Online Appendix 2 to "The Dynamics of Inequality"

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J Proof of Proposition 2

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J.1 Proof of Proposition 2: Case without lower bound ("non-ergodic").

The proof strategy is roughly as follows. We take an initial distribution that is essentially completely in the "upper tail" (above some very large $R > 0$). There, the process is basically a constant-coefficient process. Then, as in Proposition 1, the speed of convergence is δ. For an arbitrary initial distribution, there is a small perturbation, with a small mass in the upper tail, that ensures a speed arbitrarily close to δ.

Take $q(x,t) = e^{\delta t}(p(x,t) - p_\infty(x))$. Then,

$$q_t = -(\mu q)_x + (Dq)_{xx}, \qquad D(x) := \frac{\sigma^2(x)}{2}$$

with initial condition $q(x,0) = q_0(x) = p_0(x) - p_\infty(x)$.

Call $S(t)$ the solution semi-group for the equation $q_t = -(\mu q)_x + (Dq)_{xx}$, i.e. $q(x,t) = (S(t)q_0)(x)$ and $S(t+s) = S(t)S(s)$.

We define

$$\lambda(q) := -\overline{\lim}_{t\to\infty} \frac{1}{t} \log \|S(t)q\|_{L^1} \tag{94}$$

Given that for all t, $\|S(t)q\|_{L^1} \leq \|q\|_{L^1}$, we have $\lambda(q) \geq 0$.

Observe that semi-group $S(t)$ "removed a factor δ". Hence, proving the Proposition 2 here is proving that, for a given p_0, there is a $\tilde{p}_0$ arbitrarily close to p_0 such that $\lambda(\tilde{p}_0 - p_\infty) = 0$.

We start with a generally useful lemma.

Lemma 11 (The slowest convergence wins) *Suppose two distributions q, r with $\lambda(q) < \lambda(r)$. Then, $\lambda(q+r) = \lambda(q)$.*

Hence, if there are two distributions with $\lambda(q) < \lambda(r)$, the convergence rate of $q + r$ is the convergence rate of q, the slowest (if $\lambda(q) = \lambda(r)$, we could have $\lambda(q+r) < \lambda(q)$).

Proof of Lemma 11 The definition (94) implies that, for all $\varepsilon > 0$ small enough, $\exists T, \forall t \geq T$

$$\frac{1}{t} \log \|S(t)\, q\| \leq -\lambda(q) + \varepsilon$$

and there is a series (t_n), $t_n \to \infty$, such that

$$\frac{1}{t_n} \log \|S(t_n)\, q\| \geq -\lambda(q) - \varepsilon$$

There is an equivalent characterization of $\lambda(q)$ that we will use, by exponentiating and introducing constants: for all $\varepsilon > 0$ small enough, $\exists T, C > 0$, $\forall t \geq T$

$$\|S(t)\, q\| \leq Ce^{-(\lambda(q)-\varepsilon)t} \tag{95}$$

and there is a series (t_n), $t_n \to \infty$, and a constant $C' > 0$, such that

$$\|S(t_n)\, q\| \geq C' e^{-(\lambda(q)+\varepsilon)t_n} \tag{96}$$

Now, given a small $\varepsilon > 0$, the above characterization gives T_q, C_q etc. Set $T = \max(T_q, T_r)$. We have, for all $t \geq T$,

$$\begin{aligned} &\|S(t)(q+r)\| \leq \|S(t)\, q\| + \|S(t)\, r\| \leq C_q e^{-(\lambda(q)-\varepsilon)t} + C_r e^{-(\lambda(r)-\varepsilon)t} \\ &\|S(t)(q+r)\| \leq (C_q + C_r)\, e^{-(\lambda(q)-\varepsilon)t} \end{aligned} \tag{97}$$

Next, we have a series t_n such that $\|S(t_n)\, q\| \geq C'_q e^{-(\lambda(q)+\varepsilon)t_n}$. That implies:

$$\|S(t_n)(q+r)\| \geq \|S(t_n)\, q\| - \|S(t_n)\, r\| \geq C'_q e^{-(\lambda(q)+\varepsilon)t_n} - C_r e^{-(\lambda(r)-\varepsilon)t_n}$$

We suppose that ε is small enough so that $\lambda(q) + \varepsilon < \lambda(r) - \varepsilon$. For t_n large enough, $C_r e^{-(\lambda(r)-\varepsilon)t_n} \leq \frac{1}{2} C'_q e^{-(\lambda(q)+\varepsilon)t_n}$, so that

$$\|S(t_n)(q+r)\| \geq \frac{1}{2} C'_q e^{-(\lambda(q)+\varepsilon)t_n} \tag{98}$$

Letting $\varepsilon \to 0$ proves that $\lambda(q+r) = \lambda(q)$.□

We define $\|f\|_{L^{1,\infty}} := \|f\|_{L^1} + \|f\|_{L^\infty} = \int |f|\, dx + \sup_x |f(x)|$ is the sum of the L^1 and L^∞ norm, which is a norm for $L^{1,\infty} := L^1 \cap L^\infty$.

We show a Lemma which means, in some sense, that the worse case speed has to be 0.

Lemma 12 *For all $\varepsilon > 0$,*

$$\sup\left\{\|S(t)\,q_0\|_{L^1}\,e^{\varepsilon t} : t \geq 0, \|q_0\|_{1,\infty} \leq 1, \int q_0 = 0\right\} = \infty \tag{99}$$

Proof of the Lemma 12 Suppose by contradiction that (99) is not true. There is a $\varepsilon > 0$ and a $C > 0$ such that for all $\|q_0\|_{1,\infty} \leq 1, \int q_0 = 0$

$$\int_{-\infty}^{\infty} |q(x,t)|\,dx \leq Ce^{-\varepsilon t}. \tag{100}$$

We will reach a contradiction. Define,

$$q^{(R)}(x) := q(x+R), \qquad \mu^{(R)}(x) := \mu(x+R), \qquad D^{(R)}(x) := D(x+R)$$

so that $q^{(R)}$ satisfies the equation

$$q_t^{(R)} = -\left(\mu^{(R)} q^{(R)}\right)_x + \left(D^{(R)} q^{(R)}\right)_{xx}.$$

Let's consider a given distribution $Q_0(x)$ which is C^{∞} and with compact support and with $\|Q_0\|_{1,\infty} \leq 1$. Consider the particular initial condition

$$q^{(R)}(x,0) = Q_0(x). \tag{101}$$

Consider also the equation $q_t = -(\mu q)_x + (Dq)_{xx}$ with initial condition $q_0(x) := Q_0(x-R)$. Then, for all time t,

$$q(x,t) = q^{(R)}(x-R,t). \tag{102}$$

Also, it follows from (100) that $\int_{-\infty}^{\infty} |q(x,t)|\,dx \leq Ce^{-\varepsilon t}$. Given that $\int_{-\infty}^{\infty} |q(x,t)|\,dx = \int_{-\infty}^{\infty} \left|q^{(R)}(x,t)\right| dx$ we have:

$$\int_{-\infty}^{\infty} \left|q^{(R)}(x,t)\right| dx \leq Ce^{-\varepsilon t}.$$

Now, taking the limit as $R \to \infty$ and using Fatou's Lemma, we have[83] a limit function

[83] The argument for the existence relies on Prokhorov's Theorem. The family of measures $|q^R(x,t)|$ is tight, so by Prokhorov's Theorem there exists a subsequence converging weakly (in the sense of measures) to some limit. Then pass to the limit in the sense of distributions (say) the limit is a solution of the limit equation. It is unique and smooth, as in the theory of the heat equation.

$q^{(\infty)}$, with initial condition $q^{(\infty)}(x,0) = Q_0(x)$, such that:

$$\int_{-\infty}^{\infty} \left|q^{(\infty)}(x,t)\right| dx \leq Ce^{-\varepsilon t} \tag{103}$$

where $q^{(\infty)}$ satisfies

$$q_t^{(\infty)} = -\left(\bar{\mu} q^{(\infty)}\right)_x + \left(\bar{D} q^{(\infty)}\right)_{xx}. \tag{104}$$

But this is exactly the same equation as (69) in the Proof of Proposition 1. And we saw that the solution $q^{(\infty)}(x)$ does not decay exponentially (via the heat equation). This contradicts (103).□

We next refine Lemma 12 to show the existence of a particular q_* such that $\lambda(q_*) = 0$.

Lemma 13 *There is a $q_* \in L^1 \cap L^\infty$ with $\int q_* = 0$ such that $\lambda(q_*) = 0$.*

Proof of Lemma 13 From the Banach-Steinhaus theorem[84] and (99), there is a $q_* \in L^1, \|q_*\|_{L^{1,\infty}} \leq 1, \int q_* dx = 0$, such that

$$\sup_{t \geq 0} \|S(t)\, q_*\|_{L^1} e^{\varepsilon t} = \infty$$

i.e. $\lambda(q_*) \leq 0$. Given that for all q, $\lambda(q) \geq 0$, we have $\lambda(q_*) = 0$.□

Let us conclude the proof. We start from a given p_0. If $\lambda(p_0 - p_\infty) = 0$, we are all set. If $\lambda(p_0 - p_\infty) > 0$, take a q_* given by Lemma 13. We consider a nearby density $\tilde{p}_0 := (1-\varepsilon_1)\, p_0 + \varepsilon_1 p_\infty + \varepsilon_2 q_*$, with $\varepsilon_1 > 0$ arbitrarily small. To make sure that $\tilde{p}_0 \geq 0$, we impose: $\varepsilon_2 \|q_*\|_{L^\infty} < \varepsilon_1 \|p_\infty\|_{L^\infty}$. Next, we have $\tilde{p}_0 - p_\infty = (1-\varepsilon_1)(p_0 - p_\infty) + \varepsilon_2 q_*$. Applying Lemma 11 to $q = \varepsilon_2 q_*$ and $r = (1-\varepsilon_1)(p_0 - p_\infty)$ with $\lambda(q) = 0 < \lambda(r) = \lambda(p_0 - p_\infty)$, we obtain: $\lambda(q + r) = \lambda(q) = 0$, i.e., given $\tilde{p}_0 - p_\infty = q + r$,

$$\lambda(\tilde{p}_0 - p_\infty) = 0$$

Hence, we found a $\tilde{p}_0$ arbitrarily close to p_0, whose speed of convergence is 0.□

J.2 Proof of Proposition 2: Case with a lower bound ("ergodic")

We here prove the statement for the case with a lower bound on income (either a reflecting barrier or exit with reinjection). For simplicity, we first focus on the case without death $\delta = 0$ and constant coefficients $\mu(x,t) = \bar{\mu}, \sigma(x,t) = \bar{\sigma}$. The generalization to variable

[84] Here the family of continuous mappings is $S(t)e^{\varepsilon t}$ indexed by t from L^1 to L^1.

coefficients is then relatively straightforward using a "translation at infinity" argument, and the case $\delta > 0$ will also be a direct generalization. Note that the constant coefficient case with reflecting barrier has already been proven in Proposition 1. However in this section we employ a different approach involving "energy methods" that gives a treatment to a one-parameter family of models emcompassing both reflection and exit with reinjection.

J.2.1 Setting the stage: a unified one-parameter model with a lower bound

We start by remarking that one can embed the model with exit and reinjection and the model with a reflecting barrier in a one-parameter family of models.

Let $\mathcal{A}^*$ be the operator

$$\mathcal{A}^* p := \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \mu\frac{\partial p}{\partial x} + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p(0)\rho(x)$$

with $\mu < 0$ and boundary condition

$$\frac{\sigma^2}{2}p_x(0) - \mu p(0) = \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p(0) \tag{105}$$

where $\theta \in [0,1]$. This recovers the special cases of pure reflection, $\theta = 1$, and pure exit with reinjection, $\theta = 0$. When $\theta = 1$, we get $\mathcal{A}^* = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \mu\frac{\partial p}{\partial x}$ with boundary condition $\frac{\sigma^2}{2}p_x(0) - \mu p(0) = 0$, consistent with the pure reflection case. When $\theta = 0$, (105) implies the boundary condition $p(0) = 0$ and

$$\lim_{\theta \to 0}\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p(0) = \frac{\sigma^2}{2}p_x(0) \tag{106}$$

and therefore substituting into $\mathcal{A}^* p = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \mu\frac{\partial p}{\partial x} + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p(0)\rho(x)$, we have $\mathcal{A}^* p = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \mu\frac{\partial p}{\partial x} + \frac{\sigma^2}{2}p_x(0)\rho(x)$ consistent with the exit with reinjection case.

Let p_∞ be the solution[85] to

$$\mathcal{A}^* p_\infty = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p_\infty - \mu\frac{\partial}{\partial x}p_\infty + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\rho = 0 \tag{107}$$

We would like p_∞ to be the generalized stationary distribution, that is $p \geq 0$ and $\int_0^\infty p_\infty dx = 1$. Note that p_∞ multiplied by any constant c remains a solution of (107), so we can rescale p_∞ so that $\frac{1-\theta}{\theta}p_\infty(0) \geq 0$. Then $-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p_\infty + \mu\frac{\partial}{\partial x}p_\infty \geq 0$ on $[0,\infty)$, so by the strong maximum principle for uniformly elliptic operators, $p_\infty(x) > 0$ on $(0,\infty)$.

[85] A sufficient condition for the existence and uniqueness of a decaying solution is $\mu < 0$.

Integrating the equation above from 0 to x and using the boundary condition for $\mathcal{A}^*$, we obtain

$$\frac{\sigma^2}{2}\frac{\partial}{\partial x}p_\infty(x) - \mu p_\infty(x) = \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\int_x^\infty \rho(y)dy \tag{108}$$

for $x \in (0,\infty)$. If ρ has compact support, then for all large x, $p_\infty(x) = Ce^{\frac{2\mu}{\sigma^2}x}$. If $\theta \in (0,1)$ and $p_\infty(0) = 0$, then $\frac{\sigma^2}{2}\frac{\partial}{\partial x}p_\infty(x) - \mu p_\infty(x) = 0$ for all $x \in (0,\infty)$, whence $p_\infty(x) = Ce^{\frac{2\mu}{\sigma^2}x}$, contradicting $p_\infty(0) = 0$. Thus for $\theta \in (0,1]$, $p_\infty(0) > 0$. Similarly, when $\theta = 0$, $\frac{\partial}{\partial x}p_\infty(0) > 0$.

