Lectures 5 and 6
Theories of Top Inequality
Distributional Dynamics and Differential Operators

Distributional Macroeconomics

Part II of ECON 2149

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Harvard University, Spring 2018

Outline

1. Gabaix (2009) “Power Laws in Economics and Finance”
2. Literature on inequality and random growth
3. Gabaix-Lasry-Lions-Moll (2016) “The Dynamics of Inequality”
    - tools: differential operators as transition matrices
    - will be extremely useful for analysis, computation of fully-fledged heterogeneous agent models later on

Power Laws

- Definition 1: S follows a power law (PL) if there exist $k, \zeta > 0$ s.t.

$$\Pr(S > x) = kx^{-\zeta}, \quad \text{all } x$$

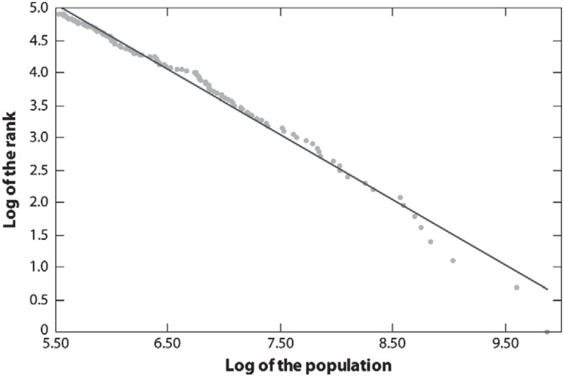
- S follows a PL $\Leftrightarrow$ S has a Pareto distribution
- Definition 2: S follows an asymptotic power law if there exist $k, \zeta > 0$ s.t.

$$\Pr(S > x) \sim kx^{-\zeta} \quad \text{as} \quad x \to \infty$$

- Note: for any f, g $f(x) \sim g(x)$ means $\lim_{x \to \infty} f(x)/g(x) = 1$
- Surprisingly many variables follow power laws, at least in tail

City Size

- Order cities in US by size (NY as first, LA as second, etc)
- Graph ln Rank ($\ln \text{Rank}_{NY} = \ln 1$, $\ln \text{Rank}_{LA} = \ln 2$) vs. ln Size
- Basically plot log quantiles $\ln \Pr(S > x)$ against $\ln x$



City Size

- Surprise 1: straight line, i.e. city size follows a PL

$$\Pr(S > x) = kx^{-\zeta}$$

- Surprise 2: slope of line ≈ -1, regression:

$$\ln \text{Rank} = 10.53 - 1.005 \ln \text{Size}$$

  i.e. city size follows a PL with exponent $\zeta \approx 1$

$$\Pr(S > x) = kx^{-1}.$$

- A power law with exponent $\zeta = 1$ is called "Zipf's law"

- Two natural questions:

  1. Why does city size follow a power law?
  2. Why on earth is $\zeta \approx 1$ rather than any other number?

Where Do Power Laws Come from?

- Gabaix's answer: random growth
- Economy with continuum of cities
- S_t^i: size of city i at time t

$$S_{t+1}^i = \gamma_{t+1}^i S_t^i, \quad \gamma_{t+1}^i \sim f(\gamma) \tag{RG}$$

- S_t^i follows random growth process $\Leftrightarrow$ $\log S_t^i$ follows random walk
- Gabaix shows: (RG) + stabilizing force (e.g. minimum size) $\Rightarrow$ power law. Use "Champernowne's equation"
- Easier: continuous time approach

Random Growth Process in Continuous Time

- Consider random growth process over time intervals of length Δt

$$S^i_{t+\Delta t} = \gamma^i_{t+\Delta t} S^i_t$$

- Assume in addition that $\gamma^i_{t+\Delta t}$ takes the particular form

$$\gamma^i_{t+\Delta t} = 1 + g\Delta t + \nu \varepsilon^i_t \sqrt{\Delta t}, \quad \varepsilon^i_t \sim \mathcal{N}(0,1)$$

- Substituting in

$$S^i_{t+\Delta t} - S^i_t = (g\Delta t + \nu \varepsilon^i_t \sqrt{\Delta t}) S^i_t$$

- Or as $\Delta t \to 0$

$$dS^i_t = gS^i_t dt + \nu S^i_t dW^i_t$$

i.e. a geometric Brownian motion!

Stationary Distribution

- Assumption: city size follows random growth process

$$dS_t^i = gS_t^i dt + \nu S_t^i dW_t^i$$

- Does this have a stationary distribution? No! In fact

$$\log S_t^i \sim \mathcal{N}((g - \nu^2/2)t, \nu^2 t)$$

$\Rightarrow$ distribution explodes.

- Gabaix insight: random growth process + stabilizing force does have a stationary distribution and that's a PL
  - Note: Gabaix uses "friction" rather than "stabilizing force"
  - use the latter because "friction" already means something else
- Simplest possible stabilizing force: $g < 0$ and minimum size $S_{\min}$
  - if process goes below $S_{\min}$ it is brought back to $S_{\min}$ ("reflecting barrier")

Stationary Distribution

- Use Kolmogorov Forward Equation

- Recall: stationary distribution satisfies

$$0 = -\frac{d}{dx}[\mu(x)f(x)] + \frac{1}{2}\frac{d^2}{dx^2}[\sigma^2(x)f(x)]$$

- Here geometric Brownian motion: $\mu(x) = gx, \sigma^2(x) = \nu^2 x^2$

$$0 = -\frac{d}{dx}[gxf(x)] + \frac{1}{2}\frac{d^2}{dx^2}[\nu^2 x^2 f(x)]$$

Stationary Distribution

- Claim: solution is a Pareto distribution, $f(x) = S_{\min}^{\zeta} x^{-\zeta-1}$
- Proof: Guess $f(x) = Cx^{-\zeta-1}$ and verify

$$\begin{aligned} 0 &= -\frac{d}{dx}[gxCx^{-\zeta-1}] + \frac{1}{2}\frac{d^2}{dx^2}[\nu^2 x^2 C x^{-\zeta-1}] \\ &= Cx^{-\zeta-1}\left[g\zeta + \frac{\nu^2}{2}(\zeta-1)\zeta\right] \end{aligned}$$

