Capital Accumulation with Interdependent Countries*

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Abstract

What are the implications of cross-country interdependencies for capital accumulation? I tackle this question by introducing bilateral international externalities into a neoclassical growth model with n otherwise standard economies. In particular, I ask under which conditions there exists a unique steady state and when such a steady state is locally stable. I present a stability proof that uses results from "Inertia Theory", a field in linear algebra. To my knowledge this is the first paper in economics to make use of these results. In contrast to stability results for n-sector growth models, the conditions guaranteeing local stability do not depend on the discount rate.

JEL codes: D90, C62

Introduction

The aggregate growth model by Ramsey (1928), Cass (1965) and Koopmans (1965) is a workhorse of modern macroeconomics. This paper studies an extension of this model to n interdependent economies. The production technologies in each country display positive externalities within and across countries. Growth models with externalities were first analyzed by Romer (1983,1986) and Lucas (1988) and an extension to a multi-country setting seems

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natural.[1] Firms or households in a given country benefit in the form of increased production if firms/households in a neighboring country have high capital stocks. Firms or households in each country choose their capital stocks taking as given the average capital stocks throughout the world. Production technologies are allowed to vary across countries.

I analyze the dynamics of the vector of capital stocks describing a competitive equilibrium in the world. In particular, I establish a sufficient condition for the existence and uniqueness of a steady state of world capital stocks. This condition says that production in each country as a function of both internal and external capital stocks must exhibit decreasing returns. Another way of putting this is that diminishing returns in production with respect to a country's own capital stock must dominate relative to the positive externalities from all other countries.

I next analyze the local stability of the system of ordinary differential equations describing a competitive equilibrium in the world. I use the linearized version of this system of differential equations. Since there are n heterogeneous countries, with a state and a costate for each, the problem boils down to analyzing the eigenvalues of a matrix of dimension $2n \times 2n$. The linearized system is saddle-path stable if n of the eigenvalues have negative real parts. The proof uses a result from a field of linear algebra known as "Inertia Theory". The results I use are from Datta (1999).[2] The inertia of a matrix is defined as the triplet of the numbers of eigenvalues with positive, negative, and zero real parts, and there is a well-developed mathematical theory for examining this inertia triplet. I mainly use an "Inertia Theorem" which is a generalization of the Lyapunov Theorem for matrices. See Gantmacher (1985) for the latter. To my knowledge this is the first paper in economics to make use of these results.[3]

Using this Inertia Theorem, I derive sufficient conditions for the vector of steady state capital stocks to be locally stable. These sufficient conditions do not depend on the discount rate. The same diminishing returns restriction that guarantees existence and uniqueness of a steady state is made use of. Two additional restrictions are needed for proving local stability: first, that spillovers between any pair of countries be symmetric; second, that spillovers be "small" in a sense made precise below. For ease of exposition, the analysis is split into two

[1] This paper can be regarded as a follow-up on Romer's papers. His focus is on a single country with external effects. Both of Romer's papers informally discuss an example with two countries, but he does not formally analyze dynamics in a multi-country setting.

[2] This excellent and easily accessible survey of the field also discusses some examples.

[3] Benhabib and Nishimura (1981) use a similar but less general result by Wielandt (1973) in a stability proof for n-sector growth models.

steps: I first treat a special Cobb-Douglas case, and then extend the results to more general production functions. The Cobb-Douglas case carries most of the intuition while the analysis is considerably easier and cleaner.

It should be possible to establish a more general stability result. The additional restrictions of spillover "symmetry" and "smallness" are sufficient but not necessary conditions for local stability. This is revealed in an "unsuccessful" attempt to find numerical counterexamples to stable behavior when imposing only the diminishing returns assumption. I leave the task of proving stability for the more general case for further research.

International capital spillovers of the form postulated here can be motivated in at least two different ways. First, capital can be interpreted as knowledge capital. Externalities in knowledge capital were first analyzed by Romer (1983, 1986) and Lucas (1988). My paper can be regarded as a follow-up on Romer's paper. He argues that knowledge or technology is nonrival and only partially excludable in the sense that it can be used by several agents at the same time, and cannot be perfectly patented or kept secret. Romer makes this argument in the context of a closed economy. An extension to a multi-country setting seems plausible: If technology or knowledge is nonrival and partially excludable it will also hardly be confined to national borders.

Second, even if capital is interpreted as physical capital, spillovers can arise as byproducts of its accumulation. Romer as well as Acemoglu (2008) discuss this possibility. Suppose that due to learning-by-doing, the stock of knowledge capital in each country is some increasing function of that country's stock of physical capital (for example, they are used in fixed proportions). Then, knowledge is essentially "embodied" in physical capital. Even if there are only knowledge spillovers, production in a given country will depend on other countries' physical capital stocks.[4]

The model presented here is also related to the existing literature on international externalities and growth. The theoretical literature is summarized by Klenow and Rodriguez-Clare (2005), for example. Usually, a "world technology frontier" is assumed from which countries receive spillovers according to whether they have high or low barriers to technology adoption.

[4]In this context, a model with two state variables - physical capital and knowledge - may be more natural. McGrattan and Prescott (2007) refer to knowledge as technology capital, and while their motivation is different, they end up analyzing a world of n countries with aggregate production functions that depend on each country's own physical capital stocks and *all* the stocks of technology capital throughout the world. A model with two types of state variables is clearly more difficult to analyze than the model in the present paper.

Often, the progress on this frontier is modeled as being purely exogenous.[5] In these models, there is therefore an intrinsic asymmetry. There are positive spillovers from the world technology frontier but where does this frontier come from in the first place? Other approaches endogenize progress on the frontier by assuming that the research effort of all countries is bundled in the frontier and then is "spilled back over" to all countries. This latter case can be seen as a special case of the model outlined here: Relabel the technology frontier as country $n+1$ and restrict bilateral knowledge spillovers between all countries 1 to n to be zero. In my opinion, it seems more realistic to see spillovers as bilateral and specific to pairs of countries.