Recall that by Assumption 4, $\rho(x) = o(e^{\frac{2\mu}{\sigma^2}x})$ as $x \to \infty$. Thus $\int_x^\infty \rho(y)dy = o(e^{\frac{2\mu}{\sigma^2}x})$ as well. Multiplying (108) by $e^{-\frac{2\mu}{\sigma^2}x}$ gives $\left(\frac{\sigma^2}{2}\frac{\partial}{\partial x}p_\infty(x) - \mu p_\infty(x)\right)e^{-\frac{2\mu}{\sigma^2}x} = o(1)$. Thus $\frac{\sigma^2}{2}\frac{\partial}{\partial x}(p_\infty(x)e^{-\frac{2\mu}{\sigma^2}x}) = o(1)$, so $p_\infty(x) \sim C_\theta e^{\frac{2\mu}{\sigma^2}x}$ as $x \to \infty$, where C_θ is a constant depending on θ. In particular, p_∞ is integrable. Also, since $p_\infty(0) = 0$ and $\frac{\partial}{\partial x}p_\infty(0) > 0$ when $\theta = 0$, $p_\infty(x) \sim \left(\frac{\partial}{\partial x}p_\infty(0)\right)x$ as $x \to 0$.

We can now rescale p_∞ so that $\int p_\infty dx = 1$, i.e. p_∞ is a probability distribution on $[0,\infty)$. As shown above p_∞ then generalizes the stationary distribution in cases of reflecting barrier and exit with reinjection.

With $\mathcal{A}^*$ defined above, let

$$\mathcal{A}u := \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}u + \mu\frac{\partial}{\partial x}u$$

where $\mu < 0$ with boundary condition

$$-\theta u_x(0) + (1-\theta)\left(u(0) - \int u(x)\rho(x)dx\right) = 0 \tag{109}$$

We can check through integration by parts that $\mathcal{A}^*$ with boundary condition (105) is indeed the adjoint of $\mathcal{A}$ with boundary condition (109). Here we also remark that if a function u satisfies $\partial_t u(x,t) = \mathcal{A}u(x,t)$ with the boundary condition for $\mathcal{A}$, and $\tilde{u}(x,t) = u(x,t) + c$ for some constant c, then $\partial_t \tilde{u} = \mathcal{A}\tilde{u}$ with the boundary condition of $\mathcal{A}$ as well. This is because the boundary condition (109) is invariant when we add a constant to u.

Intuitively, the boundary condition (109) describes the following behavior: if the process ever reaches $x = 0$, then, with probability θ, the process is reflected; and with probability $1-\theta$, the process jumps to some $x > 0$, drawn from the distribution $\rho(x)$.

J.2.2 Proof Strategy for Proposition 2

When $\theta < 1$, i.e. when we depart from the pure reflection case, one can no longer construct a self-adjoint transformation $\mathcal{B}$ of $\mathcal{A}$ as in the proof of Proposition 1. Therefore, it is no longer possible to obtain an explicit formula for the spectral gap of the operator $\mathcal{A}$.[86] We instead follow an alternative approach that works directly with the operator $\mathcal{A}$ using "energy methods" (i.e. techniques techniques involving L^2-norms of various expressions – see Evans (1998) for their usefulness in other applications).

The proof of Proposition 2 has three parts. The first part proves that the cross-sectional income distribution converges to its stationary distribution exponentially at *some* rate $\lambda > 0$. This is proved in Lemmas 14 and 15. The second part is to prove that this rate λ satisfies $\lambda \leq \frac{\mu^2}{2\sigma^2}$, which is the content of Lemma 16. The third part simply concludes the proof by combining the two previous parts.

J.2.3 Part 1: exponential convergence to stationary distribution

A Poincaré-like inequality. We first establish the following Poincaré-like inequality using energy methods.

Lemma 14 *Let p_∞ be the solution to (107). Let u be the solution to $\partial_t u = \mathcal{A}u$, supplemented with the boundary condition (109), and with the orthogonality condition $\int p_\infty u(x,0)dx = \int p_\infty u_0(x)dx = 0$. For $\theta \in (0,1]$, let*

$$\lambda := \frac{1}{2}\inf_u \left\{\sigma^2 \int u_x^2 p_\infty dx + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\left[\left(u(0) - \int u\rho dy\right)^2 + \int\left(u - \int u\rho dy\right)^2 \rho dx\right] \right.$$
$$\left. s.t. \quad \int u^2 p_\infty dx = 1, \qquad \int u p_\infty dx = 0\right\}. \tag{110}$$

and when $\theta = 0$, replace $\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)$ with $\frac{\sigma^2}{2}(p_\infty)_x(0)$.

Then

$$\int u(x,t)^2 p_\infty(x)dx \leq e^{-2\lambda t}\int u_0(x)^2 p_\infty(x)dx \ . \tag{111}$$

Remark: Note that the constraint $\int u p_\infty dx = 0$ is needed to ensure $\lambda > 0$, since without it, $\lambda = 0$ with minimizer $u \equiv 1$.[87] As we shall see in the special case $\theta = 1$, this constraint

[86]It may be possible to have an explicit bound on λ, which requires further investigation of the operator $\mathcal{B}^*$ defined later.

[87]More precisely, let $C \in \mathbb{R}$ be an arbitrary number. Notice that if u satisfies $\int u^2 p_\infty = 1$ and $\int u p_\infty = 0$

is also natural as it is closely related to the orthogonality condition when calculating the second smallest eigenvalue of a self-adjoint operator using the min-max principle.

In the pure reflection case, one indeed recovers $\lambda = \frac{\mu^2}{2\sigma^2}$. To see this, note that with $\theta = 1$

$$\lambda = \frac{1}{2}\inf_u \Big\{\sigma^2 \int u_x^2 p_\infty dx \quad \text{s.t.} \quad \int u^2 p_\infty dx = 1, \qquad \int u p_\infty dx = 0\Big\}. \tag{112}$$

The stationary distribution $\frac{\sigma^2}{2}\frac{\partial^2 p_\infty}{\partial x^2} - \mu\frac{\partial p_\infty}{\partial x} = 0$ with boundary condition $\frac{\sigma^2}{2}p'_\infty(0) - \mu p_\infty(0) = 0$ is given explicitly by $p_\infty = -\frac{e^{2\mu x/\sigma^2}}{2\mu/\sigma^2}$. Define $v(x) = p_\infty^{\frac{1}{2}}u(x)$. Then v satisfies $\int v^2 dx = 1$ and $\int v p_\infty^{\frac{1}{2}} dx = 0$.

Through an integration by parts, we have

$$\int_0^\infty (v_x)^2 dx = \int_0^\infty (u_x)^2 p_\infty dx - \frac{1}{4}\int_0^\infty (\frac{2\mu}{\sigma^2})^2 (u p_\infty^{\frac{1}{2}})^2 dx - \frac{1}{2}u^2(0)p'_\infty(0)$$

whence

$$\frac{\sigma^2}{2}\int_0^\infty (u_x)^2 p_\infty dx = \frac{\sigma^2}{2}\int_0^\infty (v_x)^2 dx + \frac{\mu^2}{2\sigma^2}\int_0^\infty v^2 dx + \frac{\mu}{2}v^2(0)$$

Note that the second term is $\frac{\mu^2}{2\sigma^2}$. The first term is positive while the third term negative, but it isn't obvious that under the constraints for v that they can cancel each other.

Recalling from Lemma 6 that v satisfies $v_t = \mathcal{B}v := \frac{\sigma^2}{2}v_{xx} - \frac{1}{2}\frac{\mu^2}{\sigma^2}v$ with boundary condition

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then $v = u + C$ satisfies $\int v p_\infty = C$ and $\int v^2 p_\infty = \int u^2 p_\infty + C^2 = 1 + C^2$. Moreover, adding a constant to u does not change the value of the terms in the definition of λ. Now let $\tilde{u} = v/\sqrt{1+C^2}$. Then $\int \tilde{u}^2 p_\infty = 1$ and $\int \tilde{u} p_\infty = C/\sqrt{1+C^2}$. It follows that

$$\lambda = \frac{1}{2}(1+C^2)\inf_{\tilde{u}}\Big\{\sigma^2\int \tilde{u}_x^2 p_\infty dx + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\left[\left(\tilde{u}(0) - \int \tilde{u}\rho dx\right)^2 + \int\left(\tilde{u} - \int \tilde{u}\rho dy\right)^2 \rho dx\right]$$
$$\text{s.t.} \quad \int \tilde{u}^2 p_\infty dx = 1, \qquad \int \tilde{u} p_\infty dx = \frac{C}{\sqrt{1+C^2}}\Big\}.$$

$v_x(0) = \frac{\mu}{\sigma^2} v(0)$, we see that

$$\begin{aligned}\int_0^\infty \mathcal{B}v \cdot v dx &= \int_0^\infty \frac{\sigma^2}{2} v_{xx} v dx - \frac{1}{2}\frac{\mu^2}{\sigma^2}\int_0^\infty v^2 dx \\ &= -\int_0^\infty \frac{\sigma^2}{2}(v_x)^2 dx - \frac{1}{2}\frac{\mu^2}{\sigma^2}\int_0^\infty v^2 dx - \frac{\sigma^2}{2} v_x(0) v(0) \\ &= -\int_0^\infty \frac{\sigma^2}{2}(v_x)^2 dx - \frac{\mu^2}{2\sigma^2}\int_0^\infty v^2 dx - \frac{\mu}{2} v^2(0) \\ &= -\frac{\sigma^2}{2}\int_0^\infty (u_x)^2 p_\infty dx\end{aligned}$$

Thus

$$\lambda = \inf_{\int v^2 = 1, \int v p_\infty^{\frac{1}{2}} = 0} -\int_0^\infty \mathcal{B}v \cdot v dx$$

Since $\mathcal{B}$ is self-adjoint with non-positive eigenvalues, and $p_\infty^{\frac{1}{2}}$ satisfies $\mathcal{B} p_\infty^{\frac{1}{2}} = 0$, the orthogonality condition $\int v p_\infty^{\frac{1}{2}} = 0$ implies, by the min-max principle, that λ is the second smallest eigenvalue of $-\mathcal{B}$, i.e. $\frac{\mu^2}{2\sigma^2}$.

Proof of Lemma 14: We first show that $\lambda > 0$ for all $\theta \in [0,1]$ when ρ vanishes for $x > 0$ large. This condition on ρ implies $p_\infty(x) = Ce^{-\frac{2\mu}{\sigma^2}x}$ for all large x. The case of general ρ follows with minor modifications. Then we show the exponential decay.

To prove $\lambda > 0$, we argue by contradiction and assume $\lambda = 0$. Since all terms in the definition of λ are non-negative, there exists a sequence $(u^{(n)})_{n \geq 1}$ such that

$$\int_0^\infty (u_x^{(n)})^2 p_\infty(x) dx \to 0$$

with $\int_0^\infty u^{(n)}(x) p_\infty(x) dx = 0$ and $\int_0^\infty (u^{(n)}(x))^2 p_\infty(x) dx = 1$.

Step 1. We show that the assumption implies that $u^{(n)}$ converge strongly in $H^1(\delta, \frac{1}{\delta})$ and uniformly on $(\delta, \frac{1}{\delta})$ to 0 for any $0 < \delta < 1$, and that $u^{(n)} p_\infty^{\frac{1}{2}}$ converge weakly in $L^2(0,\infty)$ to 0.

First recall that $p_\infty(x) > 0$ on $(0, \infty)$. For any fixed $0 < \delta < 1$, $p_\infty(x) \geq c_\delta > 0$ on $(\delta, \frac{1}{\delta})$. Thus on $(\delta, \frac{1}{\delta})$,

$$\int_\delta^{\frac{1}{\delta}} (u_x^{(n)})^2 dx \leq \frac{1}{c_\delta}\int_\delta^{\frac{1}{\delta}} (u_x^{(n)})^2 p_\infty(x) dx \leq \frac{1}{c_\delta}\int_0^\infty (u_x^{(n)})^2 p_\infty(x) dx$$

Letting $n \to \infty$, the right hand side tends to 0, so that $\lim_{n\to\infty} \int_\delta^{\frac{1}{\delta}} (u_x^{(n)})^2 dx = 0$. Thus $u_x^{(n)}$ converges strongly in $L^2(\delta, \frac{1}{\delta})$ to 0, and so $\sup_n \|u_x^{(n)}\|_{L^2(\delta,\frac{1}{\delta})} < \infty$. Moreover, $\int_0^\infty (u^{(n)}(x))^2 p_\infty(x) dx = 1$ and the positivity of p_∞ implies

$$\int_\delta^{\frac{1}{\delta}} (u^{(n)})^2 dx \le \frac{1}{c_\delta} \int_\delta^{\frac{1}{\delta}} (u^{(n)})^2 p_\infty(x) dx \le \frac{1}{c_\delta}$$

Thus,

$$\sup_n \|u^{(n)}\|_{H^1(\delta,\frac{1}{\delta})} < \infty$$

By the Banach-Alaoglu Theorem, we can extract a subsequence of $u^{(n)}$ that converges weakly in $H^1(\delta, \frac{1}{\delta})$, and by the Rellich Compact Embedding Theorem, we can extract a further subsequence that converges weakly in $H^1(\delta, \frac{1}{\delta})$, and *strongly* in $L^2(\delta, \frac{1}{\delta})$ to some function u^δ. Setting $\delta = \frac{1}{m}$ and using a standard diagonalization argument, we can conclude that there exists a function u on $(0, \infty)$, such that for any $\delta \in (0, 1)$, $u^{(n)}$ converges weakly in $H^1(\delta, \frac{1}{\delta})$ and strongly in $L^2(\delta, \frac{1}{\delta})$ to u. Note that the convergence may fail on $(0, \infty)$.

Since $u_x^{(n)}$ converges in $L^2(\delta, \frac{1}{\delta})$ to 0, $u_x = 0$, and $u \equiv A$ for some constant A.