- This is a quadratic equation with two roots $\zeta = 0$ and

$$\zeta = 1 - \frac{2g}{\nu^2}$$

- For mean to exist, need $\zeta > 1 \Rightarrow$ impose $g < 0$
- Remains to pin down C. We need

$$1 = \int_{S_{\min}}^{\infty} f(x)dx = \int_{S_{\min}}^{\infty} Cx^{-\zeta-1}dx \quad \Rightarrow \quad C = S_{\min}^{\zeta}. \square$$

Tail inequality and Zipf's Law

- "Tail inequality" (fatness of tail)

$$\eta := \frac{1}{\zeta} = \frac{1}{1 - 2g/\nu^2}$$

  is increasing in g and ν^2 (recall $g < 0$)

- Why would Zipf's Law ($\zeta = 1$) hold? We have that

$$\bar{S} = \int_{S_{\min}}^{\infty} x f(x) dx = \frac{\zeta}{\zeta - 1} S_{\min}$$

$$\Rightarrow \quad \zeta = \frac{1}{1 - S_{\min}/\bar{S}} \to 1 \quad \text{as} \quad S_{\min}/\bar{S} \to 0.$$

- Zipf's law obtains as stabilizing force becomes small

Alternative Stabilizing Force: Death

- No minimum size
- Instead: die at Poisson rate δ, get reborn at S_*
- Can show: correct way of extending KFE (for $x \neq S_*$) is

$$\frac{\partial f(x,t)}{\partial t} = -\delta f(x,t) - \frac{\partial}{\partial x}[\mu(x) f(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x) f(x,t)\right]$$

- Stationary $f(x)$ satisfies (recall $\mu(x) = gx, \sigma^2(x) = \nu^2 x^2$)

$$0 = -\delta f(x) - \frac{d}{dx}[gxf(x,t)] + \frac{1}{2}\frac{d^2}{dx^2}\left[\nu^2 x^2 f(x)\right] \qquad \text{(KFE')}$$

Alternative Stabilizing Force: Death

- To solve (KFE'), guess $f(x) = Cx^{-\zeta-1}$

$$0 = -\delta + \zeta g + \frac{\nu^2}{2}\zeta(\zeta - 1)$$

- Two roots: $\zeta_+ > 0$ and $\zeta_- < 0$. General solution to (KFE'):

$$\Rightarrow \quad f(x) = C_- x^{-\zeta_- - 1} + C_+ x^{-\zeta_+ - 1} \quad \text{for } x \neq S_*$$

- Need solution to be integrable

$$\int_0^\infty f(x)dx = f(S_*) + \int_0^{S_*} f(x)dx + \int_{S_*}^\infty f(x)dx < \infty$$

- Hence $C_- = 0$ for $x > S_*$, otherwise $f(x)$ explodes as $x \to \infty$
- And $C_+ = 0$ for $x < S_*$, otherwise $f(x)$ explodes as $x \to 0$

Alternative Stabilizing Force: Death

- Solution is a Double Pareto distribution:

$$f(x) = \begin{cases} C(x/S_*)^{-\zeta_- - 1} & \text{for } x < S_* \\ C(x/S_*)^{-\zeta_+ - 1} & \text{for } x > S_* \end{cases}$$

Generalizations and Other Stabilizing Forces

- See Appendix D of "The Dynamics of Inequality" for a pretty exhaustive list
  - death and rebirth with $S_t^i \sim \psi(S)$
  - additive term

$$dS_t^i = y\,dt + gS_t^i dt + vS_t^i dW_t^i, \quad g < 0, \; y > 0$$

  - ....

- In general, distribution will not be exactly Pareto or exactly double-Pareto
- But often, under quite weak assumptions, it will still follow asymptotic power law, i.e.

$$\Pr(S > x) \sim kx^{-\zeta} \quad \text{as} \quad x \to \infty$$

Literature: Inequality and Random Growth

- Income distribution
  - Champernowne (1953), Simon (1955), Mandelbrot (1961), Nirei (2009), Toda (2012), Kim (2013), Jones and Kim (2013), Aoki and Nirei (2014),...
- Wealth distribution
  - Wold and Whittle (1957), Stiglitz (1969), Cowell (1998), Nirei and Souma (2007), Panousi (2012), Benhabib, Bisin, Zhu (2012, 2014), Piketty and Zucman (2014), Piketty and Saez (2014), Piketty (2015), Benhabib and Bisin (2016) is nice survey
- Dynamics of income and wealth distribution
  - Aoki and Nirei (2014), Gabaix, Lasry, Lions and Moll (2016), Hubmer, Krusell, Smith (2016)

Literature: Inequality and Random Growth

From Piketty "About Capital in the Twenty-First Century" (AEA P&P, 2015)

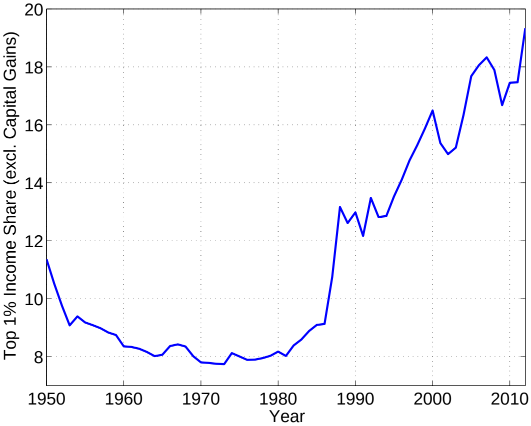
- "Technically, one can indeed show that if shocks take a multiplicative form, then the inequality of wealth converges toward a distribution that has a Pareto shape for top wealth holders [...], and that the inverted Pareto coefficient (an indicator of top end inequality) is a steeply rising function of the gap $r - g$."