There is also some empirical evidence that international externalities, in particular, are important empirically. See, for example, Klenow and Rodriguez-Clare (2005) and Conley and Ligon (2002).

Whatever the motivation, international spillovers represent the simplest possible form of cross-country interdependencies. As such, spillovers can simply be regarded as a convenient but mechanical reduced form for other kinds of cross-country interdependencies, for example international trade or international capital flows.[6] I regard the model here as a useful prototype for a variety of phenomena related to accumulation of different types of capital with interdependent countries. Hence the title.

The paper is organized as follows. Section 1 spells out the Cass model with n interdependent countries. Section 2 analyzes the dynamics of the model for a special Cobb-Douglas case. Section 3 extends the results from Section 2 to more general production functions. Section 4 concludes.

1 The Model

1.1 Preferences and Technology

There are n countries indexed by $i = 1, ..., n$. In each country, there is a continuum of *identical* households indexed by $h \in [0, 1]$. There is no population growth. Time is continuous and all

[5] See for example, Parente and Prescott (1994).

[6] In a companion paper, Moll (2007), I show that the model outlined here is isomorphic with a model of international trade and capital accumulation, namely the Eaton-Kortum model with capital as analyzed by Alvarez and Lucas (2007). Terms-of-trade effects make countries interdependent in a way very similar to capital spillovers.

variables are in per capita terms. Preferences are identical for all households and across all countries and are given by

$$\int_0^\infty e^{-\rho t} u(c_{ih}) dt, \tag{1}$$

where u is strictly increasing and strictly concave and satisfies the Inada condition $\lim_{c \to 0} u'(c) = +\infty$.

The capital stock of household h in country i is κ_{ih}. I denote the *average* capital stock in a country by $k_i = \int_0^1 \kappa_{ih} dh$. The vector $k = (k_1, ..., k_n)$ denotes the state of the world economy. I choose some compact state space $X = [\underline{x}, \overline{x}]^n \subset \mathbb{R}_+^n$ for k. I postpone the discussion of convenient choices for the lower and upper bounds $\underline{x}$ and $\overline{x}$ until Section 2 when analyzing the dynamics of the model. For now, it is sufficient to assume that the state space X is compact.

In the following section, I will look at the problem of a typical household in a given country i. For notational convenience, I will therefore drop the h subscripts on a household's own capital stock and write κ_i instead of κ_{ih}. The state for the household is therefore (κ_i, k). This state summarizes all necessary information for the household. Production of consumption goods y^i is a function of each household's own capital stock κ_i and the vector of *all* average capital stocks k. The production technology of a household in country i is given by

$$y^i = A^i(k) f^i(\kappa_i).$$

The functions f^i and A^i are bounded and continuous. The scale factor A^i depends on the vector of *average* capital stocks throughout the world. There is a positive externality from country j to country i if $A^i(k)$ is strictly increasing in its jth argument. I assume that there are positive externalities *between* all countries, and also *within* each country. The production technology is a natural generalization to n countries of the production technology in Romer (1986). The particular functional form is very similar to the one used in Azariadis and Drazen (1990). Note that the production functions are indexed by i and hence may take different forms across countries. All f^i are strictly increasing and concave in κ_i and satisfy the following conventional Inada conditions

$$f^i(0) = 0, \quad \lim_{\kappa_i \to \infty} f_\kappa^i(\kappa_i) = 0, \quad \lim_{\kappa_i \to 0} f_\kappa^i(\kappa_i) = +\infty, \tag{2}$$

where f_κ^i denotes the first derivative of f^i.

1.2 Competitive Equilibrium

In each country, households can convert production into either consumption or investment on a one-for-one basis. No international borrowing or lending is allowed. And there is no motive for borrowing or lending within a country. Hence households in a given country can only use their own resources. The law of motion for capital is then

$$\dot{\kappa}_i = A^i(k) f^i(\kappa_i) - \delta \kappa_i - c_i \tag{3}$$

where δ is the depreciation rate. In each country i, each household maximizes (1) by choice of its consumption c_i taking as given the vector of average capital stocks in the world k and the law of motion (3). Households across countries solve the following problem.

$$\begin{aligned} &\max_{c_i} \int_0^\infty e^{-\rho t} u(c_i) dt \quad \text{subject to} \\ &\dot{\kappa}_i = A^i(k) f^i(\kappa_i) - \delta \kappa_i - c_i, \quad \kappa_i(0) = \kappa_{i0}, \quad i = 1, ..., n. \end{aligned} \tag{P}$$

Definition 1 *Given* $k_0 = (k_{10}, ..., k_{n0})$*, the allocation* $c(t), k(t), t \geq 0$ *is a* competitive equilibrium *if*

(i) $k(0) = k_0$*, and*

(ii) $c_i(t), \kappa_i(t) = k_i(t), t \geq 0$ *solves* (P)*, given* k *and* $\kappa_{i0} = k_{i0}$*, for each* $i = 1, ..., n$*.*

In equilibrium, $\kappa_i = k_i$, all i. Hence it is convenient to let

$$F^i(k) \equiv A^i(k) f^i(k_i) \quad \text{and} \quad R^i(k) \equiv A^i(k) f^i_\kappa(k_i)$$

denote production and marginal product of capital respectively. Much of the remaining analysis will be concerned with the properties of the functions F^i and R^i. Denoting by $\lambda = (\lambda_1, ..., \lambda_n)$ a vector of costates, and using that in equilibrium $\kappa_i = k_i$, we obtain a system of $2n$ ordinary differential equations in (λ, k) that characterize a competitive equilibrium.

$$\begin{aligned} \dot{\lambda}_i &= [\rho + \delta - R^i(k)] \lambda_i \\ \dot{k}_i &= F^i(k) - \delta k_i - (u')^{-1}(\lambda_i), \quad i = 1, ..., n. \end{aligned} \tag{4}$$

In addition, any competitive equilibrium must satisfy the n transversality conditions

$$\lim_{t \to \infty} e^{-\rho t} \lambda_i(t) k_i(t) = 0, \quad i = 1, ..., n. \tag{5}$$

A competitive equilibrium may not exist. The issue is broadly that the positive externalities may be too big. A competitive equilibrium requires that for each $i = 1, ..., n$, $\kappa_i(t) = k_i(t)$ solves (P), given k and $\kappa_{i0} = k_{i0}$. The question of existence can therefore be viewed as the following fixed point problem. The path of external (average) capital stocks is described by a function $k(t), t \geq 0$. For any such k, the solution to (P) defines an operator on this function, $\kappa = Tk$. A competitive equilibrium is a fixed point of this operator $k^e = Tk^e$. For a given k, we expect the optimal choice Tk to be bigger (in some sense), the bigger are the externalities (the derivatives $\partial A^i(k)/\partial k_j$). It is not hard to imagine a situation where externalities are so big that it is never possible to find a fixed point.