Next, we show $u^{(n)} p_\infty^{\frac{1}{2}}$ converges weakly in $L^2(0, \infty)$ to $A p_\infty^{\frac{1}{2}}$. This is because finite linear combinations of indicator functions of finite open intervals (simple functions) are dense in $L^2(0, \infty)$, and for a indicator function $g = \chi_{(a,b)}$ with $0 < a < b$,

$$\begin{aligned} |\int_0^\infty (u^{(n)} - A) p_\infty^{\frac{1}{2}} g dx| &\le \int_a^b |u^{(n)} - A| p_\infty^{\frac{1}{2}} dx \\ &\le (\int_a^b (u^{(n)} - A)^2 dx)^{1/2} \cdot (\int_0^\infty p_\infty dx)^{1/2} \end{aligned}$$

and this tends to 0 since $u^{(n)} \to A$ in $L^2(a, b)$.

Since $u^{(n)} p_\infty = u^{(n)} p_\infty^{\frac{1}{2}} \cdot p_\infty^{\frac{1}{2}}$, and $p_\infty^{\frac{1}{2}} \in L^2(0, \infty)$, we deduce that

$$0 = \lim_{n\to\infty} \int_0^\infty u^{(n)} p_\infty dx = \int_0^\infty A p_\infty^{\frac{1}{2}} \cdot p_\infty^{\frac{1}{2}} dx = A$$

Hence, $u^{(n)}$ converges to 0 strongly in $H^1(\delta, \frac{1}{\delta})$, and thus *uniformly* on $(\delta, \frac{1}{\delta})$, since the

$H^1(\delta, \frac{1}{\delta})$ norm dominates the uniform norm in $\mathbb{R}$:

$$\begin{aligned}|f(y) - f(x)| &= |\int_x^y f'(t)dt| \\ &\le (\int_x^y |f'|^2 dt)^{1/2} |x-y|^{1/2} \\ &\le |x-y|^{1/2} \|f\|_{H^1(\delta, \frac{1}{\delta})}\end{aligned}$$

and if we let $y \in (\delta, \frac{1}{\delta})$ be such that $|f(y)| < \inf_{x \in (\delta, \frac{1}{\delta})} |f(x)| + \epsilon$, then for any $x \in (\delta, \frac{1}{\delta})$,

$$\begin{aligned}|f(x)| &\le |f(y) - f(x)| + |f(y)| \\ &\le |\frac{1}{\delta} - \delta|^{1/2} \|f\|_{H^1(\delta, \frac{1}{\delta})} + \frac{1}{|\delta - \frac{1}{\delta}|} \int_\delta^{\frac{1}{\delta}} (|f(x)| + \epsilon) dx \\ &= C_\delta(\|f\|_{H^1(\delta, \frac{1}{\delta})}) + \epsilon\end{aligned}$$

Notice that if $p_\infty(0) > 0$, i.e. $\theta > 0$, then all the above convergence are on $(0, \frac{1}{\delta})$ instead of $(\delta, \frac{1}{\delta})$ by the same argument since $p_\infty(x)$ is bounded from below on $(0, \frac{1}{\delta})$ for any $\delta \in (0, 1)$.

Step 2. We show that for any $\bar{x} \in [1, \infty)$, $\int_0^{\bar{x}} p_\infty(x)(u^{(n)})^2 dx \to 0$. Note that when $p_\infty(0) > 0$, this is clearly true by uniform convergence of $u^{(n)}$ to 0 on $(0, \frac{1}{\delta})$.

Suppose $p_\infty(0) = 0$. We know that $p_\infty(x) \sim c_0 x$ as $x \to 0_+$ for some $c_0 > 0$ and for $\epsilon > 0$, we can find δ such that

$$\begin{aligned}\int_0^{\bar{x}} x(u_x^{(n)})^2 dx &= \int_\delta^{\bar{x}} x(u_x^{(n)})^2 dx + \int_0^\delta x(u_x^{(n)})^2 dx \\ &\le \bar{x} \int_\delta^{\bar{x}} (u_x^{(n)})^2 dx + \int_0^\delta x(u_x^{(n)})^2 dx \\ &\le \bar{x} \int_\delta^{\bar{x}} (u_x^{(n)})^2 dx + \int_0^\delta \frac{p_\infty}{c_0} (u_x^{(n)})^2 dx + \epsilon\end{aligned}$$

Letting $n \to \infty$, the first two terms vanish, so that $\lim_{n\to\infty} \int_0^{\bar{x}} x(u_x^{(n)})^2 dx \le \epsilon$ for arbitrary ϵ, thus $\lim_{n\to\infty} \int_0^{\bar{x}} x(u_x^{(n)})^2 dx = 0$.

Next, we write for $x \in (0, \bar{x})$

$$
\begin{aligned}
|u^{(n)}(x)| &\leq |u^{(n)}(\bar{x})| + \int_x^{\bar{x}} |u_x^{(n)}(y)| dy \\
&\leq |u^{(n)}(\bar{x})| + (\int_0^{\bar{x}} y |u_x^{(n)}(y)|^2 dy)^{1/2} (\int_x^{\bar{x}} y^{-1} dy)^{1/2} \\
&\leq |u^{(n)}(\bar{x})| + |\log \bar{x} - \log x|^{1/2} (\int_0^{\bar{x}} y (u_x^{(n)}(y))^2 dy)^{1/2}
\end{aligned}
$$

Therefore, since $u^{(n)}(\bar{x}) \to 0$ and $\int_0^{\bar{x}} y(u_x^{(n)}(y))^2 dy \to 0$,

$$
\begin{aligned}
\int_0^{\bar{x}} x (u^{(n)}(x))^2 dx &\leq \int_0^{\bar{x}} x (|u^{(n)}(\bar{x})| + |\log \bar{x} - \log x|^{1/2} (\int_0^{\bar{x}} y (u_x^{(n)}(y))^2 dy)^{1/2})^2 dx \\
&\leq C_{\bar{x}} \epsilon_n
\end{aligned}
$$

where $\epsilon_n \to 0$ and $C_{\bar{x}}$ is a fixed constant depending on $\bar{x}$. We conclude that, in particular,

$$
\int_0^{\bar{x}} p_\infty(x) (u^{(n)})^2 dx \to 0
$$

since $u^{(n)}$ converges uniformly to 0 on $(\delta, \frac{1}{\delta})$, and $p_\infty(x) \sim c_0 x$ as $x \to 0^+$.

Step 3. We choose $\bar{x}$ large enough such that for $x \geq \bar{x}$, $\rho(x) = 0$, hence $p_\infty(x) = C e^{\frac{2\mu}{\sigma^2} x}$ for some $C > 0$ and $x > \bar{x}$.

First, notice that

$$
\begin{aligned}
u^{(n)}(\bar{x}) &\to 0 \\
\int_{\bar{x}}^\infty p_\infty (u_x^{(n)})^2 dx &\to 0 \\
\int_{\bar{x}}^\infty p_\infty (u^{(n)})^2 dx &\to 1 \\
\int_{\bar{x}}^\infty p_\infty u^{(n)} dx &\to 0
\end{aligned}
$$

using steps 1 and 2. Since $\int_0^\infty p_\infty(x) (u_x^{(n)})^2 dx \to 0$ and the integrand is nonnegative, $\int_{\bar{x}}^\infty p_\infty(x) (u_x^{(n)})^2 dx \to 0$. Since $\int_0^\infty p_\infty(x) (u^{(n)})^2 dx = 1$ and $\int_0^{\bar{x}} p_\infty(x) (u^{(n)})^2 dx \to 0$, $\int_{\bar{x}}^\infty p_\infty(x) (u^{(n)})^2 dx \to 1$. Finally, that $u^{(n)} p_\infty^{\frac{1}{2}}$ converges weakly to 0 in $L^2(0, \infty)$ implies $\int_{\bar{x}}^\infty p_\infty u^{(n)} dx \to 0$.

Next, let $v^{(n)}(x) = u^{(n)}(\bar{x} + x)$ for $x \geq 0$. By a change of variables,

$$v^{(n)}(0) \to 0$$
$$\int_0^\infty (v_x^{(n)})^2 p_\infty dx \to 0$$
$$\int_0^\infty v^{(n)} p_\infty dx \to 0$$
$$\int_0^\infty (v^{(n)})^2 p_\infty dx \to e^{-\frac{2\mu}{\sigma^2}\bar{x}}$$

and this essentially contradicts the explicit spectral gap $\lambda = \frac{\mu^2}{2\sigma^2} > 0$ shown in the pure reflection case ($\theta = 1$).

Indeed, we compute

$$\begin{aligned}
\int_0^\infty (v_x^{(n)})^2 p_\infty dx &= \int_0^\infty (v_x^{(n)} p_\infty^{\frac{1}{2}})^2 dx \\
&= \int_0^\infty ((v^{(n)} p_\infty^{\frac{1}{2}})_x - v^{(n)} (p_\infty^{\frac{1}{2}})_x)^2 dx \\
&= \int_0^\infty (v^{(n)} p_\infty^{\frac{1}{2}})_x^2 dx + \frac{2\mu}{\sigma^2} \int_0^\infty (v^{(n)} p_\infty^{\frac{1}{2}})_x \cdot v^{(n)} p_\infty^{\frac{1}{2}} dx + \frac{\mu^2}{\sigma^4} \int_0^\infty (v^{(n)})^2 p_\infty dx \\
&= \int_0^\infty (v^{(n)} p_\infty^{\frac{1}{2}})_x^2 dx - \frac{\mu}{\sigma^2} (v^{(n)}(0) p_\infty^{\frac{1}{2}}(0))^2 + \frac{\mu^2}{\sigma^4} \int_0^\infty (v^{(n)})^2 p_\infty dx
\end{aligned}$$

Therefore, $\int_0^\infty (v^{(n)})^2 p_\infty dx \to 0$ and we get a contradiction.

Step 4. When ρ does not have compact support, steps 1 and 2 remain unchanged. When ρ does not have compact support but satisfies Assumption 4, we have shown that $p_\infty(x) \sim C e^{\frac{2\mu}{\sigma^2} x}$ as $x \to \infty$. Step 3 follows with minor modifications.

Now we show the exponential decay

$$\int_0^\infty u(x,t)^2 p_\infty(x) dx \leq e^{-2\lambda t} \int_0^\infty u_0(x)^2 p_\infty(x) dx$$

In what follows we omit the limits of integration which are always assumed to be from 0 to ∞. Morever we write $u(x)$ to denote $u(x,t)$, whenever time t is implicit.

Step 5. Suppose $u(x,t)$ satisfies $\partial_t u = \mathcal{A} u$, with $\int u^2 p_\infty dx = 1$ and $\int u p_\infty dx = 0$. Start by writing the equation for u^2:

$$\frac{\partial}{\partial t} u^2 - \mathcal{A} u^2 + \sigma^2 u_x^2 = 0 \tag{113}$$

We next show that multiplying this equation by p_∞ and integrating we obtain

$$\frac{\partial}{\partial t}\int u^2 p_\infty dx+\sigma^2\int u_x^2 p_\infty dx+\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\left[\left(u(0)-\int u\rho dy\right)^2+\int\left(u-\int u\rho dy\right)^2\rho dx\right]=0. \tag{114}$$

This is shown by means of the following computations. Using integration by parts, we have

$$\begin{aligned}\int(\mathcal{A}u^2)p_\infty dx &= \int\left(\left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}+\mu\frac{\partial}{\partial x}\right)u^2\right)p_\infty dx\\ &= \int\left(-\frac{\sigma^2}{2}\frac{\partial}{\partial x}u^2-\mu u^2\right)\frac{\partial}{\partial x}p_\infty dx-\frac{\sigma^2}{2}(u^2)_x(0)p_\infty(0)-\mu u^2(0)p_\infty(0)\\ &= \int u^2\mathcal{A}^*p_\infty dx-\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\int u^2\rho dx-\frac{\sigma^2}{2}(u^2)_x(0)p_\infty(0)\\ &+\frac{\sigma^2}{2}(u^2)(0)(p_\infty)_x(0)-\mu u^2(0)p_\infty(0)\end{aligned}$$

Next, observe that $(u^2)_x(0)=2u_x(0)u(0)$. When $\theta>0$, (109) implies[88]

$$u_x(0)=\frac{1-\theta}{\theta}\left(u(0)-\int u\rho dx\right),$$

and by (105),

$$\frac{\sigma^2}{2}(p_\infty)_x(0)=\mu p_\infty(0)+\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0).$$

Inserting also (107), we obtain

$$\begin{aligned}&\int u^2\mathcal{A}^*p_\infty dx=0\\ &\frac{\sigma^2}{2}(u^2)_x(0)p_\infty(0)=\sigma^2\frac{1-\theta}{\theta}p_\infty(0)(u(0)-\int u\rho dx)u(0)\\ &\frac{\sigma^2}{2}(p_\infty)_x(0)u^2(0)=\mu p_\infty(0)u^2(0)+\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)u^2(0)\end{aligned}$$

[88] When $\theta=0$, we note that $\frac{\sigma^2}{2}(u^2)_x(0)p_\infty(0)=0$ and so need to use $u(0)-\int u\rho=0$ instead of $u_x(0)=\frac{1-\theta}{\theta}\left(u(0)-\int u\rho\right)$. The subsequent calculations follow exactly by replacing $\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)$ with $\frac{\sigma^2}{2}(p_\infty)_x$.

Summing up, we have

$$\int \mathcal{A}u^2 p_\infty dx = -\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\int u^2\rho dx - \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\cdot 2(u(0) - \int u\rho dx)u(0)$$
$$+ \mu p_\infty(0)u^2(0) + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)u^2(0) - \mu u^2(0)p_\infty(0)$$
$$= -\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\left[\int u^2\rho dx - 2u(0)\left(\int \rho u dx\right) + u^2(0)\right]$$

Furthermore,

$$u(0)^2 - 2u(0)\int u\rho dx + \int u^2\rho dx = \left(u(0) - \int u\rho dy\right)^2 + \int\left(u - \int u\rho dy\right)^2\rho dx.$$

and so

$$\int \mathcal{A}u^2 p_\infty dx = -\frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\left[\left(u(0) - \int u\rho dy\right)^2 + \int\left(u - \int u\rho dy\right)^2\rho dx\right]$$

Substituting into (113), we obtain (114).