- Idea: $\mu(x) = (r - g - \text{constant})x$

- In book this point unfortunately gets lost in discussion about how $r - g$ affects capital share
  - factor income vs personal income distribution
  - no general connection between capital share and inequality

The Dynamics of Inequality

Question



- In U.S. past 40 years have seen rapid rise in top income inequality
- Why?

Question

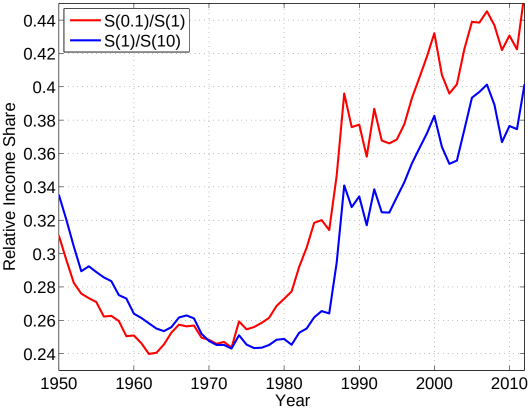
- Main fact about top inequality (since Pareto, 1896): upper tails of income (and wealth) distribution follow power laws

- Equivalently, top inequality is fractal

  1. ... top 0.01% are X times richer than top 0.1%,... are X times richer than top 1%,... are X times richer than top 10%,...

  2. ... top 0.01% share is fraction Y of 0.1% share,... is fraction Y of 1% share, ... is fraction Y of 10% share,...

Evolution of "Fractal Inequality"



- $\frac{S(0.1)}{S(1)}$ = fraction of top 1% share going to top 0.1%
- $\frac{S(1)}{S(10)}$ = analogous

This Paper

- Starting point: existing theories that explain top inequality at point in time
  - differ in terms of underlying economics
  - but share basic mechanism for generating power laws: random growth
- Our ultimate question: which specific economic theories can also explain observed dynamics of top income inequality?
  - e.g. falling income taxes? superstar effects?
- What we do:
  - study transition dynamics of cross-sectional income distribution in theories with random growth mechanism
  - contrast with data, rule out some theories, rule in others
- Today: income inequality. Paper: also wealth inequality.

Main Results

- Transition dynamics of standard random growth models too slow relative to those observed in the data
    - analytic formula for speed of convergence
    - transitions particularly slow in upper tail of distribution
    - jumps cannot generate fast transitions either
- Two parsimonious deviations that generate fast transitions
    1. heterogeneity in mean growth rates
    2. “superstar shocks” to skill prices
- Both only consistent with particular economic theories
- Rise in top income inequality due to
    - ~~simple tax stories, stories about Var(permanent earnings)~~
    - rise of “superstar” entrepreneurs or managers

A Random Growth Theory of Income Dynamics

- Continuum of workers, heterogeneous in human capital h_{it}
- die/retire at rate δ, replaced by young worker with h_{i0}
- Wage is $w_{it} = \omega h_{it}$
- Human capital accumulation involves
  - investment
  - luck
- "Right" assumptions $\Rightarrow$ wages evolve as

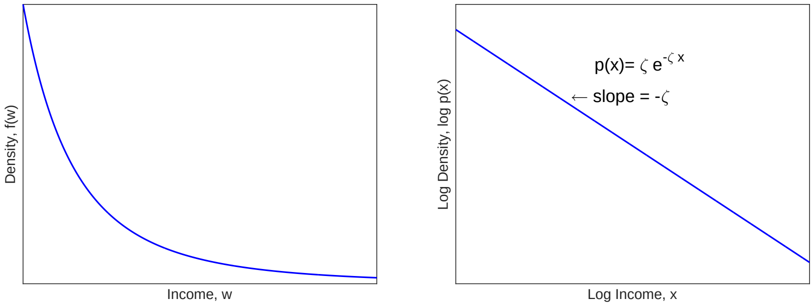
$$d \log w_{it} = \mu dt + \sigma dZ_{it}$$

  - growth rate of wage w_{it} is stochastic
  - μ, σ depend on model parameters
  - see Appendix C: log-utility + constant returns (same trick as AK-RBC model in Lecture 4)

Stationary Income Distribution

- Result: The stationary income distribution has a Pareto tail

$$\Pr(\tilde{w} > w) \sim Cw^{-\zeta}$$



- Convenient to work with log income $x_t = \log w_t$

$$\Pr(\tilde{w} > w) \sim Cw^{-\zeta} \quad \Leftrightarrow \quad \Pr(\tilde{x} > x) \sim Ce^{-\zeta x}$$

- Tail inequality $1/\zeta$ increasing in μ, σ, decreasing in δ

Stationary Income Distribution

- Have $x_{it} = \log w_{it}$ follows
$$dx_{it} = \mu dt + \sigma dZ_{it}$$
- Need additional "stabilizing force" to ensure existence of stat. dist.
  - income application: death/retirement at rate δ
  - alternative: reflecting barrier
- Distribution $p(x, t)$ satisfies ($\psi(x) =$ distribution of entry wages)
$$p_t = -\mu p_x + \frac{\sigma^2}{2} p_{xx} - \delta p + \delta \psi \qquad (*)$$
- With reflecting barrier at $x = 0$, have boundary condition
$$0 = -\mu p(0, t) + \frac{\sigma^2}{2} p_x(0, t)$$