Romer (1983, 1986) proves existence of a competitive equilibrium for the one-country case $n = 1$ of the model outlined here. He takes a slightly different route because he notes that the existence of equilibrium is very closely linked with a candidate equilibrium's qualitative behavior. In a special case, the model outlined here essentially collapses to Romer's model so that I can apply his results. Let $k_{-i} = (k_1, ..., k_{i-1}, k_{i+1}, ..., k_n)$ and suppose that the the scale factors can be written in a separable and symmetric way as

$$A^i(k) = A[b(k_i), B(k_{-i})], \quad i = 1, ..., n.$$

Also, assume that $f^i(\cdot) = f(\cdot), i = 1, ..., n$ and that all countries have the same initial conditions, $k_{i0} = x_0, i = 1, ..., n$. With these assumptions the identity of a country is immaterial, and all countries will choose the same capital paths $k_i(t) = x(t), t \geq 0$ all i. Therefore, the system of $2n$ ODEs (4) collapses to a system of only two ODEs. Defining $\mathcal{F}(x) \equiv F(x, x, ..., x)$ and $\mathcal{R}(x) = R(x, x, ..., x)$, we have

$$\begin{aligned} \dot{\lambda} &= [\rho + \delta - \mathcal{R}(x)]\lambda \\ \dot{x} &= \mathcal{F}(x) - \delta x - (u')^{-1}(\lambda). \end{aligned} \tag{6}$$

This is a simplified version of the Euler equations in Romer's analysis.[7] Romer then notes that the question of existence of a competitive equilibrium is closely linked to the qualitative behavior of trajectories described by (6). This is because a solution to (6) is a competitive equilibrium, if and only if it satisfies the transversality condition

$$\lim_{t \to \infty} e^{-\rho t} \lambda(t) x(t) = 0. \tag{7}$$

[7] Romer's analysis is slightly more complicated because he postulates a non-linear accumulation technology $\dot{x} = xh(I/x)$ where I is investment and $h(\cdot)$ is concave. In my analysis, $h(\cdot)$ is linear.

Romer notes that standard results concerning the existence of solutions to differential equations can be used to show that (6) determines a unique trajectory through any point (x, λ) in the phase plane. If one can then show that for any given value of k_0 there exists a λ_0 such that the transversality condition (7) is satisfied, one has proven the existence of a competitive equilibrium. A sufficient condition for this to be the case, is not hard to find. The ODEs in (6) are like those for the neoclassical growth model, except for the modification that they feature the private marginal product of capital $\mathcal{R}(k)$ as opposed to the social marginal product $\mathcal{F}'(k)$. Nevertheless, a simple phase-diagram analysis proves that the system is globally saddle-path stable if $\mathcal{F}(\cdot)$ is concave and $\mathcal{R}(\cdot)$ is decreasing. This implies immediately that there is a λ_0 satisfying the transversality condition.[8] In Section 3, I will discuss in more detail conditions under which $\mathcal{F}$ and $\mathcal{R}$ are, respectively, concave and decreasing. We then have

Proposition 1 *Consider the special case of identical countries,* $A^i(k) = A[b(k_i), B(k_{-i})]$ *and* $f^i = f$ *all* i*. If* $\mathcal{F}(x) \equiv F(x, x, ..., x)$ *is concave and* $\mathcal{R}(x) = R(x, x, ..., x)$ *is decreasing, then there exists a competitive equilibrium for any symmetric initial condition* $k_{i0} = x_0, i = 1, ..., n$*.*

The case of identical countries above suggests that also in the general case with heterogeneous countries, the question of existence is closely tied to the *global* stability of the system of ODEs in (4). In the next Section, I analyze the stability of this system. However, my argument is *local* in nature. A local stability proof has two implications for the question of existence of a competitive equilibrium. First, conditions for local stability are suggestive for global stability and hence for the existence of a competitive equilibrium. Second, local stability yields a local theorem for the existence of equilibrium. That is, a competitive equilibrium exists if countries' initial capital stocks are sufficiently close to their steady state values. For a similar result see Section 18.2 in Stokey et al. (1989). Proving the existence of equilibrium for the case of heterogeneous countries and arbitrary initial conditions is left for further research.

2 Cobb-Douglas Technologies

I now focus on the following two questions: (i) existence and uniqueness of a steady state, (ii) local stability of this steady state. I first consider a special case with Cobb-Douglas technologies

[8] Could a competitive equilibrium exist in the borderline case $\mathcal{R}(x) = r$ and $\mathcal{F}(x) = \phi x$? In this case, we essentially have an AK model. Under the assumption that utility is of the CRRA form with parameter σ, we could ensure that the transversality condition is satisfied by imposing $\rho > (1 - \sigma)(\phi - \delta)$. See Acemoglu (2008).

and CRRA utility. The Cobb-Douglas case is instructive because it carries all the intuition while the analysis is considerably easier and cleaner. The generalization to the functional form used in Section 1 then follows in Section 3.