Step 6. We next derive the exponential bound (111). Define

$$X(t) := \int u(x,t)^2 p_\infty(x)dx \tag{115}$$

Note that the definition of λ in (110) confines attention to functions u that are orthogonal to p_∞, $\int up_\infty dx = 0$. To check this for our $u(x,t)$ for all t, recall that u_0 is orthogonal to p_∞, $\int u_0(x)p_\infty(x)dx = 0$. Moreover, since $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$,

$$\frac{\partial}{\partial t}\int u(x,t)p_\infty(x)dx = \int \mathcal{A}u(x,t)p_\infty(x)dx$$
$$= \int u(x,t)\mathcal{A}^*p_\infty(x)dx$$
$$= 0$$

which implies that $u(x,t)$ remains orthogonal to p_∞ for all t.

From the definition of λ in (110), it then follows that for all such functions u:

$$2\lambda X(t) \leq \sigma^2\int u_x^2 p_\infty dx + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\left[\left(u(0) - \int u\rho dy\right)^2 + \int\left(u - \int u\rho dy\right)^2\rho dx\right]$$

Combining with (114), we have

$$\partial_t X(t) + 2\lambda X(t) \leq 0$$

But then also $\partial_t(e^{2\lambda t}X(t)) = e^{2\lambda t}(\partial_t X(t) + 2\lambda X(t)) \leq 0$ and thus

$$X(t) \leq e^{-2\lambda t}X(0), \quad \forall t \tag{116}$$

or equivalently (111).□

The Exponential Decay Estimate. Next, we want to show that this implies the following more general exponential decay estimate.

Lemma 15 *Consider $p(x,t)$ with $p_t = \mathcal{A}^* p$ and initial condition $p_0(x) \in L^1_{(0,\infty)}$. Then, there exists a constant C_0 such that:*

$$\int |p(x,t) - Jp_\infty|\, dx \leq C_0 e^{-\lambda t} \tag{117}$$

where $J := \int p_0(y)\, dy$ is not necessarily 1.

The role of J is to facilitate the proof of the general case when $\delta > 0$, where we need $J = 0$ instead of $J = 1$. The desired convergence result in Proposition 2 with $\delta = 0$ and constant coefficient is the special case $J = 1$.

Proof: We claim that if u and p satisfy $u_t = \mathcal{A}u$ and $p_t = \mathcal{A}^* p$, with $\int u_0^2 p_\infty dx < \infty$ and the respective boundary conditions, then we have the dual property[89]

$$\int u(x,t)p_0(x)dx = \int u_0(x)p(x,t)dx. \tag{118}$$

[89]For convenience, we reproduce the proof given earlier. Let $I(s) = \int u(x,t-s)p(x,s)dx$. Then

$$\frac{d}{ds}I(s) = -\int \partial_t u(x,t-s)p(x,s)dx + \int u(x,t-s)\partial_s p(x,s)dx =$$
$$-\int \mathcal{A}u(x,t-s)p(x,s)dx + \int u(x,t-s)\mathcal{A}^* p(x,s)dx = 0$$

Setting $s = 0$ and t gives the result.

Therefore

$$\int \left(u - \int u_0 p_\infty dy\right) p_0 dx = \int u_0 p dx - J \int u_0 p_\infty dy = \int u_0 \left(p - J p_\infty\right) dx. \tag{119}$$

Note that we are no longer requiring $\int u_0 p_\infty dx = 0$. But recall $\int u(x,t) p_\infty(x) dx = \int u_0 p_\infty dx$, so $\tilde{u} = u - \int u_0 p_\infty dy$ satisfies

$$\int \tilde{u}(x,t) p_\infty(x) dx = 0$$

for all $t \geq 0$. Moreover, $\tilde{u}$ also satisfies $\tilde{u}_t = \mathcal{A}\tilde{u}$ with the required boundary conditions for $\mathcal{A}$. So we can apply Lemma 14 to $\tilde{u}(x,t) = u(x,t) - \int u_0 p_\infty dx$.

Applying Cauchy-Schwarz to (119) and Lemma 14 to $\tilde{u}$, we have[90]

$$\begin{aligned}
\left|\int u_0 \left(p(x,t) - J p_\infty\right) dx\right| &= \left|\int \left(u(x,t) - \int u_0 p_\infty dy\right) p_0 dx\right| \\
&= \left|\int \left(\left(u(x,t) - \int u_0 p_\infty dy\right)^2 p_\infty\right)^{1/2} \frac{p_0}{p_\infty^{1/2}} dx\right| \\
&\leq \left(\int \frac{(p_0)^2}{p_\infty} dx\right)^{1/2} \left(\int \left(u(x,t) - \int u_0 p_\infty dy\right)^2 p_\infty dx\right)^{1/2} \\
&\leq C_0 e^{-\lambda t} \left(\int (u_0 - \int u_0 p_\infty dy)^2 p_\infty dx\right)^{1/2} \\
&= C_0 e^{-\lambda t} \left(\int (u_0)^2 p_\infty dx - (\int u_0 p_\infty dx)^2\right)^{1/2} \\
&\leq C_0 e^{-\lambda t} \left(\int (u_0)^2 p_\infty dx\right)^{1/2}
\end{aligned}$$

where we need to show that $\left(\int \frac{(p_0)^2}{p_\infty} dx\right)^{1/2} < \infty$.

Recall Assumption 1, which states that $\int \frac{p_0^2(x)}{\bar{p}_\infty} dx < \infty$, where $\bar{p}_\infty = -\frac{2\mu}{\sigma^2} e^{\frac{2\mu}{\sigma^2}}$ is the surrogate steady state, which coincides with the true steady state p_∞ when $\theta = 1$ and $\delta = 0$,

[90]Note that since $\int u(x,t) p_\infty(x) dx = \int u_0(x) p_\infty(x) dx$ for all $t \geq 0$, the first equality can also be written as $\left|\int u_0 \left(p(x,t) - J p_\infty\right) dx\right| = \left|\int \left(u(x,t) - \int u_0 p_\infty dy\right) (p_0 - J p_\infty) dx\right|$, resulting in $C_0 = \left(\int \frac{(p_0 - J p_\infty)^2}{p_\infty} dx\right)^{1/2}$ instead of $C_0 = \left(\int \frac{(p_0)^2}{p_\infty} dx\right)^{1/2}$. There is no fundamental difference here, but $C_0 = \left(\int \frac{(p_0 - J p_\infty)^2}{p_\infty} dx\right)^{1/2}$ will be used later to show that $\lambda \leq \frac{\mu^2}{2\sigma^2}$.

but are otherwise different. However, we have shown the asymptotic behavior of $\bar{p}_\infty$ and p_∞ are both $O(e^{\frac{2\mu}{\sigma^2}x})$ when $x \to \infty$. Moreover, recall that when $\theta > 0$, $p_\infty(0) > 0$. When $\theta = 0$, $p_\infty(x) \sim cx$ as $x \to 0$, but the boundary condition of $\mathcal{A}^*$ also requires $p_0(0) = 0$.[91] It follows that for any $\theta \in [0, 1]$,

$$\int \frac{p_0^2(x)}{\bar{p}_\infty} dx < \infty \iff \int \frac{p_0^2(x)}{p_\infty} dx < \infty$$

i.e. Assumption 1 is equivalent to $\int \frac{p_0^2(x)}{p_\infty} dx < \infty$.

Now dividing the inequality $\left|\int u_0 \left(p(x,t) - Jp_\infty\right) dx\right| \leq C_0 e^{-\lambda t} \left(\int (u_0)^2 p_\infty dx\right)^{1/2}$ by $\left(\int (u_0)^2 p_\infty dx\right)^{1/2}$, we get

$$\frac{\left|\int u_0 \left(p(x,t) - Jp_\infty\right) dx\right|}{\left(\int (u_0)^2 p_\infty dx\right)^{1/2}} \leq C_0 e^{-\lambda t}.$$

The above inequality holds for any u_0 satisfying $\int (u_0)^2 p_\infty dx < \infty$. We would like to choose $u_0 = (p(x,t) - Jp_\infty)/p_\infty$.[92] We need to check $\int (u_0)^2 p_\infty dx < \infty$, i.e.

$$\int (u_0)^2 p_\infty dx = \int \frac{(p(x,t))^2 - 2Jp_\infty p(x,t) + J^2 p_\infty^2}{p_\infty} < \infty$$

Similar to Lemma 2, $|p|$ is a subsolution associated to $\mathcal{A}^*$, i.e. $|p|_t \leq \mathcal{A}^*|p|$. Thus[93]

$$\begin{aligned}
\partial_t \int_0^\infty |p(x,t)| &\leq \int_0^\infty \frac{\sigma^2}{2}\frac{\partial^2 |p|}{\partial x^2} - \mu \frac{\partial |p|}{\partial x} + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}|p(0)|\rho(x) dx \\
&= -\frac{\sigma^2}{2}|p|_x(0) + \mu|p(0)| + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}|p(0)| \\
&= -\frac{\sigma^2}{2}|p_x(0)| + \mu|p(0)| + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}|p(0)| \\
&\leq 0
\end{aligned}$$

where the last inequality follows from the boundary condition $\frac{\sigma^2}{2}p_x(0) - \mu p(0) = \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p(0)$ for $\mathcal{A}^*$. We have shown $p(x,t)$ remains bounded in L^1 for all t.[94]

[91] When $\theta = 0$, recall that the boundary condition of $\mathcal{A}^*$ requires $p_0(0) = 0$, and $\frac{\sigma^2}{2}\frac{\partial^2 p_0}{\partial x^2} - \mu \frac{\partial p_0}{\partial x} + \frac{\sigma^2}{2}\frac{\partial p_0(0)}{\partial x}\rho = p_t(x, 0)$. Also $p_t(0,t) = 0$ for all t since $p(0,t) = 0$, so $\frac{\sigma^2}{2}\frac{\partial^2 p_0(0)}{\partial x^2} - \mu\frac{\partial p_0(0)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial p_0(0)}{\partial x}\rho(0) = 0$, or $\frac{\sigma^2}{2}\frac{\partial^2 p_0(0)}{\partial x^2} = (\mu - \frac{\sigma^2}{2}\rho(0))\frac{\partial p_0(0)}{\partial x}$, with $\mu < 0$. Thus the first derivative and the second derivative of $p_0(x)$ at $x = 0$ must have opposite signs or are both zero. In either case, we have $p_0(x) = O(x)$ when $x \to 0$.

[92] This choice comes from the first order condition of the optimization problem $\max_{u_0} \int u_0(p(x,t) - Jp_\infty) dx$ s.t $\int u_0^2 p_\infty = 1$.

[93] As before when $\theta = 0$ replace $\frac{\sigma^2}{2}\frac{1-\theta}{\theta}|p(0)|$ with $\frac{\sigma^2}{2}|p_x(0)|$.

[94] We can also show that the L^1 norm of $p(x,t) = p_0 * F_t(x)$ is bounded for all t by using Young's inequality

It remains to check $\int \frac{(p(x,t))^2}{p_\infty} < \infty$. We already know that $\int \frac{(p_0(x))^2}{p_\infty} < \infty$. To prove this for general $t > 0$, we define $q = pp_\infty^{-\frac{1}{2}}$ and show that $q_t - \mathcal{B}^* q = f$ for some uniformly elliptic operator $-\mathcal{B}^*$ and some $f(x) \in L^2$.[95] Then the inequality $\int q^2(x,t)dx = \int \frac{p^2(x,t)}{p_\infty(x)} < \infty$ will follow from L^2 estimates of second order uniformly parabolic equations and the initial condition $\int \frac{p_0^2(x)}{\bar{p}_\infty} < \infty \iff \int q_0^2 = \int \frac{p_0^2(x)}{p_\infty} < \infty$ from Assumption 1, where we again note $\bar{p}_\infty = -\frac{2\mu}{\sigma^2} e^{\frac{2\mu}{\sigma^2}x}$ is in general different from p_∞.

First recall the integrated equation for p_∞:

$$\frac{\sigma^2}{2}\frac{\partial}{\partial x}p_\infty(x) - \mu p_\infty(x) = \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\int_x^\infty \rho(y)dy$$

where $\rho(x) = o(e^{\frac{2\mu}{\sigma^2}x})$ as $x \to \infty$. Thus $r(x) := \int_x^\infty \rho(y)dy$ is also $o(e^{\frac{2\mu}{\sigma^2}x})$, so

$$\frac{\partial}{\partial x}p_\infty(x) = \frac{2}{\sigma^2}\left(\mu p_\infty(x) + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)r(x)\right),$$

which we use repeatedly below to replace first order derivative of p_∞ with p_∞ and r.

We have

$$q_t = p_t p_\infty^{-\frac{1}{2}} = p_\infty^{-\frac{1}{2}}\left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p(x) - \mu\frac{\partial}{\partial x}p(x) + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p(0)\rho(x)\right)$$

On the other hand,

$$\begin{aligned}
q_x &= (pp_\infty^{-\frac{1}{2}})_x \\
&= p_x p_\infty^{-\frac{1}{2}} - \frac{1}{2}pp_\infty^{-\frac{3}{2}}(p_\infty)_x \\
&= p_x p_\infty^{-\frac{1}{2}} - \frac{1}{\sigma^2}pp_\infty^{-\frac{3}{2}}\left(\mu p_\infty(x) + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)r(x)\right) \\
&= p_x p_\infty^{-\frac{1}{2}} - \frac{\mu}{\sigma^2}pp_\infty^{-\frac{1}{2}} - \frac{1}{2}\frac{1-\theta}{\theta}pp_\infty^{-\frac{3}{2}}p_\infty(0)r(x)
\end{aligned}$$

and Gaussian estimates of the L^1 norm of F_t.

