

Online Appendix to “The Dynamics of Inequality”

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E The Dynamics of Wealth Inequality

In this appendix we explore the implications of our results for the dynamics of wealth inequality. We first provide a brief overview of the facts, and then show how our theoretical results can be extended to a simple model of top wealth inequality. We then ask whether an increase in $r - g$, the gap between the after-tax average rate of return and the growth rate, can explain the increase in top wealth inequality observed in some datasets as suggested by Piketty (2014).

E.1 Motivating Facts: the Evolution of Top Wealth Inequality

Figure 7 presents facts about the evolution of top wealth inequality, analogous to those about top income inequality in Figure 1. Panel (a) shows the time path of the top 1%

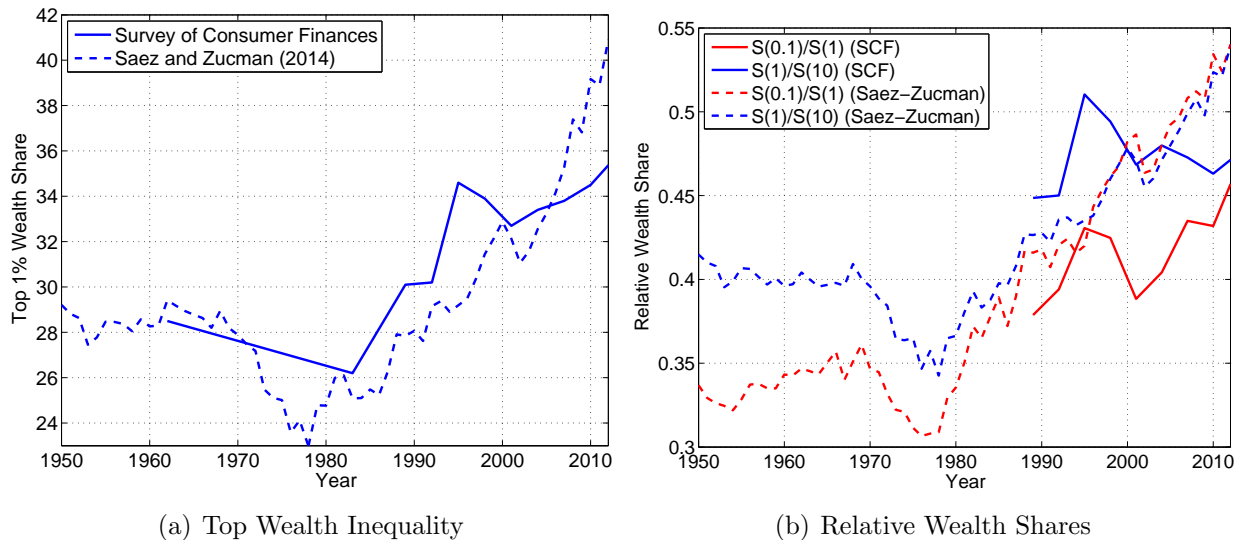


Figure 7: Evolution of Top Wealth Inequality

wealth share from two different data sources. The first is the Survey of Consumer Finances (SCF) and the second is a series constructed by Saez and Zucman (2015) by capitalizing capital income data.⁶¹ The two series suggest quite different conclusions. In particular, data

⁶¹The SCF data for 1989 to 2013 is from the Online Appendix of Saez and Zucman (2015) available at <http://gabriel-zucman.eu/files/SaezZucman2014MainData.xlsx> in Sheet DataFig1-6-7-11-12. The

from the SCF suggest a relatively gradual rise in the top 1% wealth share, whereas Saez and Zucman’s estimates suggest a much more dramatic rise, a discrepancy that has generated some controversy (see e.g. Kopczuk, 2015; Bricker, Henriques, Krimmel, and Sabelhaus, 2015).⁶² Finally, comparing Figures 1 and 7 one can also see that wealth is much more unequally distributed than income.

Panel (b) plots the evolution of relative wealth shares which are informative about the fatness of the Pareto tail of the wealth distribution as discussed in Section 2. The finding depends again on the underlying data source, with the SCF showing no clear pattern and the capitalization method suggesting a large thickening of the tail of the wealth distribution.

There are three main takeaways from this section. First, top wealth shares appear to have increased though it is unclear by how much. Second, it is ambiguous whether the thickness of the tail of the wealth distribution has increased over time. And finally, wealth is more unequally distributed than income and, relatedly, the wealth distribution has a fatter Pareto tail than the income distribution.