Consider the case, in which

$$f^i(\kappa_i) = \kappa_i^{\alpha_i}, \quad A^i(k) = \prod_{j=1}^{n} k_j^{\theta_{ij}}, \quad \text{all i,}$$

which implies

$$F^i(k) = \kappa_i^{\alpha_i}\left(\prod_{j=1}^{n} k_j^{\theta_{ij}}\right), \quad R^i(k) = \alpha_i k_i^{\alpha_i - 1}\left(\prod_{j=1}^{n} k_j^{\theta_{ij}}\right), \quad \text{all i,} \tag{8}$$

where $\alpha_i \in (0,1), \theta_{ij} > 0$ for all i, j. The term α_i captures the curvature of the production technology in country i with respect to a typical household's own capital stock. The term θ_{ij} measures the spillover from the average capital stock in countries j to production in country i. Define the two $n \times n$ matrices

$$\alpha \equiv \text{diag}(\alpha_i), \quad \text{and} \quad \theta \equiv [\theta_{ij}].$$

I will also refer to θ as the *spillover matrix*. I impose

Assumption 1

(1.a) $\alpha_i \in (0,1), \quad \theta_{ij} > 0 \quad i, j = 1, ..., n.$

(1.b) $\alpha_i + \sum_{j=1}^{n} \theta_{ij} < 1, \quad i = 1, ..., n.$

Part 1.a has already been discussed. Part 1.b is a diminishing returns assumption which will be of central importance.

2.1 Steady State

Setting $(\dot{\lambda}_i, \dot{k}_i) = (0,0)$ for all i in (4), we find that a *steady state* is a vector of capital stocks $k^* = (k_1^*, ..., k_n^*)$ that satisfies

$$R^i(k^*) = \rho + \delta, \quad i = 1, ..., n. \tag{9}$$

Next consider appropriate bounds for the state space X. Let $X = [\underline{x}, \overline{x}]^n$. Assumption 1.b implies that there are $\underline{x}$ sufficiently small and $\overline{x}$ sufficiently big, such that

$$R^i(\underline{x}, ..., \underline{x}) > \rho + \delta \quad \text{and} \quad R^i(\overline{x}, ..., \overline{x}) < \rho + \delta, \quad i = 1, ..., n. \tag{10}$$

Condition (10) is crucial for establishing the existence of a steady state. The bounds must also satisfy

$$F^i(\overline{x}, ..., \overline{x}) \leq \overline{x}, \quad i = 1, ..., n \quad \text{and} \quad \underline{x} > 0. \tag{11}$$

Upper bounds $\overline{x}$ satisfying the first part of equation (11) imply that the capital stocks remain within the state space.[9] The second restriction on the lower bounds $\underline{x}$ in (11) is made for the purely technical reason that it will be convenient below to work with the log of capital stocks. The point $k_i = 0$ is uninteresting in any case because an economy that starts with no capital, $k_{i0} = 0$, can never produce any output $f^i(0) = 0$. I focus on the question of existence and uniqueness of a positive steady state.

The next Lemma derives the implications of assumption 1 for the matrices $\alpha \equiv \text{diag}(\alpha_i)$ and $\theta \equiv [\theta_{ij}]$. It is an application of the Gershgorin Disc Theorem, stated in the Appendix.

Lemma 1 *Let assumption 1 be satisfied. Then all eigenvalues of $I - \alpha - \theta$ have positive real parts.*

Proof Apply Corollary 2 in the Appendix to $Q = I - \alpha - \theta$ and choose $\Lambda = I - \alpha$ and $P = -\theta$.□

Proposition 2 *Under assumption 1, there is a unique steady state $k^* \in X \subset \mathbb{R}^n_+$ defined by* (9).

Proof Use (8) to rewrite (9) as a system of linear equations,

$$(1 - \alpha_i) \log k_i^* - \sum_{j=1}^{n} \theta_{ij} \log k_j^* = a_i, \quad \text{where} \quad a_i = \log \left(\frac{\alpha_i}{\rho + \delta} \right), \quad i = 1, ..., n.$$

Define $h^* \equiv \log k^*$ and rewrite this in matrix notation as

$$(I - \alpha - \theta) h^* = a. \tag{12}$$

By Lemma 1, all eigenvalues of the matrix $I - \alpha - \theta$ are positive. The Levy-Desplanque theorem states that such matrices are invertible. Hence (12) and equivalently (9) has a unique solution.□

Note that associated with the unique vector of steady state capital stocks k^*, there is a unique vector of steady state shadow prices λ^*. For each i, λ_i^* is implicitly defined by setting $\dot{k}_i = 0$ in (4).

[9] The vector of capital stocks k satisfying $F^i(k) = k_i, i = 1, ..., n$ is the "maximum maintainable capital stock". See for example, Section 6.1. in Stokey et al. (1989).

2.2 Local Stability

Next consider the local stability of this steady state. Assume preferences have the CRRA form with parameter σ. Recall the system of $2n$ ODEs in (4) that characterizes the evolution of capital stocks in the world. For the Cobb-Douglas functional form in (8) and CRRA preferences this specializes to

$$\frac{d\log\lambda_i}{dt} = \frac{\dot{\lambda}_i}{\lambda_i} = \rho + \delta - R^i(k) = \rho + \delta - \alpha_i \left(\prod_{j\neq i} k_j^{\theta_{ij}}\right) k_i^{\alpha_i+\theta_{ii}-1}$$

$$\frac{d\log k_i}{dt} = \frac{\dot{k}_i}{k_i} = \frac{F^i(k)}{k_i} - \delta - \frac{(u')^{-1}(\lambda_i)}{k_i} = \left(\prod_{j\neq i} k_j^{\theta_{ij}}\right) k_i^{\alpha_i+\theta_{ii}-1} - \delta - \frac{\lambda_i^{-1/\sigma}}{k_i}, \quad i = 1, ..., n.$$

