

# Inequality and Financial Development: A Power-Law Kuznets Curve\*

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This note extends the environment in Moll (2012) to feature a stationary wealth distribution.<sup>1</sup> This is achieved by introducing “death shocks.” I also derive two more substantive results. First, I prove that the wealth distribution has a Pareto tail as is common for random growth processes with death (see for example section 3.4.2. in Gabaix, 2009), and as also seems to be relevant empirically (see for example Gabaix, 2009; Benhabib, Bisin and Zhu, 2011; Blaum, 2012). Second, I examine how tail inequality varies with the level of financial development and hence GDP. In particular, I argue that tail inequality is a hump-shaped function of financial development and hence GDP, meaning that it follows a “Kuznets curve.” This hump-shaped relationship is due to two counteracting effects of financial development: a “leverage effect” resulting in greater inequality, and a “return equalization effect” that lowers inequality. For parameterizations that are broadly consistent with the data, the downward-sloping part of the Kuznets curve is more pronounced than the upward-sloping part.

## 1 Introducing a Stationary Wealth Distribution

Consider the steady state equilibrium in section 1.5 of Moll (2012). Entrepreneurs optimally save a fraction  $s(z) = \lambda \max\{z\pi - r - \delta, 0\} + r - \rho$  of their current wealth  $a$ , where  $z$  is the entrepreneur’s productivity which follows a diffusion. Wealth accumulation therefore follows a random growth process

$$\frac{da}{a} = s(z)dt, \quad \text{and} \quad dz = \mu(z)dt + \sigma(z)dW \tag{1}$$

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<sup>1</sup>The model in Moll (2012) does not feature a stationary wealth distribution because individual wealth accumulation is given by a random growth process (that is, the logarithm of wealth follows a random walk).

Processes of this form do not possess a stationary distribution (Angeletos, 2007; Gabaix, 2009). One simple way of extending the model to feature a stationary wealth distribution is to assume that entrepreneurs die with some fixed probability  $\theta > 0$ , and upon death give birth to an offspring with some finite wealth level that potentially depends on her productivity,  $a^*(z)$ . For now, I assume that an individual's offspring inherits her parent's productivity perfectly, and that it is born with the average wealth level of that productivity type  $a^*(z) = \mathbb{E}[a|z]$ . I make this assumption solely because it is convenient, rather than it being “natural,” based on empirical evidence, founded in economic theory, or the like. In particular, this assumption has the convenient implication that the share of wealth held by a given productivity types,  $\omega(z)$ , remains unchanged. As shown by Moll (2012) all aggregates in the model only depend on these wealth shares rather than the entire joint distribution of productivity and wealth. Therefore all aggregates are unaffected by the introduction of “death shocks” and their characterization in Propositions 1 and 2 remains valid.

Future work could explore the robustness of properties of the right tail of the wealth distribution to various alternative and perhaps more natural assumptions. For example, parents could optimally choose how much to bequeath to their children and some fraction of bequests are redistributed through estate taxation as in Benhabib, Bisin and Zhu (2011). (My assumption above does mimic some desirable features of such a setup, for example that wealth is redistributed rather than destroyed upon entrepreneurs' death and that there is redistribution in the sense that some children are born with less than their parents' wealth and others with more.) That being said, my conjecture is that such alternative assumptions will not change the properties of the *tail* of the wealth distribution by much. The individuals in the tail of the wealth distribution are those that have experience a relatively long series of consecutive high productivity shocks, and so must have been alive for a relatively long time. Hence, the exact assumption about the determination of their wealth at birth should not matter much for the properties of tail of the wealth distribution.

## 2 Characterization of Tail of Wealth Distribution

While a full characterization of the entire wealth distribution is not feasible, its *right tail* can be characterized in relatively great detail. First, some useful definitions:

**Definition 1** *We say that a variable,  $a$ , follows a power law (PL) if there exist  $C > 0$  and  $\zeta > 0$  such that*

$$\Pr(\tilde{a} > a) = Ca^{-\zeta}, \quad \text{all } a.$$

*We say that a variable,  $a$ , follows an asymptotic PL if there exists  $C > 0$  and  $\zeta > 0$  such that*

$$\lim_{a \rightarrow \infty} a^\zeta \Pr(\tilde{a} > a) = C.$$

The following Proposition is an adaptation of a Proposition in Gabaix (2010) and is my main tool for characterizing wealth inequality.