E.2 A Simple Model of Top Wealth Inequality

The following simple model captures the main features of a large number of models of the upper tail of the wealth distribution.⁶³ Time is continuous and there is a continuum of individuals that are heterogeneous in their wealth \tilde{w}_{it} . At the individual level, wealth evolves as

$$d\tilde{w}_{it} = (1 - \tau)\tilde{w}_{it}dR_{it} + (y_t - c_{it}) dt$$

where τ is the capital income tax rate, dR_{it} is the rate of return on wealth which is stochastic, y_t is labor income and c_{it} is consumption. To keep things simple, we make the following assumptions. First, capital income is i.i.d. over time and, in particular, $dR_{it} = \tilde{r}dt + \tilde{\nu}dZ_{it}$, where \tilde{r} and $\tilde{\nu}$ are parameters, and Z_{it} is a standard Brownian motion, which reflects idiosyncratic returns to human capital or to financial capital (this idiosyncratic shock captures the undiversified ownership of an entrepreneur, for instance).⁶⁴ Second, we assume that indi-

SCF data for 1962 and 1983 is from Wolff (1987, Table 3). The 1962 dataset is called the “Survey of Financial Characteristics of Consumers” or SFCC.

⁶²Kopczuk (2015) notes that a third method of measuring top wealth shares, the estate-tax multiplier technique, suggests an even smaller increase in the top one percent wealth share than the SCF. Also see critique of Piketty (2014) by Auerbach and Hassett (2015).

⁶³See e.g. Wold and Whittle (1957), Benhabib, Bisin, and Zhu (2011, 2015a,b), Piketty and Zucman (2014b), Jones (2015) and Acemoglu and Robinson (2015).

⁶⁴Benhabib, Bisin, and Luo (2015) argue that, in the data, such uninsured capital income risk is the main determinant of the wealth distribution’s right tail. In Section 5.3 we additionally consider common shocks.

viduals consume an exogenous fraction θ of their wealth at every point in time, $c_{it} = \theta \tilde{w}_{it}$.⁶⁵ Third, we assume that all individuals earn the same labor income y_t , which grows deterministically at a rate g , $y_t = ye^{gt}$. Given these assumptions, it is easy to show that detrended wealth $w_{it} = \tilde{w}_{it}e^{-gt}$ follows the stochastic process

$$dw_{it} = (y + \gamma w_{it})dt + \sigma w_{it}dZ_{it}, \quad \gamma := r - g - \theta \quad (54)$$

where $r = (1 - \tau)\tilde{r}$ is the after-tax average rate of return on wealth and $\sigma = (1 - \tau)\tilde{\nu}$ is the after-tax wealth volatility. Note that (54) is a standard random growth process with the addition of an additive income term ydt . This income term acts as a stabilizing force. Many other shocks (e.g. demographic shocks or shocks to saving rates) result in a similar reduced form. From Ito's formula, the logarithm of wealth $x_{it} = \log w_{it}$ satisfies

$$dx_{it} = (ye^{-x_{it}} + \mu)dt + \sigma dZ_{it}, \quad \mu := r - g - \theta - \frac{\sigma^2}{2}. \quad (55)$$

The properties of the stationary wealth distribution are again well understood. Applying the standard results from Appendix D, one can show that the stationary wealth distribution has a Pareto tail with tail inequality

$$\eta = \frac{1}{\zeta} = \frac{\sigma^2/2}{\sigma^2/2 - (r - g - \theta)} \quad (56)$$

provided that $r - g - \theta - \sigma^2/2 < 0$. Intuitively, tail inequality is increasing in the gap between the after-tax rate of return to wealth and the growth rate $r - g$. Similarly, tail inequality is higher the lower the marginal propensity to consume θ and the higher the after-tax wealth volatility σ . Given that $r = (1 - \tau)\tilde{r}$ and $\sigma = (1 - \tau)\tilde{\nu}$, top wealth inequality is also decreasing in the capital income tax rate τ . Intuitively, a higher gap between r and g works as an “amplifier mechanism” for wealth inequality: for a given structure of shocks (σ), the long-run magnitude of wealth inequality will tend to be magnified if the gap $r - g$ is higher (Piketty and Zucman, 2014b). However, this leaves unanswered the question whether increases in top wealth inequality triggered by an increase in $r - g$ will come about quickly or take many hundreds of years to materialize.

The model can easily be extended to the case where labor income is stochastic, i.e. y in (54) follows some stochastic process. As long as the income process does not itself have a fat-tailed stationary distribution, this does not affect the tail parameter of the wealth

⁶⁵A consumption rule with such a constant marginal propensity to consume can also be derived from optimizing behavior, at least for large wealth levels w_{it} . Results available upon request.

distribution (56). Intuitively, as wealth $w_{it} \rightarrow \infty$, labor income becomes irrelevant as an income source.