Define $z_1 = \log\lambda - \log\lambda^*$ and $z_2 = \log k - \log k^*$ and *log*-linearize this system around the steady state (λ^*, k^*), to get

$$\dot{z} = Az, \quad A = \begin{bmatrix} 0 & X \\ Y & Z \end{bmatrix}, \tag{13}$$

where

$$X = -\text{diag}(R^i) \begin{bmatrix} \alpha_1+\theta_{11}-1 & \theta_{12} & \cdots & \theta_{1n} \\ \theta_{21} & \alpha_2+\theta_{22}-1 & \cdots & \theta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{n1} & \theta_{n2} & \cdots & \alpha_n+\theta_{nn}-1 \end{bmatrix}$$

$$= \text{diag}(R^i)\,(I-\alpha-\theta)$$

$$Y = \text{diag}\left(\frac{1}{\sigma}\frac{c_i}{k_i}\right)$$

$$Z = \text{diag}\left(\frac{F^i}{k_i}\right)(\alpha+\theta-I) + \text{diag}\left(\frac{c_i}{k_i}\right).$$

If there are no spillovers ($\theta = 0$), then X, Y and Z are diagonal. In this case we can rearrange the equations so that A is block diagonal, and separately look at 2×2 blocks of A corresponding to the state and costate for each country. This is not surprising: without spillovers, the model here is simply n unconnected Cass economies. Since we know that the Cass model features a stable steady state, this special case is a useful benchmark.

The stability proof makes use of the following result taken from Datta (1999).

Definition 2 *The* inertia *of a matrix A, denoted by In(A), is the triplet $(\pi(A), \nu(A), \delta(A))$ where $\pi(A), \nu(A)$ and $\delta(A)$ are, respectively, the number of eigenvalues of A with positive, negative, and zero real parts, counting multiplicities.*

A linear system of $2n$ differential equations such as (13), is saddle path stable if exactly half of the eigenvalues of A have negative real parts. Saddle path stability is therefore equivalent to the statement

$$\text{In}(A) = (n, n, 0).$$

The following theorem is very useful in establishing sufficient conditions under which indeed $\text{In}(A) = (n, n, 0)$.[10] In the following, $M \geq 0$ means that the matrix M is positive semidefinite.

Theorem 1 *(Inertia Theorem) Let $\delta(A) = 0$, and let W be a nonsingular symmetric matrix such that*

$$WA + A^T W = M \geq 0. \tag{14}$$

Then $In(A) = In(W)$.

Proof See Theorem 4.4. in Datta (1999). The theorem is originally due to Carlson and Schneider (1963).□

This Theorem is a generalization of the Lyapunov Stability Theorem for matrices (see Theorem 3.2. in Datta (1999) or Gantmacher (1985)). It is used in the same way, and its advantages and disadvantages are similar. In particular, the exercise boils down to finding a matrix W whose inertia we can easily determine.

Adding the following assumption to assumption 1 is sufficient to guarantee the existence of such a matrix W.

Assumption 2

(2.a) $\theta_{ij} = \theta_{ji}$, all $i, j = 1, ..., n$.

(2.b) The matrix $\alpha^{-1}\theta$ is symmetric and positive semi-definite.

Part 2.a imposes a symmetry restriction on spillovers: The spillover from country i to country j must be of the same size as the spillover from country j to country i. The economics of this symmetry assumption are discussed below. Part 2.b unfortunately does not have an easy economic interpretation. It holds for example if $\alpha_i = \alpha$ for all i and if within-country spillovers

[10] It is natural to ask whether corresponding techniques are also available for systems of *difference* equations. Datta (1999) also defines a unit circle inertia as the triplet of the number of eigenvalues outside, inside and on the unit circle. Theorem 4.5. in this paper is the analogue of Theorem 1 for the unit circle inertia.

are sufficiently big so that $\theta_{ii} > \sum_{j=1}^{n} \theta_{ij}$ all i (to prove this claim formally, apply the Gershgorin Disc Theorem, Theorem 2 in the Appendix). More generally, it should hold if spillovers are symmetric and sufficiently small, $\theta \approx 0$. Note that neither assumption involves the discount rate ρ.

Proposition 3 *Let assumptions 1 and 2 be satisfied. Then* $In(A) = (n, n, 0)$ *and hence* k^* *is locally saddle-path stable.*

Proof I first show that all of X, Y, Z are symmetric and positive definite. From Lemma 1, all eigenvalues of $(I - \alpha - \theta)$ have positive real parts. Together with symmetry from assumption 2, this implies that $(I - \alpha - \theta)$ is symmetric and positive definite. In steady state $R^i = \rho + \delta$, all i so that X is also symmetric and positive definite. Y is diagonal and hence trivially positive definite. Rewrite Z in terms of parameters only. In steady state

$$\frac{c_i}{k_i} = \frac{F^i}{k_i} - \delta \quad \text{and} \quad \frac{F^i}{k_i} = \frac{\rho + \delta}{\alpha_i},$$

so that

$$Z = \rho I + (\rho + \delta)\alpha^{-1}\theta,$$

which is symmetric and positive definite under assumption 2.

Next define

$$W = \begin{bmatrix} -Y & 0 \\ 0 & X \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ 0 & XZ + ZX \end{bmatrix}.$$

We have that

$$WA + A^T W = M \geq 0.$$

Applying Theorem 1 we see that

$$\text{In}(A) = \text{In}(W) = (n, n, 0)$$

where the second equality follows because W is block diagonal and its eigenvalues are those of $-Y$ and X. □

Unfortunately, I have not been able to prove the stability of k^* for the case in which assumption 2 does not hold. If assumption 2 is violated, the matrix W in the proof of proposition 3 is not symmetric and the Inertia Theorem does not apply. Is symmetry of the spillover matrix θ a natural restriction? One would expect cross-country spillovers to depend on, say, geographic distance between two countries and on their relative population sizes. While distance affects two countries symmetrically, population size does not: the spillover from a big to a small country is likely bigger than the spillover from a small to a big country. Hence, symmetry is not a natural restriction on cross-country spillovers.