**Proposition 1 (Gabaix, 2010)** *Consider the process (1) with death probability  $\theta$ . Then wealth,  $a$ , follows an asymptotic PL. The PL exponent  $\zeta$  is such that there is a function  $f(z) \geq 0$  for all  $z$  that solves*

$$0 = [\zeta s(z) - \theta]f(z) + \mu(z)f'(z) + \frac{1}{2}\sigma^2(z)f''(z). \quad (2)$$

A heuristic proof is in the Appendix. Gabaix (2010) presents an alternative proof using some results from Hansen and Scheinkman (2009).

**Solution Strategy** For given functions  $s(z), \mu(z), \sigma^2(z)$  and a death rate  $\theta$ , equation (2) is a continuous eigenvalue problem. To see this, define the operator  $\mathcal{A}_\zeta$  by

$$\mathcal{A}_\zeta f(z) = [\zeta s(z) - \theta]f(z) + \mu(z)f'(z) + \frac{1}{2}\sigma^2(z)f''(z)$$

Then (2) is  $\mathcal{A}_\zeta f(z) = 0$ . That is,  $f(z)$  is an eigenfunction of  $\mathcal{A}_\zeta$  corresponding to a zero eigenvalue

$$\mathcal{A}_\zeta f = \eta(\zeta)f \quad (3)$$

with  $\eta(\zeta) = 0$ .<sup>2</sup> Using this logic, equation (2) can be solved efficiently using the following two-step procedure: (i) for a given  $\zeta$ , solve (3) and denote the principal eigenvalue by  $\eta(\zeta)$ ; (ii) find  $\zeta > 0$  such that  $\eta(\zeta) = 0$ . This two-step procedure works nicely in practice because the function  $\eta(\zeta)$  turns out to be monotone away from  $\zeta = 0$ . Appendix B provides more detail.

### 3 Results: Power Law Kuznets Curve

In section 2 of Moll (2012), I have solved the model numerically under the assumption that the logarithm for productivity follows an Ornstein-Uhlenbeck process

$$d \log z = -\nu \log z + \sigma dW. \quad (4)$$

and using specific parameter values for  $\nu, \sigma$  and the remaining parameters of the model. I here solve (2) numerically under the same assumptions and parameter values plus a one percent death probability,  $\theta = 0.01$ . In particular, we are interested in how the tail parameter  $\zeta$  varies with the

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<sup>2</sup>We further know from the Perron-Frobenius theorem that  $\mathcal{A}_\zeta$  has a unique principal eigenvalue with corresponding eigenfunction  $f(z)$  which is positive everywhere. Moreover, there are no other positive eigenfunctions except positive multiples of  $f$ , i.e. all other eigenfunctions must have  $f(z) < 0$  for some  $z$ . Since we require  $f(z) \geq 0$  for all  $z$ ,  $\eta$  and  $f$  are the principal eigenvalue and eigenfunction of  $\mathcal{A}_\zeta$ .

level of financial development,  $\lambda$ . Rather than directly reporting  $\zeta$  as a function of  $\lambda$ , I instead report the relationship between two related objects that have more natural counterparts in the data: the share of total wealth held by the top one percent of the population corresponding to  $\zeta$ , and the external finance to GDP ratio corresponding to  $\lambda$ . First, for a Pareto distribution with tail parameter  $\zeta$ , the share of total wealth going to the top  $p \times 100$  percent is given by

$$\text{Share}(p; \zeta) = p^{1-1/\zeta}.$$

Second,  $\lambda$  is related to the external-finance to GDP ratio  $D/Y$

$$\frac{D}{Y} = \frac{D}{K} \frac{K}{Y} = \left(1 - \frac{1}{\lambda}\right) \frac{\alpha}{\rho + \delta}. \quad (5)$$

For example, in 1997, the US had an external finance to GDP ratio of 2.53 (see Table 1 in Moll (2012) based on Beck, Demirguc-Kunt and Levine, 2000) and in 2001 the top 1% of the richest households in the U.S. held around 33% of wealth (Wolff, 2006).