[95] One should compare the definition of $q = pp_\infty^{-\frac{1}{2}}$ to that of $v = up_\infty^{\frac{1}{2}}$ used earlier. Note also that our choice of notation $\mathcal{B}^*$ is not coincidental. Indeed, as we shall see in the special case $\theta = 1$, $\mathcal{B}^*$ with appropriate boundary conditions is the adjoint of the operator $\mathcal{B}$ defined in Lemma 6.

Differentiating again,

$$q_{xx} = (p_{xx} - \frac{\mu}{\sigma^2}p_x)p_\infty^{-\frac{1}{2}} - \frac{1}{2}(p_x - \frac{\mu}{\sigma^2}p)p_\infty^{-\frac{3}{2}}(p_\infty)_x - \frac{1}{2}\frac{1-\theta}{\theta}p_\infty(0)\left(pp_\infty^{-\frac{3}{2}}r(x)\right)_x$$

Thus, after some calculations, we obtain

$$\begin{aligned} q_t = &\frac{\sigma^2}{2}q_{xx} - \frac{1}{2}\frac{\mu^2}{\sigma^2}q + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)(r(x)p_\infty^{-1})q_x + \frac{\sigma^2}{4}\frac{1-\theta}{\theta}p_\infty(0)\cdot q\left(p_\infty^{-1}r(x)\right)_x \\ &+ \left(\frac{\sigma^2}{8}(\frac{1-\theta}{\theta})^2 p_\infty^2(0)r^2(x)p_\infty^{-2}\right)q + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}q(0)\rho p_\infty^{-\frac{1}{2}}p_\infty^{\frac{1}{2}}(0) \end{aligned}$$

So $q_t = \mathcal{B}^*q + f(x)$ where

$$\begin{aligned} \mathcal{B}^*q = &\frac{\sigma^2}{2}q_{xx} + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)(r(x)p_\infty^{-1})q_x \\ &+ \left[\frac{\sigma^2}{4}\frac{1-\theta}{\theta}p_\infty(0)\cdot\left(p_\infty^{-1}r(x)\right)_x - \frac{\mu^2}{2\sigma^2} + \left(\frac{\sigma^2}{8}(\frac{1-\theta}{\theta})^2 p_\infty^2(0)r^2(x)p_\infty^{-2}\right)\right]q \end{aligned}$$

and

$$f(x) = \frac{\sigma^2}{2}\frac{1-\theta}{\theta}q(0)p_\infty^{\frac{1}{2}}(0)\rho(x)p_\infty^{-\frac{1}{2}}(x)$$

In general $\mathcal{B}^*$ is not self-adjoint. However, when $\theta = 1$, $q_t = \mathcal{B}^*q = \frac{\sigma^2}{2}q_{xx} - \frac{\mu^2}{2\sigma^2}q$ with boundary condition $q_x(0) = \frac{\mu}{\sigma^2}q(0)$. This is the dual structure to Lemma 6 since $\mathcal{B}$ defined there is given by $\mathcal{B}v = \frac{\sigma^2}{2}v_{xx} - \frac{\mu^2}{2\sigma^2}v$ with boundary condition $v_x(0) = \frac{\mu}{\sigma^2}v(0)$. Thus in this case $\mathcal{B}$ is self-adjoint. Also, if ρ has compact support, then $r(x) = 0$ for all large x, so $\mathcal{B}^*q(x) = \frac{\sigma^2}{2}q_{xx}(x) - \frac{\mu^2}{2\sigma^2}q(x)$ for large x.

Moreover, because $\rho, r = o(e^{\frac{2\mu}{\sigma^2}x})$ and $p_\infty = O(e^{\frac{2\mu}{\sigma^2}x})$, the coefficients of q_x and q in $\mathcal{B}^*q$ are both bounded. Recall that Assumption 1 is equivalent to $\int q_0^2 = \int \frac{p_0^2}{p_\infty} < \infty$. Since $q_t = \mathcal{B}^*q + f(x)$ and $-\mathcal{B}^*$ is uniformly elliptic, by energy estimates for uniformly parabolic operators[96],

$$\begin{aligned} \|q(x,t)\|_{L^2} &\lesssim \|f\|_{L^2} + \|q(x,0)\|_{L^2} \\ &\leq C(\|\rho p_\infty^{-\frac{1}{2}}\|_{L^2} + \|q(x,0)\|_{L^2}) < \infty \end{aligned}$$

as was to be shown.

[96]See for example Evans (1998) Page 376.

Putting $u_0 = (p(x,t) - Jp_\infty)/p_\infty$ into $\frac{\left|\int u_0 (p(x,t) - Jp_\infty) dx\right|}{\left(\int (u_0)^2 p_\infty dx\right)^{1/2}} \leq C_0 e^{-\lambda t}$, we obtain

$$\left(\int \frac{(p(x,t) - Jp_\infty)^2}{p_\infty} dx\right)^{1/2} = \frac{\left|\int u_0 \left(p(x,t) - Jp_\infty\right) dx\right|}{\left(\int (u_0)^2 p_\infty dx\right)^{1/2}} \leq C_0 e^{-\lambda t}.$$

Finally, by Cauchy-Schwarz inequality

$$\int |p(x,t) - Jp_\infty|\, dx \leq \left(\int \frac{(p(x,t) - Jp_\infty)^2}{p_\infty} dx\right)^{1/2} \underbrace{\left(\int p_\infty dx\right)^{1/2}}_{=1} \leq C_0 e^{-\lambda t},$$

which is the desired result.□

J.2.4 Part 2: the rate of convergence cannot be larger than $\frac{\mu^2}{2\sigma^2}$

Note that as an intermediate step in the proof of Lemma 15, we have shown

$$\int \frac{(p(x,t) - Jp_\infty(x))^2}{p_\infty(x)} dx \leq e^{-2\lambda t} \int \frac{(p_0(x) - Jp_\infty(x))^2}{p_\infty(x)} dx \tag{120}$$

for all continuous initial probability density p_0 with $p_t = \mathcal{A}^* p$ and $\mathcal{A}^* p_\infty = 0$, where λ is defined in Lemma 14 and $J = \int p_0$. In fact, the inequality with $J = 0$ implies the inequality with $J \neq 0$. To see this, note that if $\int p_0 \neq 0$, then defining $\tilde{p}_0(x) = p_0(x) - Jp_\infty(x)$ we see $\int \tilde{p}_0 = 0$. Moreover, if $\tilde{p}(x)$ is the solution to $\tilde{p}_t = \mathcal{A}^* \tilde{p}$ with initial condition $\tilde{p}_0$, then $\tilde{p} = p - Jp_\infty$ where p solves $p_t = \mathcal{A}^* p$ with initial condition p_0. So the inequality with $J = 0$ applied to $\tilde{p}$ tells us $\int \frac{(\tilde{p})^2}{p_\infty(x)} dx \leq Ce^{-2\lambda t} \int \frac{(\tilde{p}_0(x))^2}{p_\infty(x)} dx$, i.e. $\int \frac{(p - Jp_\infty)^2}{p_\infty(x)} dx \leq Ce^{-2\lambda t} \int \frac{(p_0 - Jp_\infty)^2}{p_\infty(x)} dx$.

In the pure reflection case $\theta = 1$ with no death $\delta = 0$, we know that $\lambda = \frac{\mu^2}{2\sigma^2}$. Now we show that under the sufficient condition $\mu < 0$ for the existence of a unique steady state p_∞, the validity of (120) for all p_0 and $p(x,t)$ satisfying $p_t = \mathcal{A}^* p$ with initial condition p_0 implies $\lambda \leq \frac{\mu^2}{2\sigma^2}$ for all $\theta \in [0,1]$ in the generalized model incorporating reflecting barrier ($\theta = 1$) and exit with reinjection ($\theta = 0$), with $\delta = 0$.

Lemma 16 *For all $\theta \in [0,1]$, the constant λ defined in Lemma 14 satisfies $\lambda \leq \frac{\mu^2}{2\sigma^2}$.*

Proof: We argue by contradiction and suppose that $\lambda > \frac{\mu^2}{2\sigma^2}$. Then the bound (120) holds for some $C > 0$ and all $p_0, p(x,t)$ with this $\lambda > \frac{\mu^2}{2\sigma^2}$. We first prove the lemma in the case when ρ has compact support.

Define as before $q(x) = p(x) p_\infty^{-\frac{1}{2}}(x)$. Denote the upper end of the support of ρ by $\bar{x}$. As before, $\mathcal{A}^* p = p_t$ implies $q_t = \mathcal{B}^* q + \frac{\sigma^2}{2} \frac{1-\theta}{\theta} q(0) \rho p_\infty^{-\frac{1}{2}} p_\infty^{\frac{1}{2}}(0)$ with $\mathcal{B}^*$ defined before, and that for $x > \bar{x}$, $\mathcal{B}^* q = \frac{\sigma^2}{2} q_{xx} - \frac{\mu^2}{2\sigma^2} q$, so that for $x > \bar{x}$

$$q_t - \frac{\sigma^2}{2} q_{xx} + \frac{\mu^2}{2\sigma^2} q = 0 \tag{121}$$

Given the reasoning above, it suffices to derive a contradiction for the inequality with $J = 0$. With $J = 0$, the inequality implies

$$\left(\int_0^\infty q(x,t)^2 dx \right)^{\frac{1}{2}} \leq c_0 e^{-\lambda t}. \tag{122}$$

We now obtain a contradiction to (122). Let φ be a positive, smooth function with compact support and $\int \varphi(x) dx = 1$. Then let $q_0^R(x) = \varphi(x - R)$, and let $q^R(x,t)$ be the solution to the equation $q_t = \mathcal{B}^* q + \frac{\sigma^2}{2} \frac{1-\theta}{\theta} q(0) \rho p_\infty^{-\frac{1}{2}} p_\infty^{\frac{1}{2}}(0)$ with initial condition $q_0^R(x)$. On the other hand, define $\tilde{q}$ to be the solution to (121) on $\mathbb{R}$. Note that q^R solves (121) for $x > \bar{x}$, but not in general, whereas $\tilde{q}$ is defined to solve (121) on the entire real line, so $\tilde{q}_t \neq \mathcal{B}^* \tilde{q} + \frac{\sigma^2}{2} \frac{1-\theta}{\theta} \tilde{q}(0) \rho p_\infty^{-\frac{1}{2}} p_\infty^{\frac{1}{2}}(0)$ for $x \leq \bar{x}$. The key to the translation at infinity method is that for fixed t, $q^R(x + R, t)$ converges locally in $L^2(x)$ to $\tilde{q}(x,t)$ as $R \to \infty$. Given this, since (122) implies $\left(\int_0^\infty q^R(x + R, t)^2 dx \right)^{\frac{1}{2}} \leq c_0 e^{-\lambda t}$ for all R, we have

$$\left(\int_0^\infty \tilde{q}(x,t)^2 dx \right)^{\frac{1}{2}} \leq c_0 e^{-\lambda t}$$

To show the local convergence, note that intuitively, we are translating the initial condition p_0 further and further to the right by R, then letting it evolve to a certain point t in time according to $q_t = \mathcal{B}^* q + \frac{\sigma^2}{2} \frac{1-\theta}{\theta} q(0) \rho p_\infty^{-\frac{1}{2}} p_\infty^{\frac{1}{2}}(0)$ (which is just $q_t - \frac{\sigma^2}{2} q_{xx} + \frac{\mu^2}{2\sigma^2} q$ for $x > \bar{x}$), then translating it back to the left by R. As R gets larger this looks more and more like $q_t - \frac{\sigma^2}{2} q_{xx} + \frac{\mu^2}{2\sigma^2} q = 0$ starting with p_0. More precisely, for every fixed t and $x \leq \bar{x}$, $q^R(x,t) \to 0$. By Dini's Theorem this implies $q^R(x,t) \to 0$ uniformly on $[0, \bar{x}]$. By writing $q^R(x,t) = q^R(x,t) \chi_{[0,\bar{x}]} + q^R(x,t)(1 - \chi_{[0,\bar{x}]})$, with $\chi_{[0,\bar{x}]}$ the indicator function, we can show that for every $x > 0$, $q^R(x + R, t) \to \tilde{q}(x,t)$ pointwise. Thus on any compact $I \subset (0, \infty)$, $q^R(x + R, t) \to \tilde{q}(x,t)$ in L^2, by dominated convergence theorem.

In summary, we have found a function $\tilde{q}(x,t)$ which is a solution to $q_t - \frac{\sigma^2}{2} q_{xx} + \frac{\mu^2}{2\sigma^2} q = 0$ on $\mathbb{R}$ with $\varphi(x) \equiv q_0(x)$ as initial condition, and which satisfies $\left(\int_0^\infty \tilde{q}(x,t)^2 dx \right)^{\frac{1}{2}} \leq c_0 e^{-\lambda t}$. For simplicity of notation, denote $\tilde{q}(x,t)$ simply by $q(x,t)$ from now on.