I briefly argue that it should be possible to establish a stability theorem under assumption 1 only. I have tried to find numerical counterexamples to stable behavior of the linearized system (13). In particular, I have considered the case $n = 4$ countries implying that A is of dimension 8×8. Using a random number generator, I generated $100,000$ matrices A imposing only assumption 1 but not assumption 2. I then calculated their eigenvalues numerically. In all of these cases, I obtained $\text{In}(A) = (4, 4, 0)$. The code is available on request. Establishing a stability proof under assumption 1 only is left for future research.

3 More General Production Functions

In this Section, I argue that the results established in the preceding Section for a Cobb-Douglas production technology are not really special at all. Under appropriate generalizations of the assumptions used there, they carry over easily to the general form of the production function used in Section 1. Define

$$\alpha_i(k_i) \equiv 1 + \frac{\partial \log f^i_\kappa(k_i)}{\partial \log k_i} \quad \text{and} \quad \theta_{ij}(k) \equiv \frac{\partial \log A^i(k)}{\partial \log k_j}. \tag{15}$$

These elasticities are the natural generalizations of the parameters α_i and θ_{ij} from the Cobb-Douglas example. Essentially none of the analysis in the preceding section has to be changed as long as one pays special attention to the fact that these elasticities are no longer constant and instead are functions of the capital stocks k. I will keep the discussion as brief as possible and emphasize where the analysis in the preceding section has to be altered.

3.1 Steady State

The following is the appropriate generalization of assumption 1 from the Cobb-Douglas case.

Assumption 3

(3.a) All f^i are strictly increasing and strictly concave. All A^i are strictly increasing in all their arguments.

(3.b) Each F^i is homogeneous of degree $s < 1$,

$$F^i(\psi k) = \psi^s F^i(k), \quad \text{all } \psi > 0 \quad i = 1, ..., n.$$

(3.c) $\lim_{x \to 0} R^i(x, ..., x) = +\infty, \quad \lim_{x \to \infty} R^i(x, ..., x) = 0, \quad i = 1, ..., n.$

Part 3.b is the appropriate generalization of the diminishing returns assumption 1.b from the Cobb-Douglas case. Because F^i is homogeneous of degree s, all its partial derivatives are homogeneous of degree $s - 1$.[11] That is

$$R^i(\psi k) = \psi^{s-1} R^i(k), \quad \text{all } \psi > 0 \quad i = 1, ..., n. \tag{16}$$

With the definitions in (15)

$$\frac{\partial \log R^i(k)}{\partial \log k_i} = \alpha_i(k) + \theta_{ii}(k) - 1, \qquad \frac{\partial \log R^i(k)}{\partial \log k_j} = \theta_{ij}(k), \quad j \neq i$$

Differentiating (16) with respect to ψ, we see that assumption 1.b implies

$$\alpha_i(k_i) + \sum_{j=1}^{n} \theta_{ij}(k) < 1, \quad i = 1, ..., n$$

This last way of expressing assumption 3.b shows that it is the appropriate generalization of assumption 1.b from the Cobb-Douglas case.

Part 3.c is an Inada condition. It is needed to ensure the existence of a steady state. That diminishing returns are generally not sufficient for the existence of a steady state has been demonstrated by Jones and Manuelli (1990) whose model features sustained growth in the presence of decreasing returns. I have already imposed some Inada conditions on $f^i, i = 1, ..., n$ in (2) that guarantee that the first order conditions hold with equality. These are only restrictions on the functions f^i but not on the scale factors A^i. Instead a joint restriction is needed. In the Cobb-Douglas example, the Inada condition 3.c was automatically satisfied as part of assumption 1.b. Here, it is important to ensure that we can pick bounds $\underline{x}, \overline{x}$ that satisfy (10).

Proposition 4 *Under assumption 3, there is a unique steady state $k^* \in X \subset \mathbb{R}^n_+$ defined by* (9).

[11] The sufficient condition for the existence of an equilibrium in Proposition 1 was that $\mathcal{F}$ is concave and $\mathcal{R}$ is decreasing. As the argument here shows, both restrictions are immediately implied if F is homogeneous of degree $s < 1$.

Proof Define $h = \log(k), i = 1, ..., n$ and define the operator $T^*(h) = [T_1^*(h), ..., T_n^*(h)]'$ by

$$f_\kappa^i(\exp T_i^*(h))A^i(\exp h) = \rho + \delta. \tag{17}$$

The logarithm of the steady state h^* is then a fixed point of T^*. Figure 1 features a graphical representation of equation (17). My goal here is to show that T^* is a contraction under the