  Derivation: $\int_0^\infty p(x, t) dx = 1$ for all t, and hence from $(*)$
$$0 = \int_0^\infty p_t dx = \left[-\mu p + \frac{\sigma^2}{2} p_x \right]_0^\infty$$

Stationary Income Distribution

- Stationary Distribution $p_\infty(x)$ satisfies

$$0 = -\mu p_x + \frac{\sigma^2}{2} p_{xx} - \delta p + \delta \psi$$

- Find solution via guess-and-verify: plug in $p(x) = Ce^{-\zeta x}$

$$0 = \mu\zeta + \frac{\sigma^2}{2}\zeta^2 - \delta + \delta \frac{\psi(x)}{Ce^{-\zeta x}}$$

- Assume $\lim_{x\to\infty} \psi(x)/e^{-\zeta x} = 0 \Rightarrow$ last term drops for large x & ζ solves

$$0 = \mu\zeta + \frac{\sigma^2}{2}\zeta^2 - \delta$$

  with positive root

$$\zeta = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2}$$

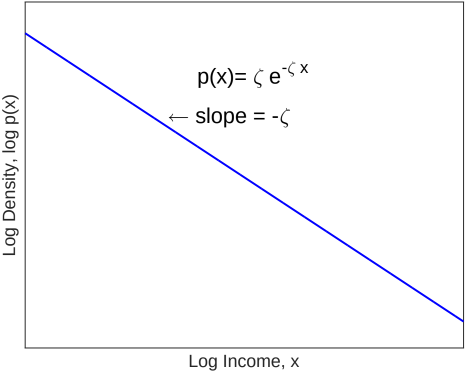
- Tail inequality $\eta = 1/\zeta$ increasing in μ, σ, decreasing in δ

Other Theories of Top Inequality

- We confine ourselves to theories that generate power laws
  - random growth
  - models with superstars (assignment models) – more later
- Example of theories that do not generate power laws, i.e. do not generate fractal feature of top income inequality:
  - theories of rent-seeking (Benabou and Tirole, 2015; Piketty, Saez and Stantcheva, 2014)
  - someone should write that "rent-seeking $\Rightarrow$ power law" paper

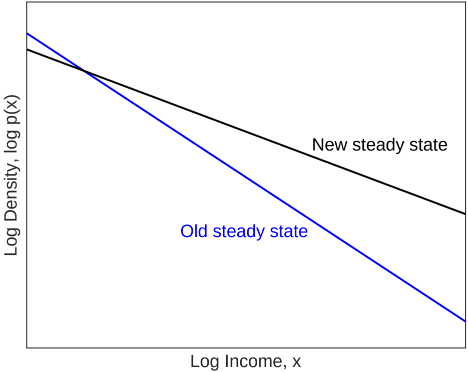
Transitions: The Thought Experiment

- Suppose economy is in Pareto steady state



Transitions: The Thought Experiment

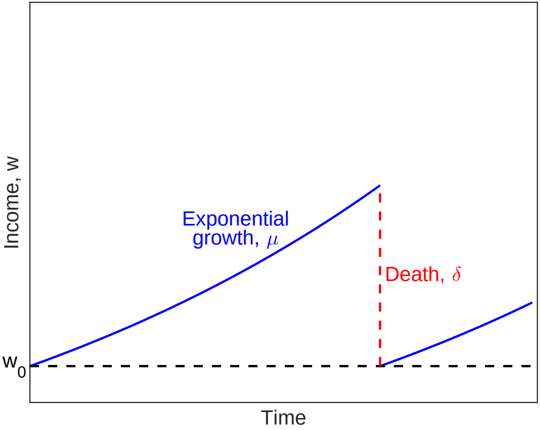
- Suppose economy is in Pareto steady state
- At $t = 0$, $\sigma \uparrow$. Know: in long-run $\rightarrow$ higher top inequality



- What can we say about the speed at which this happens?
- Which part of the distribution moves first?

Instructive Special Case: $\sigma = 0$ ("Steindl Model")

- In special case $\sigma = 0$, can solve full transition dynamics
  - w_t grows at rate μ, gets reset to $w_0 = 1$ at rate δ



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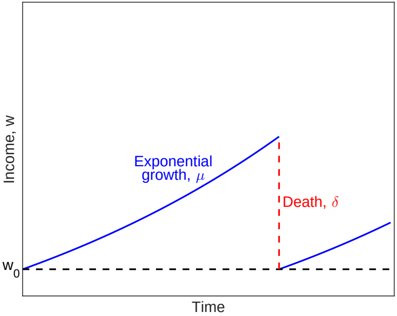
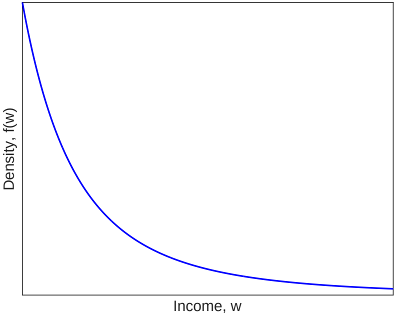
  - stationary distribution $f(w) = \zeta w^{-\zeta}, \zeta = \delta/\mu$