Using the fact that (121) is the Kolmogorov Forward equation for a Brownian motion with death rate $\frac{\mu^2}{2\sigma^2}$, we obtain[97]

$$q(x,t) = e^{-\frac{\mu^2}{2\sigma^2}t}\left(q_0(x) * \frac{e^{-\frac{x^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}\right) \tag{123}$$

where $*$ is the convolution operator. Therefore, for $x \geq 0$,

$$q(x,t) = \frac{e^{-\frac{\mu^2}{2\sigma^2}t}}{\sqrt{2\pi\sigma^2 t}}\int_{-\infty}^{\infty} q_0(y)e^{-\frac{(x-y)^2}{2\sigma^2 t}}dy \geq \frac{e^{-\frac{\mu^2}{2\sigma^2}t}}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{x^2}{2\sigma^2 t}}\int_0^{\infty} q_0(y)e^{\frac{xy}{\sigma^2 t}}e^{-\frac{y^2}{2\sigma^2 t}}dy \geq c\frac{e^{-\frac{\mu^2}{2\sigma^2}t}}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{x^2}{2\sigma^2 t}}$$

for some $c > 0$. Thus

$$\begin{aligned}\lim_{t\to\infty}\left(\int_0^{\infty} q(x,t)^2 dx\right)^{\frac{1}{2}} e^{\frac{\mu^2}{2\sigma^2}t}\sqrt{2\pi\sigma^2 t}t^{-1/4} &\geq \lim_{t\to\infty} c(\int_0^{\infty} e^{-\frac{x^2}{2\sigma^2 t}}t^{-1/2}dx)^{1/2} \\ &= c(\int_0^{\infty} e^{\frac{-z^2}{2\sigma^2}}dz)^{1/2} > 0\end{aligned}$$

We now obtain a contradiction to the asymptotic property

$$\lim_{t\to\infty}\left(\int_0^{\infty} q(x,t)^2 dx\right)^{\frac{1}{2}} e^{\frac{\mu^2}{2\sigma^2}t}\sqrt{2\pi\sigma^2}t^{\frac{1}{4}} > 0 \tag{124}$$

We have shown $\left(\int_0^{\infty} q(x,t)^2 dx\right)^{\frac{1}{2}} \leq c_0 e^{-\lambda t}$ with $\lambda > \frac{\mu^2}{2\sigma^2}$, so $\left(\int_{-\infty}^{\infty} q(x,t)^2 dx\right)^{\frac{1}{2}} e^{\frac{\mu^2}{2\sigma^2}t}\sqrt{2\pi\sigma^2}t^{\frac{1}{4}} \leq c_0 e^{-\lambda t + \frac{\mu^2}{2\sigma^2}t}\sqrt{2\pi\sigma^2}t^{\frac{1}{4}} \to 0$ as $t \to \infty$. However, this contradicts the behavior (124) thereby proving the result for ρ with compact support.

When ρ does not have compact support, the inequality

$$\left(\int_0^{\infty} q(x,t)^2 dx\right)^{\frac{1}{2}} \leq c_0 e^{-\lambda t}$$

[97] This can be seen by taking the Fourier transform of the equation. Alternatively, the fundamental solution of the operator $\partial_t - \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\mu^2}{2\sigma^2}$ is given by $\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{x^2}{2\sigma^2 t} - \frac{\mu^2}{2\sigma^2}t}$, the heat kernel multiplied by $e^{-\frac{\mu^2}{2\sigma^2}t}$.

still holds if $J = 0$. This time q satisfies

$$
\begin{aligned}
q_t &= \mathcal{B}^* q + \frac{\sigma^2}{2}\frac{1-\theta}{\theta} q(0)\rho p_\infty^{-\frac{1}{2}} p_\infty^{\frac{1}{2}}(0) \\
&= \frac{\sigma^2}{2} q_{xx} + \frac{\sigma^2}{2}\frac{1-\theta}{\theta} p_\infty(0)(r(x)p_\infty^{-1}) q_x \\
&+ \left[\frac{\sigma^2}{4}\frac{1-\theta}{\theta} p_\infty(0)\cdot\left(p_\infty^{-1} r(x)\right)_x - \frac{\mu^2}{2\sigma^2} + \left(\frac{\sigma^2}{8}(\frac{1-\theta}{\theta})^2 p_\infty^2(0) r^2(x) p_\infty^{-2}\right)\right] q \\
&+ \frac{\sigma^2}{2}\frac{1-\theta}{\theta} q(0)\rho p_\infty^{-\frac{1}{2}} p_\infty^{\frac{1}{2}}(0)
\end{aligned}
$$

when $\theta > 0$, and we replace $\frac{\sigma^2}{2}\frac{1-\theta}{\theta} p_\infty(0)$ with $\frac{\sigma^2}{2}(p_\infty(0))_x$ when $\theta = 0$, and where $r(x) = \int_x^\infty \rho$. Note $r(x)p_\infty^{-1}(x) \to 0$, $\left(p_\infty^{-1} r(x)\right)_x \to 0$, $\rho p_\infty^{-\frac{1}{2}} p_\infty^{\frac{1}{2}}(0) \to 0$ as $x \to \infty$. Thus the translation at infinity argument applies to q, and we can conclude that again $\lambda > \frac{\mu^2}{2\sigma^2}$ results in a contradiction.[98] $\square$

J.2.5 Part 3: conclusion of proof of Prop. 2

Combining Lemmas 15 and 16, we see that when $\delta = 0$ and $\mu < 0$,

$$
-\lim_{t\to\infty} \frac{1}{t} \log \int_0^\infty |p(x,t) - p_\infty(x)| dx \geq \lambda \quad \text{with} \quad \lambda \leq \frac{\mu^2}{2\sigma^2}.
$$

Now it remains to show that the rate of convergence $\lambda = \frac{\mu^2}{2\sigma^2}$ to p_∞ is generically attained.

Now we show that given any p_0, an arbitrarily small perturbation will result in a rate of convergence at most $\frac{\mu^2}{2\sigma^2}$. Then we will have shown that for any initial distribution p_0 satisfying Assumption 1, the rate of convergence is at least λ, with $\lambda \leq \frac{\mu^2}{2\sigma^2}$, and an arbitrarily small perturbation of p_0 will have a rate of convergence at most λ.

The case $\theta = 1$ was already shown in Section F.2.1. For $\theta \in (0,1)$, we recall that

$$
\mathcal{A}^* p := \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \mu\frac{\partial p}{\partial x} + \frac{\sigma^2}{2}\frac{1-\theta}{\theta} p(0)\rho(x)
$$

with boundary condition $\frac{\sigma^2}{2} p_x(0) - \mu p(0) = \frac{\sigma^2}{2}\frac{1-\theta}{\theta} p(0)$. Given f_0 a smooth compactly supported function, uniformly bounded by 1 and with mass concentrated near the origin, with f solving $f_t = \mathcal{A}^* f$ with initial condition $f_0(x)$, we let $\eta(x,t) = f_x - (\frac{2}{\sigma^2}\mu + \frac{1-\theta}{\theta}) f$, so that

[98] Alternatively we can also use the fact that the fundamental solution $F(t,x)$ of $\partial_t - \mathcal{B}^*$ is bounded below by $\alpha e^{-\frac{\mu^2}{2\sigma^2} t} \frac{e^{-\frac{x^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}$ for some $\alpha > 0$. See for example the paper Aronson (1967). Thus $q(x,t) = q_0 * F \geq \alpha c \frac{e^{-\frac{\mu^2}{2\sigma^2} t}}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{x^2}{2\sigma^2 t}}$

$\eta(0,t) = 0$, and thus $\eta(x,t)$ solves $\frac{\sigma^2}{2}\frac{\partial^2\eta}{\partial x^2} - \mu\frac{\partial\eta}{\partial x} = \eta_t$. We can again extend η to the real line by reflecting around the origin, and we have come back to the case when $\theta = 1$. Thus $-\lim_{t\to\infty}\frac{1}{t}\log\int_0^\infty|\eta(x,t)|dx \le \frac{\mu^2}{2\sigma^2}$. By perturbing any initial distribution p_0 by $\varepsilon\eta_0(x)$, we will obtain a rate of convergence at most $\frac{\mu^2}{2\sigma^2}$.

When $\theta = 0$, the boundary condition for $\mathcal{A}^*$ is already $p(0) = 0$, and $\mathcal{A}^*p = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \mu\frac{\partial p}{\partial x} + \frac{\sigma^2}{2}p_x(0)\rho(x)$. Given f_0 a smooth compactly supported function with f^R solving $f_t = \mathcal{A}^*f$ with initial condition $f_0(x-R)$. Using the translation at infinity argument, we can let $R\to\infty$ and obtain a function $f(x,t)$ that satisfies $f(0,t) = 0$ and $\frac{\sigma^2}{2}\frac{\partial^2 f(x,t)}{\partial x^2} - \mu\frac{\partial f(x,t)}{\partial x} = 0$, and we can then apply the case $\theta = 1$ as before. Thus the rate of convergence $\lambda = \frac{\mu^2}{2\sigma^2}$ is generically attained.

When $\delta > 0$, p satisfies

$$p_t = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \mu\frac{\partial p}{\partial x} + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p(0)\rho(x) - \delta p(x) + \delta\psi(x)$$

where δ is the rate of "death" and "rebirth", and $\psi(x)$ is the distribution of income from which a newborn worker is drawn following the death of an individual. The boundary condition for p remains unchanged:

$$\frac{\sigma^2}{2}p_x(0) - \mu p(0) = \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p(0).$$

Let p_∞ satisfy

$$\frac{\sigma^2}{2}\frac{\partial^2 p_\infty}{\partial x^2} - \mu\frac{\partial p_\infty}{\partial x} + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}p_\infty(0)\rho(x) - \delta p_\infty(x) + \delta\psi(x) = 0$$

and define

$$\tilde{p}(x,t) := e^{\delta t}(p - p_\infty)$$

so that $\tilde{p}$ satisfies

$$\tilde{p}_t = \frac{\sigma^2}{2}\frac{\partial^2\tilde{p}}{\partial x^2} - \mu\frac{\partial\tilde{p}}{\partial x} + \frac{\sigma^2}{2}\frac{1-\theta}{\theta}\tilde{p}(0)\rho(x)$$

with boundary condition $\frac{\sigma^2}{2}\tilde{p}_x(0) - \mu\tilde{p}(0) = \frac{\sigma^2}{2}\frac{1-\theta}{\theta}\tilde{p}(0)$. Thus the results for $\delta = 0$ applies to

$\tilde{p}$. Since $J = \int \tilde{p}_0 = \int p_0 - \int p_\infty = 0$, Lemma 15 with $J = 0$ applied to $\tilde{p}$ gives

$$\lambda = - \lim_{t\to\infty} \frac{1}{t} \log \|\tilde{p}\|$$

which is equivalent to

$$\lambda + \delta = - \lim_{t\to\infty} \frac{1}{t} \log \|p(x,t) - p_\infty(x)\|$$

as was to be shown.

J.2.6 Extension of Prop. 2 to income-dependent coefficients $\mu(x, t)$ and $\sigma(x, t)$

So far we have proven 2 when μ and σ are constant. We now extend this to the case when $\mu(x,t)$ and $\sigma(x,t)$ depend on x and t,[99] but satisfy the conditions given in Assumption 2. Essentially, we need to extend the results Lemmas 14, 15, 16 to the variable coefficient case.

Extension of Lemma 14 First we show that the exponential decay similar to that established in Lemma 14 still holds, provided that we make modifications to terms with σ and μ, and we define $\mathcal{A}^*$, which now depends on time, as

$$\mathcal{A}^*(t)p(x) = \left(\frac{\sigma^2(x,t)}{2}p\right)_{xx} - (\mu(x,t)p)_x + \frac{\sigma^2(0,t)}{2}\frac{1-\theta}{\theta}p(0)\rho(x)$$

with the time-dependent boundary condition $\left(\frac{\sigma^2(x,t)}{2}p\right)_x(0) - \mu(0,t)p(0) = \frac{\sigma^2(0,t)}{2}\frac{1-\theta}{\theta}p(0)$, where we note that σ and μ depend on t. We also define

$$\mathcal{A}(t)u(x) = \frac{\sigma^2(x,t)}{2}\frac{\partial^2}{\partial x^2}u + \mu(x,t)\frac{\partial}{\partial x}u$$

with boundary condition $-\theta u_x(0) + (1-\theta)\left(u(0) - \int u(x)\rho(x)dx\right) = 0$. We check that with the above extended definitions, $\mathcal{A}(t)$ and $\mathcal{A}^*(t)$ are indeed adjoints of each other for fixed t. Let $\mathcal{A}_\infty$ and $\mathcal{A}^*_\infty$ denote the operators with time-independent coefficients $\tilde{\sigma}(x)$ and $\tilde{\mu}(x)$, and let $p_\infty(x)$ be the time-independent steady state solution $\mathcal{A}^*_\infty(t)p_\infty(x) = 0$. As before, p_∞ can be rescaled to be a probability distribution function, and $p_\infty(x) \sim Ce^{\frac{2\bar{\mu}}{\bar{\sigma}^2}x}$ as $x \to \infty$. For simplicity of notation, we will sometimes write μ, σ instead of $\mu(x,t)$ and $\sigma(x,t)$ when there is no confusion.

[99] The intermediate case when $\mu(x)$ and $\sigma(x)$ depend on x but not on t requires much less work, and can be obtained by directly adapting the arguments for the constant coefficient case. It is the time-dependent case that requires some new ideas.

We start from the equation satisfied by u^2:

$$\frac{\partial}{\partial t}u^2 - \mathcal{A}(t)u^2 + \sigma^2(u_x)^2 = 0$$

where we note the time dependence of $\mathcal{A}(t)$.

Now multiply the equation by p_∞ and perform integration by parts, but pay attention to the boundary conditions:

$$\int (\mathcal{A}u^2)p_\infty dx = \int \left(\left(\frac{\sigma^2(x,t)}{2}\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial x}\right)u^2\right)p_\infty dx \tag{125}$$

$$= \int -\frac{\partial}{\partial x}u^2\frac{\partial}{\partial x}(\frac{\sigma^2}{2}p_\infty) - u^2\frac{\partial}{\partial x}(\mu p_\infty)dx - \frac{\sigma^2}{2}(0,t)(u^2)_x(0)p_\infty(0)$$
$$- \mu(0,t)u^2(0)p_\infty(0) \tag{126}$$
$$= \int u^2\mathcal{A}^*(t)p_\infty dx - \frac{\sigma^2(0,t)}{2}\frac{1-\theta}{\theta}p_\infty(0)\int u^2\rho dx - \frac{\sigma^2(0,t)}{2}(u^2)_x(0)p_\infty(0)$$
$$+ (u^2)(0)(\frac{\sigma^2(x,t)}{2}p_\infty)_x(0) - \mu(0,t)u^2(0)p_\infty(0)$$

Unlike before, $\int u^2\mathcal{A}^*(t)p_\infty dx \neq 0$, but $\int u^2(x,t)dx$ is bounded by $\int u_0^2 dx$ for all t[100], and by our assumptions on the uniform convergence of $\mu(x,t)$ and $\sigma(x,t)$ and the fact that $p_\infty(x) \in L^\infty_{[0,\infty)}$, $\mathcal{A}^*(t)p_\infty(x) \to \mathcal{A}^*_\infty p_\infty(x) = 0$ uniformly as $t \to \infty$, so that $\int u^2\mathcal{A}^*(t)p_\infty dx \to 0$ as $t \to \infty$. Moreover, when $\theta > 0$, $(p_\infty)_x, (p_\infty)_{xx} \leq Cp_\infty$, for some universal constant, and when $\theta = 0$, this is true for x away from 0, since $p_\infty(0) = 0$. Therefore, we have the estimate

$$|\int u^2\mathcal{A}^*(t)p_\infty dx| \leq D_1(t)(\int u^2p_\infty dx + \int_0^1 u^2dx)$$

where $\int_0^1 u^2dx$ is only needed when $\theta = 0$ and $p_\infty(0) = 0$. Now a Hardy-inequality type argument gives

$$\int_0^1 u^2dx \leq C_1\left(\int u_x^2p_\infty dx + \int u^2p_\infty dx\right)$$

for some universal constant C_1. Indeed, write

$$\frac{d}{dx}(xu^2) = 2xuu_x + u^2$$

[100]Again by energy estimates of uniformly parabolic equations.