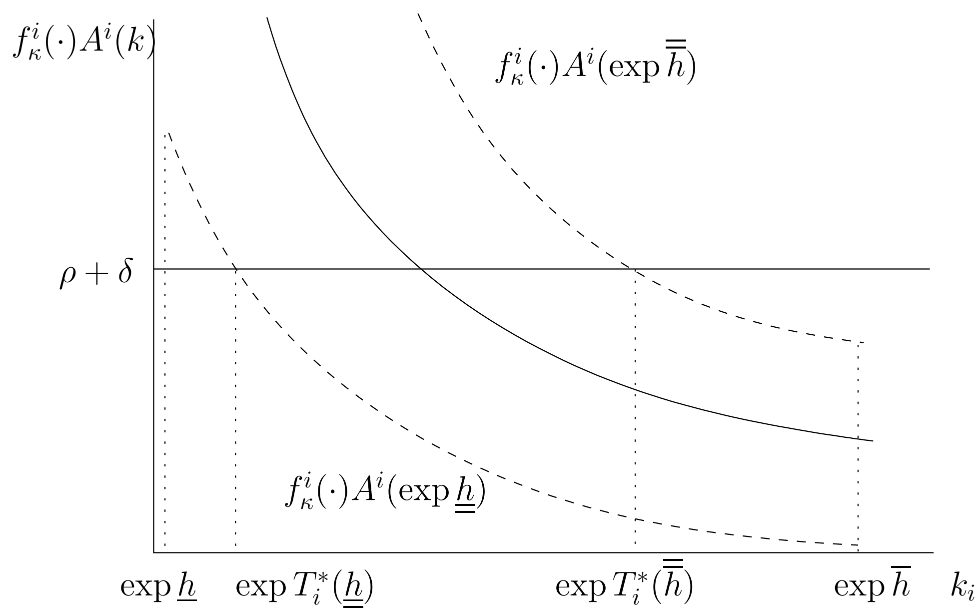

Figure 1: The operator T^*

sup-norm $||h|| = \max_i |h_i|$ by verifying Blackwell's sufficient conditions. First, note that by assumption the state space $X = [\underline{x}, \overline{x}]^n$ is compact. Therefore, also $H = [\underline{h}, \overline{h}]^n$ is compact where $\underline{h} = \log \underline{x}$ and $\overline{h} = \log \overline{x}$. Furthermore, since $H \subset \mathbb{R}^n$ it is a space on which the sup-norm is well defined. I next show that $T^* : H \to H$. Define $\underline{\underline{h}} = (\underline{h}, ..., \underline{h})$ and $\overline{\overline{h}} = (\overline{h}, ..., \overline{h})$, and rewrite (10) as

$$f_\kappa^i(\exp \underline{h})A^i(\exp \underline{\underline{h}}) > \rho + \delta, \quad i = 1, ..., n,$$
$$f_\kappa^i(\exp \overline{h})A^i(\exp \overline{\overline{h}}) < \rho + \delta, \quad i = 1, ..., n.$$

It follows immediately from (17) that (in vector form) $T^*(\underline{\underline{h}}) > \underline{\underline{h}}$ and $T^*(\overline{\overline{h}}) < \overline{\overline{h}}$, which implies that $T^* : H \to H$.

Using the definition of the elasticities in (15), the derivatives $\partial T_i^*(h)/\partial h_j$ are defined by

$$(\alpha_i(\exp T_i^*(h)) - 1) \times \frac{\partial T_i^*(h)}{\partial h_j} + \theta_{ij}(\exp h) = 0.$$

These derivatives are positive because by assumption $\alpha_i(k_i) < 1$ and $\theta_{ij}(k) > 0$, for all k. This establishes monotonicity of the operator T^*. Summing over j,

$$(1 - \alpha_i(\exp T_i^*(h))) \sum_{j=1}^{n} \frac{\partial T_i^*(h)}{\partial h_j} = \sum_{j=1}^{n} \theta_{ij}(\exp h). \tag{18}$$

Equations (18) and assumption 3.b together imply that $\sum_{j=1}^{n} \partial T_i^*(h)/\partial h_j < 1$, for all h. Now let $d > 0$ and apply the mean value theorem to obtain

$$T_i^*(h + d) = T_i^*(h) + \sum_{j=1}^{n} \frac{\partial T_i^*(h^d)}{\partial h_j} d,$$

where $h^d = h + d(1 - v)$, for some $v \in (0, 1)$.[12] Because $\beta \equiv \sum_{j=1}^{n} \frac{\partial T_i^*(h^a)}{\partial h_j} \in (0, 1)$, T^* has the discounting property

$$T_i^*(h + d) \le T_i^*(h) + d\beta.$$

The contraction mapping theorem then implies the existence of a unique fixed point h^* for T^*. □

3.2 Local Stability

As in the Cobb-Douglas case, the stability Proposition is a local argument. Generalizing it to non-constant elasticities $\alpha_i(k_i)$ and $\theta_{ij}(k)$ is trivial. This is because, a local argument is only concerned with the behavior of the system at the steady state. All results go through, when we replace α_i and θ_{ij} by the steady state values $\alpha_i(k_i^*)$ and $\theta_{ij}(k^*)$. For completeness, I also allow any functional form for the utility function $u(c)$. I define its elasticity by

$$\sigma(c) \equiv -\frac{u''(c)c}{u'(c)}.$$

The reader can verify that the log-linearization of (4), also produces a $2n \times 2n$ system of linear ODEs which takes the form in (13). Using $\alpha_i(k_i^*)$, $\theta_{ij}(k^*)$ and $\sigma(c_i^*)$, the matrices X and Y remain unaltered. We need to change Z slightly to

$$Z = \text{diag}(R^i) + \text{diag}\left(\frac{F^i}{k_i}\right)(\theta - I) + \text{diag}\left(\frac{c_i}{k_i}\right).$$

The equivalent of assumption 3 is

[12]Notice that the notation $h + d$ is slightly misleading here because $h \in H \subset \mathbb{R}^n$, but d is a scalar. It means that the scalar d is added to each element of h. This notation only follows the conventional way of defining the discounting property as part of Blackwell's conditions. $T^* : H \to H$ can be seen as an operator on a function $h : \{1, ..., n\} \to \mathbb{R}$ where above we write $h_i = h(i)$ for notational convenience. Then $(h + d)(i)$ is the function defined by $(h + d)(i) = h(i) + d$. See Stokey et al. (1989) p.54.

Assumption 4

(2.a) $\theta_{ij}(k) = \theta_{ji}(k)$, all k, all $i, j = 1, ..., n$.

(2.b) The matrix $\text{diag}\left(\frac{F^i(k)}{k_i}\right)\theta(k)$ is symmetric and positive semi-definite for all k.

We then have the following Proposition that is stated without proof.

Proposition 5 *Let assumptions 3 and 4 be satisfied. Then $In(A) = (n, n, 0)$ and hence k^* is locally saddle-path stable.*

4 Conclusion

I have presented an extension of the Cass model to an n-country world in which there are positive bilateral externalities within and across countries. A sufficient condition for the existence and uniqueness of a steady state is that production in each country exhibit diminishing returns as a function of both internal and external capital stocks. This steady state is locally stable under the same restriction and an additional one imposing "symmetry" and "smallness" of spillovers. In contrast to stability results for n-sector growth models (Benhabib and Nishimura, 1981; Boldrin and Montrucchio, 1986), the conditions guaranteeing local stability do not depend on the discount rate.