Figure 1 plots the top wealth share as a function of the external-finance to GDP ratio. (the right panel is simply a more “zoomed in” version of the left panel). It can be seen that wealth

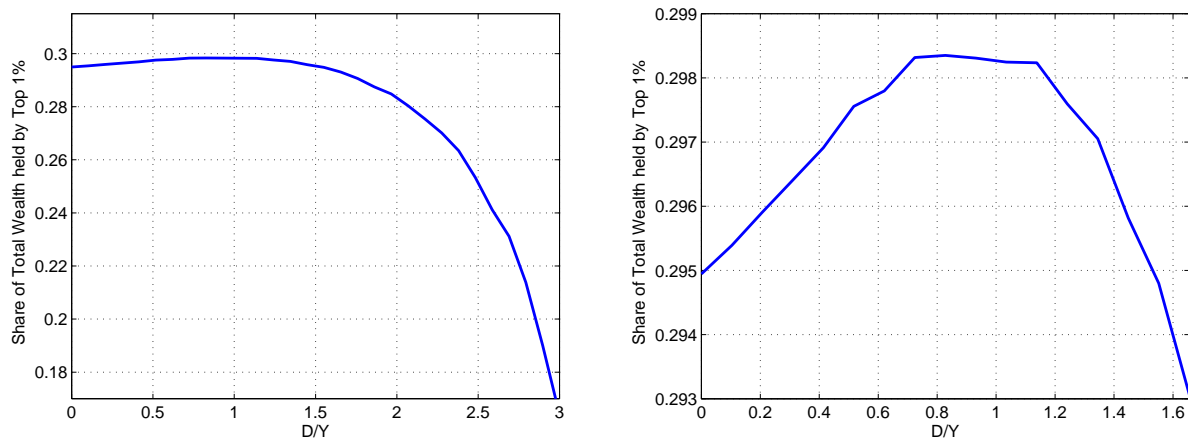


Figure 1: Financial Development and Tail Inequality

inequality is a hump-shaped function of the degree of financial development, that is it follows a Kuznets curve. Furthermore, the downward-sloping part of the Kuznets curve is steeper than the upward-sloping part. The exact shape and relative steepness of the two parts of the curve depend on parameter values. For example, the upward-sloping part of the curve is much steeper for higher values of the autocorrelation of idiosyncratic productivity shocks (in the Figure, I used  $Corr = 0.85$  as in the benchmark parameterization of Moll (2012)). But the curve is hump-shaped for all different parameter values I have tried.

The intuition for why wealth inequality follows a Kuznets curve is relatively straightforward.

The key to understanding it lies in how the savings rate,  $s(z)$ , depends on the degree of financial development,  $\lambda$ . This relationship is plotted in Figure 2. The savings rate as a function of