Integrating this expression from 0 to 1, we find

$$\begin{aligned}\int_0^1 u^2 dx &= \int_0^1 \frac{d}{dx}(xu^2)dx - 2\int_0^1 xuu_x dx \\ &\le u^2(1) + 2\left(\int_0^1 xu^2 dx\right)^{1/2}\left(\int_0^1 x(u_x)^2 dx\right)^{1/2} \\ &\le C\int_0^1 p_\infty(u^2+u_x^2)dx + C\left(\int_0^1 p_\infty u^2 dx\right)^{1/2}\left(\int_0^1 p_\infty (u_x)^2 dx\right)^{1/2} \\ &\le C\int_0^1 p_\infty(u^2+u_x^2)dx\end{aligned}$$

where we use Sobolev embedding in dimension 1 to bound the uniform norm of u by its $H^1(\mathbb{R})$ norm.

Therefore we have the estimate

$$|\int u^2 \mathcal{A}^*(t) p_\infty dx| \le D_1(t)(\int u^2 p_\infty dx + \int u_x^2 p_\infty dx) \tag{127}$$

where $D_1(t)$ is independent of u, and satisfies $D_1(t) \to 0$ as $t \to \infty$.

Another challenge comes from the other terms, because p_∞ does not satisfy the boundary conditions of $\mathcal{A}^*(t)$ with time-dependent $\sigma^2(x,t)$ and $\mu(x,t)$. We have to replace them by $\sigma^2(x)$ and $\mu(x)$, and show that the error is vanishingly small as $t \to \infty$. This is done next.

Observe that $(u^2)_x(0) = 2u_x(0)u(0)$. When $\theta > 0$, the boundary condition

$$-\theta u_x(0) + (1-\theta)\left(u(0) - \int u(x)\rho(x)dx\right) = 0$$

for $\mathcal{A}$ implies[101]

$$u_x(0) = \frac{1-\theta}{\theta}\left(u(0) - \int u\rho dx\right),$$

and the boundary condtion of ${\mathcal{A}_\infty}^*$ gives

$$(\frac{\tilde{\sigma}^2(x)}{2}p_\infty)_x(0) = \tilde{\mu}(0)p_\infty(0) + \frac{\tilde{\sigma}^2(0)}{2}\frac{1-\theta}{\theta}p_\infty(0).$$

where again note coefficients are time-independent.

[101]When $\theta = 0$, we note that $(\frac{\sigma^2}{2}u^2)_x(0)p_\infty(0) = 0$ and so need to use $u(0) - \int u\rho = 0$ instead of $u_x(0) = \frac{1-\theta}{\theta}\left(u(0) - \int u\rho\right)$. The subsequent calculations follow exactly by replacing $\frac{\sigma^2(0,t)}{2}\frac{1-\theta}{\theta}p_\infty(0)$ with $(\frac{\sigma^2}{2}p_\infty)_x(0,t)$.

Thus

$$\frac{\tilde{\sigma}^2(0)}{2}(u^2)_x(0)p_\infty(0) = \tilde{\sigma}^2(0)\frac{1-\theta}{\theta}p_\infty(0)(u(0) - \int u\rho)u(0)$$
$$(u^2)(0)(\frac{\tilde{\sigma}^2(x)}{2}p_\infty)_x(0) = \tilde{\mu}(0)p_\infty(0)u^2(0) + \frac{\tilde{\sigma}^2(0)}{2}\frac{1-\theta}{\theta}p_\infty(0)u^2(0)$$

We see that in 125 instead of $(u^2)(0)(\frac{\tilde{\sigma}^2(x)}{2}p_\infty)_x(0)$ we have $(u^2)(0)(\frac{\sigma^2(x,t)}{2}p_\infty)_x(0)$, and similarly for other terms. Thus we will replace all the terms with time-dependent coefficients with those with time-independent coefficients:

$$-\frac{\sigma^2(0,t)}{2}\frac{1-\theta}{\theta}p_\infty(0)\int u^2\rho dx \to -\frac{\tilde{\sigma}^2(0)}{2}\frac{1-\theta}{\theta}p_\infty(0)\int u^2\rho dx$$
$$-\frac{\sigma^2(0,t)}{2}(u^2)_x(0)p_\infty(0) \to -\frac{\tilde{\sigma}^2(0)}{2}(u^2)_x(0)p_\infty(0)$$
$$(u^2)(0)(\frac{\sigma^2(x,t)}{2}p_\infty)_x(0) \to (u^2)(0)(\frac{\tilde{\sigma}^2(x)}{2}p_\infty)_x(0)$$
$$-\mu(0,t)u^2(0)p_\infty(0) \to -\tilde{\mu}(0)u^2(0)p_\infty(0)$$

Note also that when $\theta > 0$, $p_\infty(0) > 0$, so that $u^2(0,t) \leq C\int u^2(x,t)p_\infty(x)dx$ for some universal constant C and all t. Moreover, $|\int u\rho dx|^2 \leq (\int u^2\rho dx) \leq C\int u^2 p_\infty$. If we let $R(t)$ denote the sum of the differences of these terms, i.e. the "error" term, using these bounds by $\int u^2 p_\infty$, we have the estimate

$$|R(t)| \leq D_2(t)\int u^2 p_\infty dx \tag{128}$$

here $D_2(t)$ is independent of u, and $D_2(t) \to 0$ as $t \to \infty$, using the convergence of $\sigma(0,t)$ and $\mu(0,t)$, as well as $(\partial_x\sigma(x,t))(0) \to (\partial_x\tilde{\sigma}(x))(0)$.[102]

Combining the above, we have

$$\int(\mathcal{A}u^2)p_\infty dx = -\frac{\tilde{\sigma}^2(0)}{2}\frac{1-\theta}{\theta}p_\infty(0)\left[\left(u(0) - \int u\rho dy\right)^2 + \int\left(u - \int u\rho dy\right)^2\rho dx\right]$$
$$+ R(t) + \int u^2\mathcal{A}^*(t)p_\infty dx$$

[102] When $\theta = 0$, note that $u(0) = \int u\rho$, and $p_\infty(0) = 0$, so the estimate still holds.

Thus

$$\frac{d}{dt}\int u^2 p_\infty(x)dx + \int \sigma^2(x,t)u_x^2 p_\infty dx$$
$$+\frac{\tilde{\sigma}^2(0)}{2}\frac{1-\theta}{\theta}p_\infty(0)\left[\left(u(0)-\int u\rho dy\right)^2+\int\left(u-\int u\rho dy\right)^2\rho dx\right]$$
$$-R(t)-\int u^2\mathcal{A}^*(t)p_\infty dx = 0$$

Now we would like to replace u with $u-\int up_\infty dx$, in order to have the orthogonality condition $\int up_\infty dx = 0$ for all t. This requires bounding the errors, which we denote collectively by $F(t)$, by terms involving $\int u^2 p_\infty$ again, and show that they vanish as $t\to\infty$. This is not hard: For the $\int u^2p_\infty dx$ term in $|R(t)| \le D_2(t)\int u^2 p_\infty dx$ and $|\int u^2\mathcal{A}^*(t)p_\infty dx| \le D_1(t)(\int u^2 p_\infty dx + \int_0^1 u^2 dx)$,

$$\begin{aligned}\int u^2 p_\infty dx - \int (u-\int up_\infty dy)^2 p_\infty dx &= (\int up_\infty dx)^2 \\ &\le \int u^2 p_\infty dx\end{aligned}$$

and for $\frac{d}{dt}\int u^2 p_\infty(x)dx$,

$$\begin{aligned}\frac{d}{dt}\int u^2 p_\infty(x)dx - \frac{d}{dt}\int (u-\int up_\infty dy)^2 p_\infty dx &= 2\int up_\infty dx\cdot\int u_t p_\infty dx \\ &= 2\int up_\infty dx\cdot\int u\mathcal{A}^*(t)p_\infty dx \\ &\le D_3(t)\int u^2 p_\infty dx\end{aligned} \tag{129}$$

where again $D_3(t)$ is independent of u and satisfies $D_3(t)\to 0$ as $t\to\infty$. Terms involving u_x do not change because $\int u(x,t)p_\infty dx$ depends only on t.

Therefore we have

$$\frac{d}{dt}\int (u-\int up_\infty dy)^2 p_\infty(x)dx$$
$$+\int\sigma^2(x,t)u_x^2p_\infty dx+\frac{\tilde{\sigma}^2(0)}{2}\frac{1-\theta}{\theta}p_\infty(0)\left[\left(u(0)-\int u\rho dy\right)^2+\int\left(u-\int u\rho dy\right)^2\rho dx\right]$$
$$-R(t)-\int u^2\mathcal{A}^*(t)p_\infty dx + F(t) = 0$$

Using $\sigma^2(x,t) \geq \gamma\bar{\sigma}^2$ and the estimates (127), (128), and (129) involving D_1, D_2, and D_3, this gives

$$\frac{d}{dt}\int (u - \int u p_\infty dy)^2 p_\infty(x) dx$$
$$+\gamma\bar{\sigma}^2 \int u_x^2 p_\infty dx + \frac{\gamma\bar{\sigma}^2}{2}\frac{1-\theta}{\theta} p_\infty(0)\left[\left(u(0) - \int u\rho dy\right)^2 + \int \left(u - \int u\rho dy\right)^2 \rho dx\right]$$
$$\leq (D_1 + D_2 + D_3)\int u^2 p_\infty dx + D_3 \int u_x^2 p_\infty dx$$

Noting that $\int (u - \int u p_\infty dy) dx = 0$, we can use the reasoning established in 14 to conclude that the terms in the second line are bounded above by

$$\gamma\bar{\sigma}^2 \int u_x^2 p_\infty dx + \frac{\gamma\bar{\sigma}^2}{2}\frac{1-\theta}{\theta} p_\infty(0)\left[\left(u(0) - \int u\rho dy\right)^2 + \int \left(u - \int u\rho dy\right)^2 \rho dx\right]$$
$$\leq \bar{\lambda}\gamma \int (u - \int u p_\infty dy)^2 p_\infty dx$$

where $\bar{\lambda}$ is defined in Lemma 14 with σ replaced by $\bar{\sigma}$.

Finally, we obtain the desired Grönwall type inequality

$$\frac{d}{dt}\int (u - \int u p_\infty dy)^2 p_\infty(x) dx + \gamma(t)\int (u - \int u p_\infty dy)^2 p_\infty(x) dx \leq 0$$

where $\gamma(t) \to \bar{\lambda}\gamma > 0$. Thus

$$\int (u - \int u p_\infty dy)^2 p_\infty(x) dx \leq C e^{-\lambda t} \int (u_0 - \int u_0 p_\infty dy)^2 p_\infty(x) dx$$

for $\lambda = \bar{\lambda}\gamma > 0$.

Extension of Lemma 15 Next, to obtain the exponential convergence

$$\int |p(x,t) - J p_\infty(x)| dx \leq C e^{-\lambda t},$$

recall that

$$\int u(x,t) p_0(x) dx = \int u_0(x) p(x,t) dx$$

This dual property still holds, using the Feynman-Kac formula, for example. More precisely,

$$\begin{aligned}\int u(x,t)p_0(x)dx &= \int E[u_0(X_t) \mid X_0 = x]p_0(x)dx \\ &= \int\int \mathbb{P}[X_t = y \mid X_0 = x]u_0(y)dyp_0(x)dx \\ &= \int\int \mathbb{P}[X_t = x \mid X_0 = y]p_0(x)dxu_0(y)dy \\ &= \int E[p_0(X_t) \mid X_0 = y]u_0(y)dy \\ &= \int u_0(y)p(y,t)dy\end{aligned}$$

Thus the calculations in Lemma 15 follows with minor modifications:

$$\int \left(u - \int u_0 p_\infty dy\right) p_0 dx = \int u_0 p dx - J\int u_0 p_\infty dy = \int u_0 \left(p - Jp_\infty\right) dx$$

and

$$\begin{aligned}\left|\int u_0 \left(p(x,t) - Jp_\infty\right) dx\right| &= \left|\int \left(u(x,t) - \int u_0 p_\infty dy\right) p_0 dx\right| \\ &= \left|\int \left(\left(u(x,t) - \int u_0 p_\infty dy\right)^2 p_\infty\right)^{1/2} \frac{p_0}{p_\infty^{1/2}} dx\right| \\ &\leq \left(\int \frac{(p_0)^2}{p_\infty} dx\right)^{1/2} \left(\int \left(u(x,t) - \int u_0 p_\infty dy\right)^2 p_\infty dx\right)^{1/2} \\ &\leq C_0 e^{-\lambda t} \left(\int (u_0 - \int u_0 p_\infty dy)^2 p_\infty dx\right)^{1/2} \\ &= C_0 e^{-\lambda t} \left(\int (u_0)^2 p_\infty dx - (\int u_0 p_\infty dx)^2\right)^{1/2} \\ &\leq C_0 e^{-\lambda t} \left(\int (u_0)^2 p_\infty dx\right)^{1/2}\end{aligned}$$

where we note that we have $\int \left(u(x,t) - \int u_0 p_\infty dy\right)^2 p_\infty dx$ instead of $\int \left(u(x,t) - \int u p_\infty dy\right)^2 p_\infty dx$ in the inequality. This is a minor issue, because going back to the Grönwall argument above, we notice that this alteration causes C_0 to depend on $\int u_0^2 p_\infty dx$. In later steps, we only use $u_0 = \frac{p - Jp_\infty}{p_\infty}$, and by energy estimates of uniformly parabolic equations, we see that

$\int u_0^2 p_\infty dx \leq C \int \frac{p_0^2}{p_\infty}$, uniformly bounded. The rest of the proof for exponential convergence is the same as in Lemma 15. To show that the rate of convergence λ is generically obtained, use translation at infinity to reduce to the constant-coefficient case.