E.3 Dynamics of Wealth Inequality: Theoretical Results

We now show how our theoretical results can be extended to wealth dynamics. The addition of the labor income term y in (55) introduces some difficulties for extending Proposition 1. However, note that for large wealth levels this term becomes negligible, which makes it possible to derive a tight *upper bound* on the speed of convergence of the cross-sectional distribution.

Proposition 9 (Speed of convergence for wealth dynamics) *Consider the wealth process (55). Under Assumption 1, and if $\mu < 0$, the cross-sectional distribution $p(x, t)$ converges to its stationary distribution. The rate of convergence $\lambda := -\lim_{t \rightarrow \infty} \frac{1}{t} \log \|p(x, t) - p_\infty(x)\|$ satisfies*

$$\lambda \leq \frac{1}{2} \frac{\mu^2}{\sigma^2} + \delta$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function, and with equality for $|\mu|$ below a threshold $|\mu^*|$.

We conjecture that with $\mu > 0$, $\lambda = \delta$, as in Proposition 1.

With the process (55), it is not possible to obtain an exact formula for the speed of convergence. However, the speed of convergence is bounded above and, in particular, is equal to or less than the speed with a reflecting barrier from Proposition 1. It is also no longer possible to characterize the corresponding Kolmogorov Forward equation by using Laplace transforms (due to the presence of the term ye^{-xt}). Numerical experiments nevertheless confirm our results from section 4.2 that the speed of convergence in the tail can be substantially lower than the average speed of convergence characterized in Proposition 9.

E.4 Wealth Inequality and Capital Taxes

In this section we ask whether an increase in $r - g$, the gap between the (average) after-tax rate of return on wealth and the economy's growth rate, can explain the increase in wealth inequality observed in some data sets, as suggested by Piketty (2014). To do so, we first construct a measure of the time series of $r - g$. This requires three data inputs: on the average pre-tax rate of return, on capital income taxes, and on a measure of the economy's growth rate. We use data on the average before-tax rate of return from Piketty and Zucman (2014a), the series of top marginal capital income tax rates from Auerbach and Hassett (2015) and data on the growth rate of per capita GDP of the United States from the Penn

World Tables. Panel (a) of Figure 8 plots our time series for $r - g$, displaying a strong upward trend starting in the late 1970s, which coincides with the time when top wealth inequality started to increase (Figure 1).⁶⁶ The figure therefore suggests that, a priori, the theory using variations in $r - g$ is a potential candidate for explaining increasing wealth inequality.