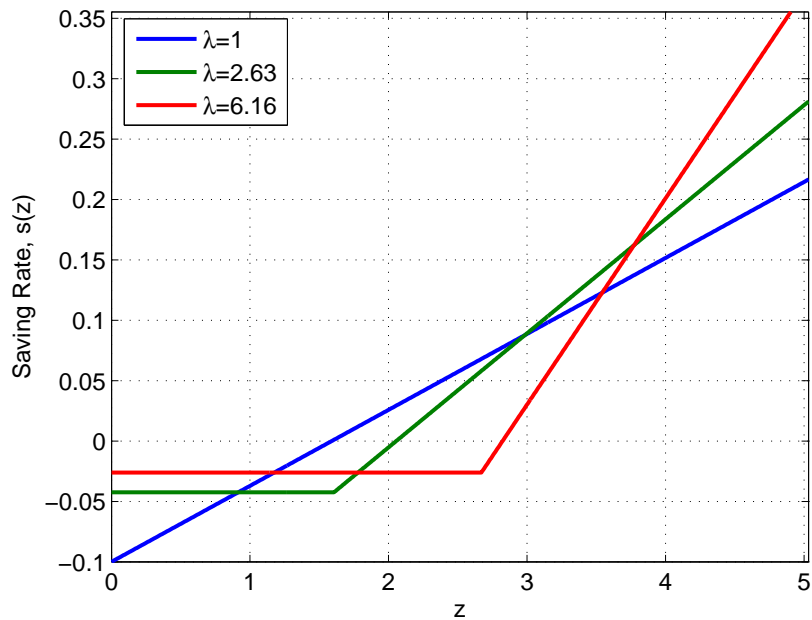


Figure 2: Saving Rate for Different  $\lambda$

productivity generally has two parts, corresponding to whether an entrepreneurs is active or not. Inactive entrepreneurs rent their capital out to active ones so that their return and hence their savings rate does not depend on their productivity. This corresponds to the flat part of the savings rate. In contrast, the savings rate of active entrepreneurs is increasing in their productivity because higher productivity means higher profits. An increase in  $\lambda$  has two effects. First, the savings rate of active entrepreneurs becomes more sensitive to their productivity. This is because active entrepreneurs obtain higher leverage resulting in higher profits and savings. I term this the “leverage effect.” But second, the productivity threshold for being active increases. Therefore, more individuals obtain the same return on their wealth (the interest rate) which is independent of their productivity. I term this the “return equalization effect.”

The “leverage effect” is a force towards higher inequality: higher leverage results in higher wealth concentration for individuals experiencing a sequence of good shocks. But the “return equalization effect” is a force towards lower inequality: more individuals earn the interest rate on their wealth which does not depend on their productivity so that a sequence of good shocks does not lead to higher wealth concentration. For instance, in the limit as  $\lambda \rightarrow \infty$ , only the highest productivity type is active but everyone earns the interest rate equal to his marginal product of capital. Therefore, wealth inequality does not depend on the stochastic process for productivity.

For low  $\lambda$  the first effect dominates, whereas for high  $\lambda$  the second effect dominates, explaining the hump-shaped Kuznets-type relationship between inequality and wealth. These same forces should also be present in more general environments, say with decreasing returns and occupational choice as in Buera and Shin (2010) or Buera, Kaboski and Shin (2011).

## A Proof of Proposition 1

The Kolmogorov forward equation for the stationary joint distribution of productivity and wealth,  $g(a, z)$  is

$$0 = -\theta g(a, z) - \frac{\partial}{\partial a} [g(a, z)s(z)a] - \frac{\partial}{\partial z} [g(a, z)\mu(z)] + \frac{1}{2} \frac{\partial^2}{\partial z^2} [\sigma^2(z)g(a, z)]. \quad (6)$$

Guess that, for large  $a$  (away from the injection point) the joint distribution of ability and wealth takes the form

$$g(a, z) \propto a^{-\zeta-1} \phi(z).$$

Plugging the guess into (6)

$$0 = -\theta a^{-\zeta-1} \phi(z) - \frac{\partial}{\partial a} [a^{-\zeta-1} \phi(z)s(z)a] - \frac{\partial}{\partial z} [a^{-\zeta-1} \phi(z)\mu(z)] + \frac{1}{2} \frac{\partial^2}{\partial z^2} [\sigma^2(z)a^{-\zeta-1} \phi(z)].$$

Dividing through by  $a^{-\zeta-1}$ , we have

$$0 = [\zeta s(z) - \theta] \phi(z) - \frac{\partial}{\partial z} [\phi(z)\mu(z)] + \frac{1}{2} \frac{\partial^2}{\partial z^2} [\sigma^2(z)\phi(z)]. \quad (7)$$

Further, we know that the stationary productivity distribution satisfies the Kolmogorov forward equation

$$0 = -\frac{\partial}{\partial z} [\mu(z)\psi(z)] + \frac{1}{2} \frac{\partial^2}{\partial z^2} [\sigma^2(z)\psi(z)] \quad (8)$$

and hence

$$C = -\mu(z)\psi(z) + \frac{1}{2} \frac{\partial}{\partial z} [\sigma^2(z)\psi(z)] \quad (9)$$

where  $C = 0$ , otherwise  $\psi(z)$  is not integrable. Define  $f(z) \equiv \phi(z)/\psi(z)$ . Plugging  $\phi(z) = \psi(z)f(z)$  into (7)