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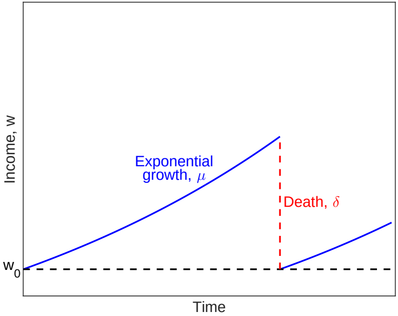
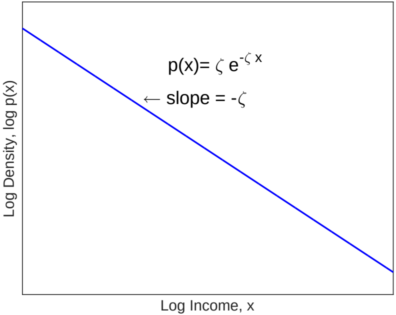
- In special case $\sigma = 0$, can solve full transition dynamics
  - w_t grows at rate μ, gets reset to $w_0 = 1$ at rate δ
  - stationary distribution of $x_t = \log w_t$: $p(x) = \zeta e^{-\zeta x}, \zeta = \delta/\mu$





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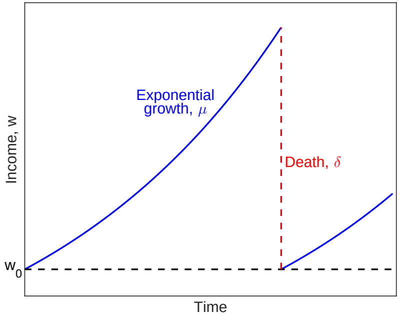
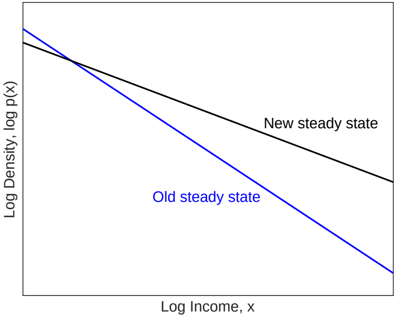
- In special case $\sigma = 0$, can solve full transition dynamics
  - w_t grows at rate μ, gets reset to $w_0 = 1$ at rate δ
  - at $t = 0$, $\mu \uparrow$. Know from $\zeta = \delta/\mu$: in long-run, top inequality $\uparrow$





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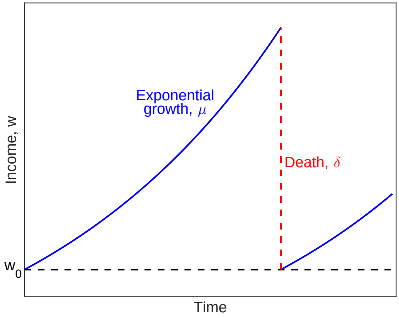
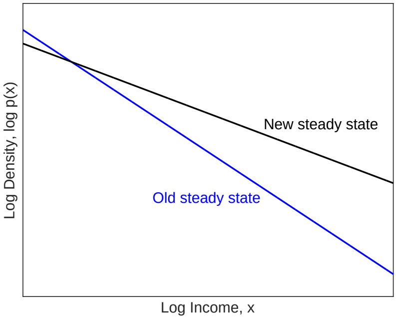




- What can we say about the speed at which this happens?
- Which parts of the distribution move first?

Transition in Steindl Model

- Denote
  - old steady state distribution: $p_0(x) = \alpha e^{-\alpha x}$
  - new steady state distribution: $p_\infty(x) = \zeta e^{-\zeta x}$
- Can show: for $t, x > 0$ density satisfies

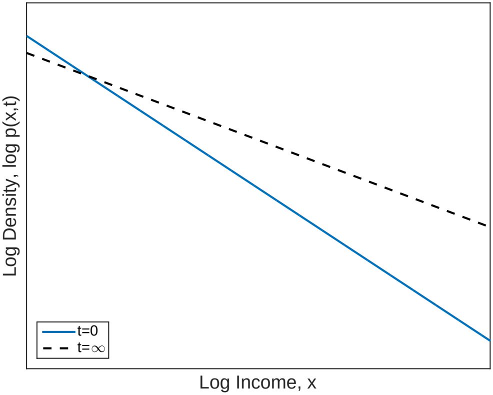
$$\frac{\partial p(x,t)}{\partial t} = -\mu \frac{\partial p(x,t)}{\partial x} - \delta p(x,t), \quad p(x,0) = \alpha e^{-\alpha x} \qquad (*)$$

- Result: the solution to $(*)$ is

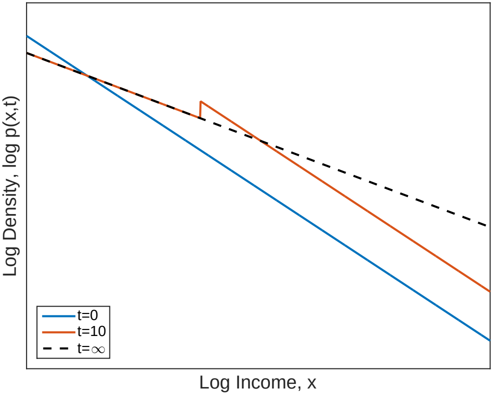
$$p(x,t) = \zeta e^{-\zeta x} \mathbf{1}_{\{x \le \mu t\}} + \alpha e^{-\alpha x + (\alpha - \zeta) t} \mathbf{1}_{\{x > \mu t\}}$$