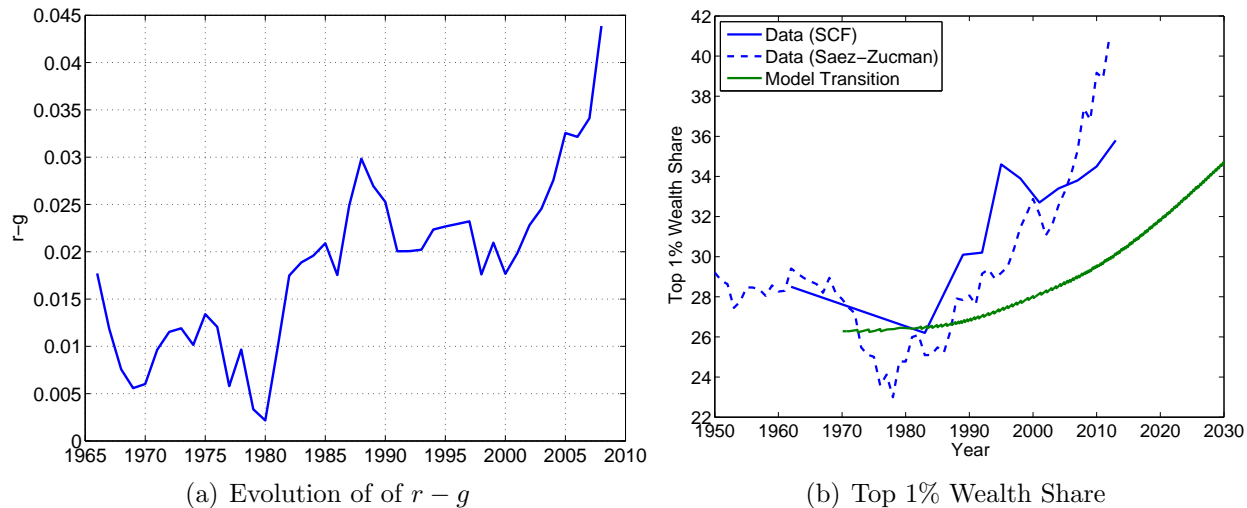


Figure 8: Dynamics of Wealth Inequality in the Baseline Model

We now ask whether the simple model of wealth accumulation from Section 3 has the potential to explain the different data series for wealth inequality in Figure 1. To this end, recall equation (54) and note that the dynamics of this parsimonious model are described by two parameter combinations only, $r - g - \theta$, where θ is the marginal propensity to consume out of wealth, and the cross-sectional standard deviation of the return to capital, σ . Our exercise proceeds in three steps. First, we obtain an estimate for σ . We use $\sigma = 0.3$, which is on the upper end of values estimated or used in the existing literature.⁶⁷ Second, given σ and our data for $r - g$ in 1970, we calibrate the marginal propensity to consume θ so as to match the tail inequality observed in the data in 1970, $\eta = 0.6$. Third, we feed the time path for $r - g$ from panel (a) of Figure 8 into the calibrated model.

Before comparing the model's prediction to the evolution of top wealth inequality in the data, we make use of our analytic formulas from Section 4 to calculate measures of the speed of convergence. To this end, revisit the average speed of convergence in Proposition 1, and

⁶⁶We have tried a number of alternative exercises with different data series for the return on capital and taxes, e.g. we set the pre-tax r equal to the yields of 10-year government bonds as in Auerbach and Hassett (2015) and Piketty and Zucman (2014b). Results are very similar and available upon request.

⁶⁷Overall, good estimates of σ are quite hard to come by and relatively dispersed. Campbell (2001) provides the only estimates for an exactly analogous parameter using Swedish wealth tax statistics on asset returns. He estimates an average σ of 0.18. Moskowitz and Vissing-Jorgensen (2002) argue for a σ of 0.3.

in particular the formula in terms of inequality (13). To operationalize this formula, we use the tail exponent observed in 2010 in the SCF of $\eta = 0.65$ together with our other parameter values.⁶⁸ With these numbers in hand, we obtain a half-life of

$$t_{1/2} \geq \frac{\log(2) \times 8 \times \eta^2}{\sigma^2} = \frac{\log(2) \times 8 \times (0.65)^2}{0.3^2} \approx 26 \text{ years.}$$

That is, on average, the distribution takes 26 years to cover half the distance to the new steady state. Panel (b) of Figure 8 displays the results of our experiment using the parameter values just discussed. The main takeaway is that the baseline random growth model cannot even explain the gradual rise in top wealth inequality found in the SCF. It fails even more obviously in explaining the rise in top wealth inequality found by Saez and Zucman (2015).

E.5 Fast Dynamics of Wealth Inequality

What, then, explains the dynamics of wealth inequality observed in the data? The lessons from Section 5 still apply. In particular, processes of the form (26) that feature deviations from Gibrat’s law in the form of “type dependence” or “scale dependence” have the potential to deliver fast transitions. We view both as potentially relevant for the case of wealth dynamics. Wealth dynamics at the individual level depend on both rates of returns and saving rates, and heterogeneity or wealth-dependence in either would result in such deviations from Gibrat’s law.

With regard to rates of returns, Fagereng, Guiso, Malacrino, and Pistaferri (2016), using high-quality Norwegian administrative data, find evidence for type dependence across the entire support of the wealth distribution. Additionally, they find some evidence for scale dependence particularly above the 95th percentile of the wealth distribution, mostly because wealthier people take more risk and compensated in the form of higher returns. This is also consistent with evidence in Bach, Calvet, and Sodini (2015) using Swedish administrative data, as well as Kacperczyk, Nosal, and Stevens (2014). With regard to saving rates, scale dependence may arise because the saving rates of the super wealthy relative to those of the wealthy may change over time (Saez and Zucman, 2015).⁶⁹

⁶⁸Ideally, one would use an estimate of tail inequality in the new stationary distribution η . Since λ is decreasing in inequality, we use the tail exponent observed in 2010 in the SCF of $\eta = 0.65$, which provides an upper bound on the speed of convergence λ . Since inequality in the new stationary distribution may be even higher, true convergence may be even slower.

⁶⁹Finally, it is natural to ask whether the extension to multiple distinct growth regimes of Section 5.2 can generate fast transition dynamics in response to the increase in $r - g$ from Section E.4. Numerical experiments suggest that the answer is no.

