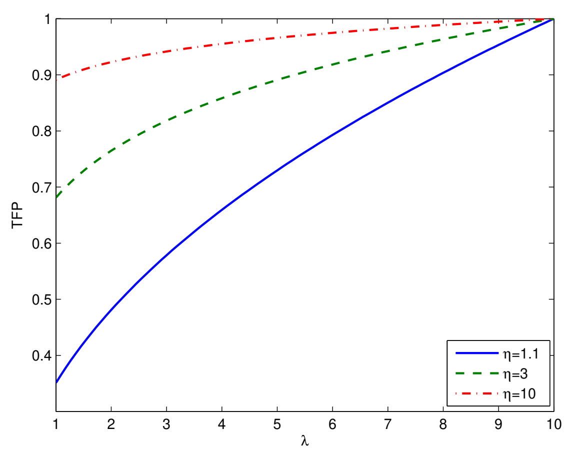

Figure 1: Total Factor Productivity

Note: TFP (12) relative to TFP for $\lambda = 10$. TFP losses are larger, the fatter is the tail of the productivity distribution (the smaller is η). The capital share is given by $\alpha = 0.3$.

of comparison. It can be seen from the Figure and the elasticity of Z with respect to λ that productivity losses from financial frictions are largest if the distribution of idiosyncratic productivities has a thick tail. This is intuitive. A thick tail implies that there are some extremely high-productivity entrepreneurs and that it is highly desirable from the point of view of society to direct capital towards them. With underdeveloped financial markets, this is however not possible so that productivity losses are large. While this example is intended to highlight the qualitative rather than quantitative implications of the model, I remark that the productivity loss from shutting down credit markets, $\lambda = 1$, relative to having good credit markets, $\lambda = 10$, varies considerably. It may be anywhere between ten and more than sixty percent depending on the value of η.

The Pareto example also delivers a simple expression for the rental rate R. Since

$$\zeta = \frac{\underline{z}}{\mathbb{E}[z|z \geq \underline{z}]} = 1 - \frac{1}{\eta} < 1,$$

we have that

$$R = \alpha \left(1 - \frac{1}{\eta}\right) Z K^{\alpha-1} L^{1-\alpha} < \alpha Z K^{\alpha-1} L^{1-\alpha}.$$

Note again the presence of the tail parameter η. A thicker tail of the productivity distribution (lower η) lowers the rental rate. This is intuitive because a low rental rate is a symptom of badly working credit markets, as discussed above.

Proofs

Proof of Lemma 3 The problem of an entrepreneur can be written in recursive form:

$$v(a, z) = \max_{a'} \log[A(z)a - a'] + \beta \mathbb{E} v(a', z')$$

where (from the linearity of profits – Lemma 1) $A(z) = \lambda \max\{z\pi - r - \delta, 0\} + 1 + r$. The proof proceeds with a guess and verify strategy. Guess that the value function takes the form $v(a, z) = V(z) + B \log a$, and substitute into the Bellman equation. In particular, note that the $\mathbb{E}v(a', z') = B \log a' + \mathbb{E}V(z')$. From the first order conditions

$$a' = \frac{\beta B}{1 + \beta B} A(z)a, \quad c = \frac{1}{1 + \beta B} A(z)a.$$

The Bellman equation becomes

$$A(z) + B \log a = \log \left[\frac{1}{1 + \beta B} A(z)a\right] + \beta \left[\mathbb{E}V(z') + B \log \frac{\beta B}{1 + \beta B} A(z)a\right]$$

Collecting the terms involving $\log a$, we can see that $B = 1/(1 - \beta)$ and $a' = \beta A(z)a$ as claimed.□

Proof of Proposition 6 As in the main text, the capital market clearing condition (5) can be written as

$$\lambda \int_{\underline{z}}^{\infty} \psi(z)dz = 1 \quad \text{or} \quad \lambda(1 - \Psi(\underline{z})) = 1. \tag{13}$$

The law of motion for aggregate capital is derived by integrating (4) over all entrepreneurs:

$$K_{t+1} = \beta \int_0^{\infty} [\lambda \max\{z\pi_t - r_t - \delta, 0\} + 1 + r_t] \psi(z)dz K_t \tag{14}$$

Using that the distribution $\psi(z)$ integrates to one and capital market clearing (13),

$$\frac{1}{\beta}\frac{K_{t+1}}{K_t} = \lambda\pi_t X + (1-\delta), \quad X \equiv \int_{\underline{z}}^{\infty} z\psi(z)dz. \tag{15}$$

Next, consider the labor market clearing condition. As in the main text (Proofs of Propositions 1 and 2), we have that labor market clearing (6) implies $\pi_t = \alpha(\lambda X)^{\alpha-1}K_t^{\alpha-1}L^{1-\alpha}$. Substituting into (15) and rearranging, we get

$$K_{t+1} = \beta\left[\alpha Z K_t^{\alpha} L^{1-\alpha} + (1-\delta)K_t\right], \quad Z = (\lambda X)^{\alpha},$$

which is equation (8) in Proposition 6.□