where $\mathbf{1}_{\{.\}}$ = indicator function

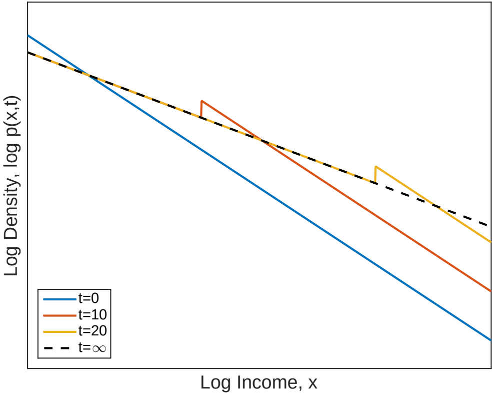
Transition in Steindl Model



Transition in Steindl Model

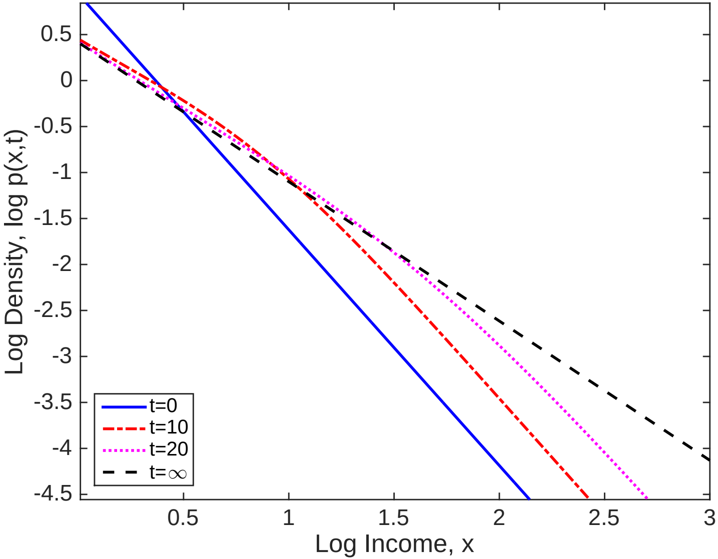


Transition in Steindl Model



- transition is slower in upper tail: it takes time $\tau(x) = x/\mu$ for the local PL exponent to converge to its steady state value ζ
- related to slow transition: crazy (age,income) distribution (Luttmer)

General Case



General Case

- Recall Kolmogorov Forward equation for $p(x,t)$

$$p_t = -\mu p_x + \frac{\sigma^2}{2} p_{xx} - \delta p + \delta \psi$$

- Question: at what speed does $p(x,t)$ converge to $p_\infty(x)$?
- need a "distance measure"
- Use L^1 norm:

$$||p(x,t) - p_\infty(x)|| := \int_{-\infty}^{\infty} |p(x,t) - p_\infty(x)| dx$$

- measures average distance between p and p_∞
- Later: more general distance measures

General Case: Average Speed of Convergence

- **Proposition:** $p(x,t)$ converges to stationary distrib. $p_\infty(x)$
  - rate of convergence
$$\lambda := -\lim_{t\to\infty} \frac{1}{t} \log ||p(x,t) - p_\infty(x)|| \tag{$*$}$$
  - without reflecting barrier
$$\lambda = \delta$$
  - with reflecting barrier
$$\lambda = \frac{1}{2}\frac{\mu^2}{\sigma^2}\mathbf{1}_{\{\mu<0\}} + \delta$$
- Intepretation of $(*)$: exponential convergence at rate λ
$$||p(x,t) - p_\infty(x)|| \sim k e^{-\lambda t} \quad \text{as } t \to \infty$$
- Half life is $t_{1/2} = \ln(2)/\lambda \Rightarrow$ precise quantitative predictions

Before proving this, let's take a step back...

- ... and take a somewhat different perspective on the Kolmogorov Forward equation
    - exploit heavily analogy to finite-state processes
- This will also be extremely useful for computations
- Let's focus on case with reflecting barrier at $x = 0$ and $\delta = 0$
- Kolmogorov Forward equation is

$$p_t = -\mu p_x + \frac{\sigma^2}{2} p_{xx}$$

with boundary condition

$$0 = -\mu p(0, t) + \frac{\sigma^2}{2} p_x(0, t)$$

Key: operator in KFE = transpose of transition matrix

- Just for a moment, suppose x_{it} = finite-state Poisson process
- $x_{it} \in \{x_1, ..., x_N\} \Rightarrow$ distribution = vector $\mathbf{p}(t) \in \mathbb{R}^N$
- Dynamics of distribution

  $$\dot{\mathbf{p}}(t) = \mathbf{A}^\top \mathbf{p}(t),$$

  where $\mathbf{A} = N \times N$ transition matrix
- Key idea: KFE is exact analogue with continuous state
- Can write in terms of differential operator $\mathcal{A}^*$

  $$p_t = \mathcal{A}^* p, \quad \mathcal{A}^* p = -\mu p_x + \frac{\sigma^2}{2} p_{xx}$$

  with boundary condition $0 = -\mu p(0) + \frac{\sigma^2}{2} p_x(0)$
- $\mathcal{A}^*$ analogue of transpose of transition matrix $\mathbf{A}^\top$

This can be made more precise...

- Definition: the inner product of two functions v and p is $< v, p >= \int_0^\infty v(x)p(x)dx$ (analogue of $\mathbf{v} \cdot \mathbf{p} = \sum_{i=1}^N v_i p_i$)
- Definition: the adjoint of an operator $\mathcal{A}$ is the operator $\mathcal{A}^*$ satisfying

$$< \mathcal{A}v, p >=< v, \mathcal{A}^* p >$$

  Note: adjoint = analogue of matrix transpose $\mathbf{Av} \cdot \mathbf{p} = \mathbf{v} \cdot \mathbf{A}^\top \mathbf{p}$
- Definition: An operator $\mathcal{B}$ is self-adjoint if $\mathcal{B}^* = \mathcal{B}$
- Definition: the infinitesimal generator of a Brownian motion is the operator $\mathcal{A}$ defined by

$$\mathcal{A}v = \mu v_x + \frac{\sigma^2}{2} v_{xx}$$

  with boundary condition $v_x(0) = 0$
  - same operator shows up in HJB equations, e.g.

$$\rho v = u + \mu v_x + \frac{\sigma^2}{2} v_{xx}, \quad u = \text{period return}$$