The stability theorem made use of a field of linear algebra known as Inertia Theory. The result used here should generally be useful when analyzing the stability of higher-order differential equations in economics. "Higher order" here means of order higher than only *three*. Closed form solutions for roots of characteristic polynomials are either not available or extremely messy (quartic). Another method for determining the inertia of a matrix is the Routh-Hurwitz method that has, for example, been used for a three-dimensional matrix by Benhabib and Perli (1994). This method becomes difficult to apply in higher-dimensional cases simply because it requires writing down the characteristic polynomial (and solving for the coefficients). The Inertia Theorem used here does not require writing down a polynomial and instead directly examines the matrix of interest. One drawback is that one must find *symmetric* matrices W and M that satisfy equation (14) to apply the Inertia Theorem. For this reason the Inertia Theorem might actually be more attractive for standard optimal control problems, such as optimal n-sector growth models. In those models the matrix of interest typically features blocks of Hessian matrices which are necessarily symmetric. See Moll (2008) for such an application.

Because the model presented above features external effects, the competitive equilibrium will in general not be Pareto optimal. This begs for a comparison of the capital paths obtained from the analysis above to the solution to a social planner's problem as in Romer (1986). In the case of n countries the analysis of a planner's problem actually becomes rather complicated. One cannot simply compare the value of the private and social marginal product of capital as in Romer's paper. This is because the social marginal value of country i's capital now depends also on the shadow prices and production technologies in all other countries in the world. Because the social planner can choose in which country to invest, she faces a more complicated resource allocation problem than a household/firm who will always only invest in its own capital. In fact, the planning problem is equivalent to an n-sector growth model. We know that optimal solutions to such problems can display unstable and chaotic behavior for high discount rates.[13] Somewhat surprisingly, this implies that for the same parameter values, the competitive equilibrium may feature a stable steady state while the Pareto optimum may not.

Growth models with interdependent countries may help answer some important questions in development economics. For example, consider the following puzzles raised by Lucas (1990) and Hall and Jones (1999): “Why doesn't capital flow from rich to poor countries?” and “Why do some countries produce so much more output per worker than others?” As the authors note, these two puzzles are essentially two sides of the same coin, namely that capital shares are only about 1/3. A higher capital share of, say, 2/3 would make both puzzles disappear. Some authors have argued that the effective capital share is higher than 1/3 if capital is interpreted more broadly and externalities in knowledge capital are taken into account. To reach an effective capital share of 2/3 from a physical capital share of 1/3, however, the spillover effect would have to equal the capital share. In the context of a closed economy this seems unnaturally high. If one starts thinking in a multi-country context this is not naturally true anymore. For example, suppose there are $n = 100$ identical countries and that the spillover effect from each is 1/300 (which is clearly low relative to a physical capital share of 1/3). The combined spillover effect is then $100 \times 1/300 = 1/3$ which increases the effective capital share to 2/3. This calculation is of course overly simplistic. The main point I would like the reader to take away from it is that some puzzles in growth theory may arise precisely because each country is treated as an

[13] See for example Benhabib and Nishimura (1981) and Boldrin and Montrucchio (1986).

isolated island. An analysis of an n-country world might provide more satisfactory answers - after all there are $n = 194$ countries in the world.

Appendix - The Gershgorin Disc Theorem and Corollaries

In what follows I state the Gershgorin disc theorem and useful corollaries. All results except corollary 2 are more or less directly copied from Horn and Johnson (1985) (Theorems 6.1.1 and Section 6.3). Some general notation is useful:

- **C**: the complex numbers
- M_m: m-by-m complex matrices

Theorem 2 *Let $B = [b_{ij}] \in M_m$, and let*

$$r_i'(B) \equiv \sum_{j \neq i} |b_{ij}|, \quad 1 \leq i \leq m$$

denote the deleted absolute row sums of B. Then all eigenvalues of B are located in the union of m discs

$$G(B) \equiv \bigcup_{i=1}^{m} \{z \in \mathbf{C} : |z - b_{ii}| \leq r_i'(B)\}.$$

Furthermore, if a union of k of these n discs forms a connected region that is disjoint from all the remaining $m - k$ discs, then there are precisely k eigenvalues of B in this region.

Proof see Horn and Johnson (1985), pp. 344-345. □

Corollary 1 *Let $\Lambda = diag(\lambda_1, ..., \lambda_m)$, let $P = [p_{ij}] \in M_m$, and consider the perturbed matrix $Q = \Lambda + P$. Then all the eigenvalues of Q are contained in the union of m discs*

$$G(Q) \equiv \bigcup_{i=1}^{m} \{z \in \mathbf{C} : |z - \lambda_i| \leq r_i(P)\}, \qquad r_i(P) = \sum_{j=1}^{m} |p_{ij}|.$$

Proof By Theorem 2, the eigenvalues of $Q = \Lambda + P$ are contained in the discs

$$\left\{ z \in \mathbf{C} : |z - \lambda_i - p_{ii}| \leq r_i'(P) = \sum_{j \neq i} |p_{ij}| \right\}, \quad i = 1, ..., m$$

which are contained in the discs

$$\left\{ z \in \mathbf{C} : |z - \lambda_i| \leq r_i(P) = \sum_{j=1}^{m} |p_{ij}| \right\}, \quad i = 1, ..., m. \ \square$$

Corollary 2 *Let $\Lambda = diag(\lambda_1, ..., \lambda_m)$, let $P = [p_{ij}] \in M_m$, and consider the perturbed matrix $Q = \Lambda + P$. Assume that $\lambda_1, ..., \lambda_m$ are positive real numbers and that*

$$\lambda_i > r_i(P) = \sum_{j=1}^{m} |p_{ij}|, \quad i = 1, ..., m. \tag{19}$$

Then all eigenvalues of Q have positive real parts.

Proof Corollary 1 states that the eigenvalues of $Q = \Lambda + P$ are contained in the m discs around λ_i with radius $r_i(P)$. (19) ensures that all m discs lie entirely within the right half of the complex plane. Hence all eigenvalues of Q have positive real parts. □

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