E.6 Proof of Proposition 9

The proof follows steps exactly analogous to the proof of Proposition 1, particularly the “ergodic case” in Appendix A.2. The only difference to the earlier proof is that some of the arguments need to be adjusted to account for the presence of the term $ye^{-x}dt$ in the wealth process (55). We again present the proof for the case $\delta = 0$. Denote the drift of wealth by

$$b(x) = \mu + ye^{-x}.$$

Consider the Kolmogorov Forward equation corresponding to (55)

$$p_t = \mathcal{A}^*p, \quad \mathcal{A}^*p = -(b(x)p)_x + \frac{\sigma^2}{2}p_{xx} \quad (57)$$

and its adjoint

$$\mathcal{A}u = b(x)u_x + \frac{\sigma^2}{2}u_{xx}.$$

As in the proof of Proposition 1, the strategy is again to construct a self-adjoint transformation \mathcal{B} of \mathcal{A} , which is again found as the operator corresponding to $v = up_\infty^{1/2}$ where p_∞ is the stationary distribution corresponding to (57). To find p_∞ , define $B(x) := -ye^{-x} + \mu x$ such that $B'(x) = b(x)$ and write

$$0 = -(B'p)' + \frac{\sigma^2}{2}p'' \quad \Rightarrow \quad \frac{p'}{p} = \frac{2B'}{\sigma^2} \quad \Rightarrow \quad p_\infty(x) \propto e^{2B(x)/\sigma^2}.$$

Since $\mu < 0$ and $y > 0$, $B(x) \rightarrow -\infty$ as $x \rightarrow \pm\infty$. Hence $p_\infty(x) \rightarrow 0$ for $x \rightarrow \pm\infty$ and there is a well-defined stationary distribution.

The rest of the proof establishes analogous versions of Lemmas 6 and 7.

Lemma 9 *Consider u satisfying $u_t = \mathcal{A}u$ and the corresponding stationary distribution, $p_\infty(x) = ce^{2B(x)/\sigma^2}$. Then $v = up_\infty^{1/2} := ue^{B(x)/\sigma^2}$ satisfies*

$$v_t = \mathcal{B}v, \quad \mathcal{B}v = \frac{\sigma^2}{2}v_{xx} - \frac{1}{2\sigma^2}(\mu^2 + y^2e^{-2x} + 2ye^{-x}\mu - \sigma^2ye^{-x})v. \quad (58)$$

Furthermore, \mathcal{B} is self-adjoint.

Proof.

We have $v_t = e^{B(x)/\sigma^2}b(x)u_x + \frac{\sigma^2}{2}e^{B(x)/\sigma^2}u_{xx}$. We need to check that the right hand side

is equal to $\frac{\sigma^2}{2}v_{xx} - \frac{1}{2\sigma^2}(\mu^2 + y^2e^{-2x} + 2ye^{-x}\mu - \sigma^2ye^{-x})v$. For this, we need to calculate

$$\begin{aligned} v_x &= u_x e^{B(x)/\sigma^2} + \frac{b(x)}{\sigma^2} e^{B(x)/\sigma^2} u \\ v_{xx} &= u_{xx} e^{B(x)/\sigma^2} + 2\frac{b(x)}{\sigma^2} u_x e^{B(x)/\sigma^2} + \frac{-ye^{-x}}{\sigma^2} e^{B(x)/\sigma^2} u + \left(\frac{b(x)}{\sigma^2}\right)^2 e^{B(x)/\sigma^2} u \\ b(x)^2 &= \mu^2 + 2\mu ye^{-x} + y^2 e^{-2x} \end{aligned}$$

and so

$$\frac{\sigma^2}{2}v_{xx} = \frac{\sigma^2}{2}u_{xx}e^{B(x)/\sigma^2} + b(x)u_x e^{B(x)/\sigma^2} + \frac{-ye^{-x}}{2}e^{B(x)/\sigma^2}u + \frac{\mu^2 + 2\mu ye^{-x} + y^2e^{-2x}}{2\sigma^2}e^{B(x)/\sigma^2}u$$

which gives the desired result. \square

Lemma 10 *The first eigenvalue of \mathcal{B} is $\lambda_1 = 0$. The second eigenvalue satisfies the following properties: there exists $-\infty < \mu^* < 0$ such that $\lambda_2 = -\frac{1}{2}\frac{\mu^2}{\sigma^2}$ for $|\mu| \leq |\mu^*|$ and $\lambda_2 \leq -\frac{1}{2}\frac{\mu^2}{\sigma^2}$ for all $\mu < 0$. All remaining eigenvalues satisfy $|\lambda| > |\lambda_2|$. Put differently, the spectral gap of \mathcal{B} satisfies $|\lambda_2| \leq \frac{1}{2}\frac{\mu^2}{\sigma^2}$.*

The conclusion of the proof (from spectral gap to L^1 -norm) is unchanged from that of Proposition 1.

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