  - will call it “HJB operator”, plays role of transition matrix

$\mathcal{A}^*$ is adjoint of $\mathcal{A}$ (& vice versa)

- Result: $\mathcal{A}^*$ in the Kolmogorov Forward equation is the adjoint of $\mathcal{A}$
- Proof:

$$
\begin{aligned}
\langle v, \mathcal{A}^* p\rangle &= \int_0^\infty v\left(-\mu p_x + \frac{\sigma^2}{2} p_{xx}\right) dx \\
&= \left[-v\mu p + \frac{\sigma^2}{2} v p_x\right]_0^\infty - \int_0^\infty \left(-\mu v_x p + \frac{\sigma^2}{2} v_x p_x\right) dx \\
&= \left[-v\mu p + \frac{\sigma^2}{2} v p_x - \frac{\sigma^2}{2} v_x p\right]_0^\infty + \int_0^\infty \left(\mu v_x p + \frac{\sigma^2}{2} v_{xx} p\right) dx \\
&= v(0)\left(\mu p(0) - \frac{\sigma^2}{2} p_x(0)\right) + \frac{\sigma^2}{2} v_x(0) p(0) + \langle \mathcal{A} v, p\rangle \\
&= \langle \mathcal{A} v, p\rangle .
\end{aligned}
$$

- key step is to use integration by parts and boundary conditions

Carries over to any diffusion process

- ... with x-dependent μ and σ
- "HJB operator" (infinitesimal generator)

$$\mathcal{A}v = \mu(x)\frac{\partial v}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2 v}{\partial x^2}$$

  with appropriate boundary conditions

- "Kolmogorov Forward operator"

$$\mathcal{A}^* p = -\frac{\partial}{\partial x}(\mu(x)p) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x)p\right)$$

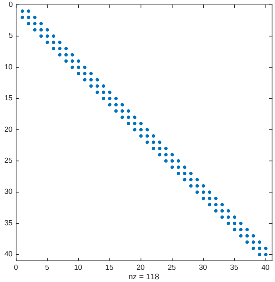
  with appropriate boundary conditions

- Result: $\mathcal{A}^*$ is adjoint of $\mathcal{A}$
- Proof: integration by parts just like previous slide

Computation of Kolmogorov Forward Equations

- That operator in KFE = transpose of transition matrix is very useful for computations
- Use finite difference method $p_i^n = p(x_i, t^n)$
- Key: already know how to discretize $\mathcal{A}$
- recall from Lectures 3 and 4 that discretize HJB equation as

$$\rho v = u + \mu v_x + \frac{\sigma^2}{2} v_{xx} \quad \text{as} \quad \rho \mathbf{v} = \mathbf{u} + \mathbf{A}\mathbf{v}$$



Computation of Kolmogorov Forward Equations

- By same logic: correct discretization of $\mathcal{A}^*$ is $\mathbf{A}^\top$
- Discretize
$$p_t = -\mu p_x + \frac{\sigma^2}{2} p_{xx} \quad \text{or} \quad p_t = \mathcal{A}^* p \tag{KFE}$$
as (explicit scheme)
$$\frac{\mathbf{p}^{n+1} - \mathbf{p}^n}{\Delta t} = \mathbf{A}^\top \mathbf{p}^n$$
or slightly better (implicit scheme)
$$\frac{\mathbf{p}^{n+1} - \mathbf{p}^n}{\Delta t} = \mathbf{A}^\top \mathbf{p}^{n+1} \quad \Rightarrow \quad \mathbf{p}^{n+1} = \left(\mathbf{I} - \Delta t \mathbf{A}^\top\right)^{-1} \mathbf{p}^n$$
- can also obtain these finite-difference schemes directly from (KFE), i.e. without using "operator in KFE = transpose of transition matrix"
  - Section 2 in `https://benjaminmoll.com/HACT_Numerical_Appendix/`
- but if have already computed $\mathbf{A}$ for HJB equation, no need to do discretization again – get (KFE) for free!

Back to the proof of average-speed proposition

- To gain intuition, suppose again finite-state process $\mathbf{p}(t) \in \mathbb{R}^N$ with

$$\dot{\mathbf{p}}(t) = \mathbf{A}^{\mathsf{T}}\mathbf{p}(t)$$

  - assume $\mathbf{A}$ is diagonalizable
  - denote eigenvalues by $0 = |\lambda_1| < |\lambda_2| < \ldots < |\lambda_N|$
  - corresponding eigenvectors by $(\mathbf{v}_1, \ldots, \mathbf{v}_N)$
- Theorem: $\mathbf{p}(t)$ converges to $\mathbf{p}_\infty$ at rate $|\lambda_2|$ ("spectral gap")
- Proof sketch: decomposition

$$\mathbf{p}(0) = \sum_{i=1}^{N} c_i \mathbf{v}_i \quad \Rightarrow \quad \mathbf{p}(t) = \sum_{i=1}^{N} c_i e^{\lambda_i t} \mathbf{v}_i$$

- Example: symmetric two-state Poisson process with intensity ϕ

$$\mathbf{A} = \begin{bmatrix} -\phi & \phi \\ \phi & -\phi \end{bmatrix}, \quad \Rightarrow \quad \lambda_1 = 0, \quad |\lambda_2| = 2\phi$$

Intuitively, speed $|\lambda_2| \nearrow$ in switching intensity ϕ

Proof of proposition (reflecting barrier, $\delta = 0$)

- Generalize this idea to continuous-state process
- Analyze Kolmogorov Forward equation

$$p_t = \mathcal{A}^* p, \quad \mathcal{A}^* p = -\mu p_x + \frac{\sigma^2}{2} p_{xx}$$

  in same exact way as $\dot{\mathbf{p}}(t) = \mathbf{A}^\top \mathbf{p}(t)$

- Proof has two steps:
  1. realization that speed = second eigenvalue (spectral gap) of operator $\mathcal{A}^*$
  2. analytic computation: spectral gap given by

$$|\lambda_2| = \frac{1}{2}\frac{\mu^2}{\sigma^2}$$

Analytic Computation of Spectral Gap

- Discrete eigenvalue problem

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

- Continuous eigenvalue problem

$$\mathcal{A}\varphi = \lambda\varphi$$

or

$$\mu\varphi'(x) + \frac{\sigma^2}{2}\varphi''(x) = \lambda\varphi(x)$$

with boundary condition $\varphi'(0) = 0$.