$$\begin{aligned} 0 = & [\zeta s(z) - \theta] \psi(z) f(z) - \frac{\partial}{\partial z} [\mu(z)\psi(z)f(z)] + \frac{1}{2} \frac{\partial^2}{\partial z^2} [\sigma^2(z)\psi(z)f(z)] \\ & [\zeta s(z) - \theta] \psi(z) f(z) - \left[ \frac{\partial}{\partial z} [\mu(z)\psi(z)] f(z) + \mu(z)\psi(z) f'(z) \right] \\ & + \frac{1}{2} \left[ \frac{\partial^2}{\partial z^2} [\sigma^2(z)\psi(z)] f(z) + 2 \frac{\partial}{\partial z} [\sigma^2(z)\psi(z)] f'(z) + \sigma^2(z)\psi(z) f''(z) \right]. \end{aligned}$$

Collecting terms

$$\begin{aligned} 0 = & \left( [\zeta s(z) - \theta] f(z) + \mu(z) f'(z) + \frac{1}{2} \sigma^2(z) f''(z) \right) \psi(z) \\ & + 2 \left( -\mu(z)\psi(z) + \frac{1}{2} \frac{\partial}{\partial z} [\sigma^2(z)\psi(z)] \right) f'(z) \\ & + \left( -\frac{\partial}{\partial z} [\mu(z)\psi(z)] + \frac{1}{2} \frac{\partial^2}{\partial z^2} [\sigma^2(z)\psi(z)] \right) f(z). \end{aligned}$$

The last two lines equal zero by (8) and (9). Therefore (2) holds.  $\square$

## B Numerical Solution

I solve (2) using a finite difference method. Approximate the derivatives as

$$f'(z_i) \approx \frac{f_{i+1} - f_{i-1}}{2\Delta z}, \quad f''(z_i) \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta z)^2},$$

(2) becomes

$$0 = [\zeta s_i - \theta]f_i + \mu_i \frac{f_{i+1} - f_{i-1}}{2\Delta z} + \frac{1}{2}\sigma_i^2 \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta z)^2}.$$

Rearranging

$$0 = a_i f_{i-1} + b_i f_i + c_i f_{i+1}, \quad a_i = -\frac{\mu_i}{2\Delta z} + \frac{\sigma_i^2}{2(\Delta z)^2}, \quad b_i = \zeta s_i - \theta - \frac{\sigma_i^2}{(\Delta z)^2}, \quad c_i = \frac{\mu_i}{2\Delta z} + \frac{\sigma_i^2}{2(\Delta z)^2}. \quad (10)$$

**Boundary Conditions for Ornstein-Uhlenbeck Process** (4). For the process (4), we have

$$\mu(z) = (-\nu \log z + \sigma^2/2)z, \quad \sigma^2(z) = \sigma^2 z^2$$

and we further assume that the process is reflected at some  $\bar{z}$ . The boundary condition corresponding to such a reflecting barrier is  $f'(\bar{z}) = 0$  and so

$$0 = f'(z_I) \approx \frac{f_{I+1} - f_{I-1}}{2\Delta z} \Rightarrow f_{I-1} = f_{I+1}.$$

Finally, it does not matter what we impose at the lower boundary  $z_1 = 0$ . To see this consider equation (10) for  $i = 1$ . Given that  $z_1 = 0$  and  $\mu(0) = \sigma^2(0) = 0$ , we have that

$$a_1 = -\frac{\mu_1}{2\Delta z} + \frac{\sigma_1^2}{2(\Delta z)^2} = 0.$$

Therefore, we can just drop that term from equation (10) for  $i = 1$ .

Together with the boundary conditions, equation (10) can be arranged in matrix form as

$$\mathbf{A}_\zeta f = 0, \quad \mathbf{A}_\zeta \equiv \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 \\ 0 & a_3 & b_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & a_{I-1} + c_{I-1} & b_{I-1} \end{bmatrix}$$

We solve it using the two-step procedure described above: (i) solve the eigenvalue problem  $\mathbf{A}_\zeta f = \eta(\zeta)f$ , (ii) solve  $\eta(\zeta) = 0$ .

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