Extension of Lemma 16 Finally, we extend Lemma 16 and show that if the exponential convergence

$$\int \frac{p^2(x,t)}{p_\infty} dx \leq Ce^{-\lambda t} \int \frac{p_0^2(x,t)}{p_\infty} dx$$

holds for all p, then λ is no larger than $\frac{\bar{\mu}^2}{2\bar{\sigma}^2}$.

In this case, the asymptotic behavior $p_\infty \sim Ce^{\frac{2\bar{\mu}}{\bar{\sigma}^2}x}$ implies that for any fixed $\varepsilon > 0$, there exists $\bar{x} > 0$ such that

$$1 - \varepsilon \leq \frac{p_\infty}{e^{\frac{2\bar{\mu}}{\bar{\sigma}^2}x}} \leq 1 + \varepsilon$$

for all $x \geq \bar{x}$. Therefore, we have

$$\begin{aligned} \int_{\bar{x}}^\infty \frac{p^2(x,t)}{e^{\frac{2\bar{\mu}}{\bar{\sigma}^2}x}} dx \leq (1+\varepsilon) \int_{\bar{x}}^\infty \frac{p^2(x,t)}{p_\infty(x)} dx &\leq (1+\varepsilon)e^{-2\lambda t} \int_{\bar{x}}^\infty \frac{p_0^2(x,t)}{p_\infty(x)} dx \\ &\leq \frac{1+\varepsilon}{1-\varepsilon} e^{-2\lambda t} \int_{\bar{x}}^\infty \frac{p_0^2(x,t)}{e^{\frac{2\bar{\mu}}{\bar{\sigma}^2}x}} dx \end{aligned}$$

An inequality of the form above can only hold when $\lambda \leq \frac{\bar{\mu}^2}{2\bar{\sigma}^2}$. This is because $\tilde{p}(x) = pe^{-\frac{\bar{\mu}}{\bar{\sigma}^2}x}$ solves

$$\partial_t \tilde{p} - (\frac{\tilde{\sigma}^2(x)}{2}\tilde{p})_{xx} + (b\tilde{p})_x + a\tilde{p} + \frac{\tilde{\sigma}^2(0)}{2}\frac{1-\theta}{\theta}\tilde{p}(0)\rho(x) = 0$$

where $b = \tilde{\mu}(x) - \frac{\bar{\mu}}{\bar{\sigma}^2}\tilde{\sigma}^2$ and $a = -\frac{\tilde{\sigma}^2(x)}{2\bar{\sigma}^4}\bar{\mu}^2 + \frac{\tilde{\mu}(x)}{\bar{\sigma}^2}\bar{\mu}$. Note that as $x \to \infty$, $b \to 0$ and $a \to \frac{\bar{\mu}^2}{2\bar{\sigma}^2}$, and of course $\rho(x) \to 0$. Then the translation at infinity argument used before shows that the inequality $\int_{\bar{x}}^\infty \frac{p^2(x,t)}{e^{\frac{2\bar{\mu}}{\bar{\sigma}^2}x}} dx \leq \frac{1+\varepsilon}{1-\varepsilon} e^{-2\lambda t} \int_{\bar{x}}^\infty \frac{p_0^2(x,t)}{e^{\frac{2\bar{\mu}}{\bar{\sigma}^2}x}} dx$ implies

$$\int_{\bar{x}}^\infty \tilde{p}^2 dx \leq \frac{1+\varepsilon}{1-\varepsilon} e^{-2\lambda t} \int_{\bar{x}}^\infty \tilde{p}_0^2 dx$$

where $\tilde{p}$ satisfies the equation $\partial_t p - \frac{\bar{\sigma}^2}{2} p_{xx} + \frac{\bar{\mu}^2}{2\bar{\sigma}^2} p = 0$ on the entire real line $\mathbb{R}$ with initial condition $\tilde{p}_0 = p_0 e^{-\frac{\bar{\mu}}{\bar{\sigma}^2}x}$, which yields a contradiction if $\lambda > \frac{\bar{\mu}^2}{2\bar{\sigma}^2}$. Proposition 2 for variable coefficients $\sigma(x,t)$ and $\mu(x,t)$ under Assumption 2 is complete. □

K Closed Form Expressions for the Finite-Time Distributions

Here are some expressions that we found useful in exploring income dynamics. They generalize the Steindl model in the paper.

Closed form with Pareto initial distribution. We here generalize the Steindl case of Lemma 1 to the case $\sigma > 0$. We consider the reflecting barrier case, with death rate $\delta \geq 0$, and rebirth at the barrier point, $x = 0$.

We now obtain the general form of the transition function.

Proposition 11 (Closed form for the transitions of reflected Brownian motion) *Take the reflected Brownian motion with drift and death rate $\delta \geq 0$ (agents who just died are reborn at $x = 0$). Suppose that the counter-CDF $P(x,t) := \mathbb{P}(x_t \geq x)$ starts from $P(x,0) = e^{-\alpha x}$. Then, the counter-CDF at time $t \geq 0$ is*

$$P(x,t) = e^{-\zeta x} + G^{\alpha}(x,t) - G^{\zeta}(x,t) \tag{130}$$

where

$$G^{\alpha}(x,t) = e^{\left(-\delta+\mu\alpha+\frac{1}{2}\alpha^2\sigma^2\right)t}\left[e^{-\alpha x}\Phi\left(\frac{-\left(\alpha\sigma^2+\mu\right)t+x}{\sigma\sqrt{t}}\right) - e^{\left(\alpha+\frac{2\mu}{\sigma^2}\right)x}\Phi\left(\frac{-\left(\alpha\sigma^2+\mu\right)t-x}{\sigma\sqrt{t}}\right)\right] \tag{131}$$

and Φ is the CDF of a standard Gaussian variable.

Proof: One can verify by calculation that: $\left(-\partial_t - \mu\partial_x + \frac{\sigma^2}{2}\partial_{xx} - \delta\right)G^{\alpha} = 0$, and

$$G^{\alpha}\left(x,0^+\right) = e^{-\alpha x} \text{ for all } x > 0,$$
$$G^{\alpha}(0,t) = 0 \text{ for all } t > 0.$$

□

We can verify that we obtain Lemma 1 as a particular case when $\sigma \to 0$. [103]

[103] Indeed, take the Steindl case, with $\sigma \to 0$. Then, $\mu > 0$, and $G^{\alpha}(x,t) = e^{(-\delta+\alpha\mu)t-\alpha x}1_{x>\mu t}$ so

$$P(x,t) = e^{-\zeta x} + G^{\alpha}(x,t) - G^{\zeta}(x,t) = e^{-\zeta x} + e^{-\delta t}\left(e^{\alpha\mu t-\alpha x} - e^{\zeta\mu t-\zeta x}\right)1_{x>\mu t}$$

so (using $\zeta\mu = \delta$), if $x > \mu t$,

$$P(x,t) = e^{-\zeta x} + e^{-\delta t}e^{\alpha\mu t-\alpha x} - e^{-\zeta x} = e^{-\delta t}e^{\alpha\mu t-\alpha x} = e^{-\alpha x+(\alpha-\zeta)\mu t}.$$

Finally, Harrison (1985, p.15, 49) gives a formula for the Brownian motion reflected at 0: $F(x,t;x_0) := \mathbb{P}(X_t \geq x \mid X_0 = x_0)$:

$$F(x,t;x_0) = \Phi\left(\frac{-x + x_0 + \mu t}{\sigma\sqrt{t}}\right) + e^{2\mu x/\sigma^2}\Phi\left(\frac{-x - x_0 - \mu t}{\sigma\sqrt{t}}\right) \tag{132}$$

for $x_0, x, t \geq 0$. Given an initial density p_0, this gives $P(x,t) = \int_0^\infty F(x,t;y)\, p_0(y)\, dy$. Equation (130) is more explicit when starting from an exponential p_0.

Model without lower bounds. We characterize the time path of the distribution for $\sigma > 0$ and for more general initial conditions $p_0(x)$:

Proposition 12 (Closed form solution for the general model without a lower bound) *In the case without a lower bound, the density can be expressed as:*

$$p(x,t) = p_\infty(x) + e^{-\delta t}\mathbb{E}\left[p_0(x - G_t) - p_\infty(x - G_t)\right] \tag{133}$$

where $G_t := \mu t + \sigma Z_t$, and the expectation is taken over the stochastic realizations of G_t. If there are jumps, the expressions are the same, except that $G_t := \mu t + \sigma Z_t + \sum_{i=1}^{N_t} g_i$, where N_t denotes the number of jumps g_i between 0 and t.

Proof: Take the case with no jumps. Call $q(x,t) = e^{-\delta t}(p(x,t) - p_\infty(x))$. Then, $q_t = -\mu q_x + \frac{\sigma^2}{2} q_{xx}$: this corresponds to the process $dG_t = \mu dt + \sigma dZ_t$, with no death. By the Feynman-Kac formula, $q(x,t) = \mathbb{E}[q_0(x - G_t)]$, i.e. (133). The case with jumps is similar. $\square$

Note that when $\sigma = 0$, this leads to the Steindl case (23) for an exponential initial distribution.

Reflecting barrier. Here is an explicit formula for the case with a reflecting barrier.

Proposition 13 (Explicit formula with a reflecting barrier) *Consider the process with a reflecting barrier at 0. We have the following explicit formula for $P(x,t) = \mathbb{P}(X_t \leq x)$:*

$$P(x,t) = e^{-\frac{\mu^2}{2\sigma^2}t + \frac{\mu}{\sigma^2}x}\mathbb{E}\left[\tilde{Q}_0(x + \sigma Z_t)\right]$$

where $\tilde{Q}_0(x) = sign(x)\, P_0(|x|, 0)\, e^{-\frac{\mu}{\sigma^2}|x|}$, and Z_t is a standard Brownian motion.

Proof: We normalize $\sigma^2 = 1$ for notational simplicity (the general case is easy from dimensional analysis). Given $p_t = -\mu p_x + \frac{1}{2} p_{xx}$, define $P(x,t) = \mathbb{P}(X_t \leq x) = \int_0^x p(y,t)\, dy$, we have:

$$P_t = -\mu P_x + \frac{1}{2} P_{xx}$$

and $P(0,t) = 0$. Next, define $Q(x,t) := e^{-\beta x} P(x,t)$ for a β to be determined soon. From $P(x,t) = e^{\beta x} Q(x,t)$, we calculate:

$$\begin{aligned} P_x &= e^{\beta x}\left(Q_x + \beta Q\right), \\ P_{xx} &= e^{\beta x}\left(Q_{xx} + 2\beta Q_x + \beta^2 Q\right), \end{aligned}$$

hence

$$e^{\beta x} Q_t = P_t = -\mu P_x + \frac{1}{2} P_{xx} = e^{\beta x}\left(\frac{1}{2} Q_{xx} + (\beta - \mu) Q_x + \left(\frac{1}{2}\beta^2 - \mu\beta\right) Q\right).$$

So, set $\beta = \mu$. This gives

$$Q_t = \frac{1}{2} Q_{xx} - \frac{\mu^2}{2} Q \tag{134}$$

and $Q(0,t) = 0$.

We next define:

$$\tilde{Q}(x) := Q(x)\, 1_{x \geq 0} - Q(-x)\, 1_{x<0} = sign(x)\, Q(|x|),$$

then $\tilde{Q}$ is defined for all $x \in \mathbb{R}$, not just for $x \in \mathbb{R}_+$, as Q is. Furthermore:

$$\tilde{Q}_t = \frac{1}{2} \tilde{Q}_{xx} - \frac{\mu^2}{2} \tilde{Q} \tag{135}$$

and $\tilde{Q}(0,t) = 0$.

Now, set $q(x,t) := e^{\frac{\mu^2}{2} t} \widetilde{Q}(x,t)$. We have:

$$\begin{aligned} q_t &= e^{\frac{\mu^2}{2} t}\left(\frac{\mu^2}{2} \widetilde{Q} + \widetilde{Q}_t\right) = e^{\frac{\mu^2}{2} t}\left(\frac{\mu^2}{2} \widetilde{Q} + \frac{1}{2} \widetilde{Q}_{xx} - \frac{\mu^2}{2} \widetilde{Q}\right) = e^{\frac{\mu^2}{2} t} \frac{1}{2} \widetilde{Q}_{xx} = \frac{1}{2} q_{xx}, \\ q_t &= \frac{1}{2} q_{xx}, \end{aligned} \tag{136}$$

Hence, $q(x,t)$ just follows the heat equation. Note that the speed of convergence of q is slower than any exponential: $\lambda_q = 0$, so indeed, the speed of convergence of $\tilde{Q}$ is: $\lambda_Q = \frac{-\mu^2}{2}$, i.e. in dimensional units, $\lambda_Q = \frac{-\mu^2}{2\sigma^2}$.

We also obtain: $q(x,t) = E[q_0(x+Z_t)]$ from Feynman-Kac formula, so that:

$$\tilde{Q}(x,t) = e^{-\frac{\mu^2}{2}t}\mathbb{E}\left[\tilde{Q}_0(x+Z_t)\right],$$
$$P(x,t) = e^{\mu x}Q(x,t) = e^{\mu x}\tilde{Q}(x,t) = e^{-\frac{\mu^2}{2}t+\mu x}\mathbb{E}\left[\tilde{Q}_0(x+Z_t)\right].$$

In the statement, we add the dimensions in σ^2. □

Appendix References

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