- In principle, could analyze that one directly, but...

Analytic Computation of Spectral Gap

- Definition: an operator $\mathcal{B}$ is *self-adjoint* if $\mathcal{B}^* = \mathcal{B}$
- Result: all eigenvalues of a self-adjoint operator are real
- want to analyze eigenvalues of $\mathcal{A}$
  - but problem: $\mathcal{A}$ is not self-adjoint
  - eigenvalues could have imaginary parts
- Solution: construct self-adjoint transformation $\mathcal{B}$ of $\mathcal{A}$ as follows
  1. Consider stationary distribution p_∞ satisfying
  $$0 = \mathcal{A}^* p \quad \Rightarrow \quad p_\infty = e^{(2\mu/\sigma^2)x}$$
  2. Consider $u = v p_\infty^{1/2} = v e^{(\mu/\sigma^2)x}$. Can show u satisfies
  $$u_t = \mathcal{B}u, \qquad \mathcal{B}u := \frac{\sigma^2}{2} u_{xx} - \frac{1}{2}\frac{\mu^2}{\sigma^2} u$$
  with boundary condition $u_x(0) = \frac{\mu}{\sigma^2} u(0)$.
- To see that $\mathcal{B}$ is self-adjoint: $< \mathcal{B}u, p > = < u, \mathcal{B}p >$ using same steps as before (integration by parts)

Eigenvalues of $\mathcal{B}$

The first eigenvalue of $\mathcal{B}$ is $\lambda_1 = 0$ and the second eigenvalue is $\lambda_2 = -\frac{1}{2}\frac{\mu^2}{\sigma^2}$. All remaining eigenvalues satisfy $|\lambda| > |\lambda_2|$

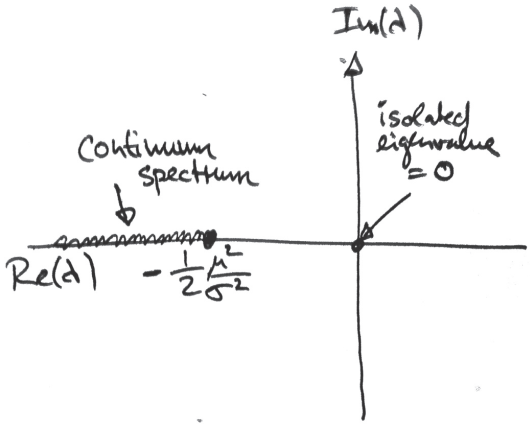


Figure: Spectrum of $\mathcal{B}$ in complex plane

Proof of Lemma

- Consider eigenvalue problem

$$\mathcal{B}\varphi = \lambda\varphi$$

$$\frac{\sigma^2}{2}\varphi''(x) - \frac{1}{2}\frac{\mu^2}{\sigma^2}\varphi(x) = \lambda\varphi(x) \tag{E}$$

  with boundary condition $\varphi'(0) = \frac{\mu}{\sigma^2}\varphi(0)$

- Can show: for $\lambda \in \left(-\frac{1}{2}\frac{\mu^2}{\sigma^2}, 0\right)$ all solutions to (E) satisfying boundary condition explode as $|x| \to \infty$. See appendix of paper.

- Intuition why rate of convergence of $\mathcal{B}$ is $\frac{1}{2}\frac{\mu^2}{\sigma^2}$
  - recall $\mathcal{B}u := \frac{\sigma^2}{2}u_{xx} - \frac{1}{2}\frac{\mu^2}{\sigma^2}u$
  - consider case $\sigma \approx 0$: $\frac{1}{2}\frac{\mu^2}{\sigma^2}$ term large relative to $\frac{\sigma^2}{2}$

$$u_t = \mathcal{B}u \approx -\frac{1}{2}\frac{\mu^2}{\sigma^2}u \quad \Rightarrow \quad u(x,t) \approx u_0(x)e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}t}$$

  i.e. operator $\mathcal{B}$ features exponential decay at rate $\frac{1}{2}\frac{\mu^2}{\sigma^2}$

Transition in Upper Tail

- Distribution $p(x,t)$ satisfies a Kolomogorov Forward Equation

$$p_t = -\mu p_x + \frac{\sigma^2}{2} p_{xx} - \delta p + \delta \psi \qquad (*)$$

- Can solve this, but not particularly instructive
- Instead, use so-called Laplace transform of p

$$\widehat{p}(\xi, t) := \int_{-\infty}^{\infty} e^{-\xi x} p(x,t)\, dx = \mathbb{E}\left[e^{-\xi x}\right]$$

- $\widehat{p}$ has natural interpretation: $-\xi$th moment of income/wealth $w_{it} = e^{x_{it}}$
    - e.g. $\widehat{p}(-2,t) = \mathbb{E}[w_{it}^2]$
- only works in case without reflecting barrier/lower bound

Transition in Upper Tail

- Proposition: The Laplace transform of p, $\widehat{p}$ satisfies

$$\widehat{p}(\xi, t) = \widehat{p}_\infty(\xi) + (\widehat{p}_0(\xi) - \widehat{p}_\infty(\xi))\, e^{-\lambda(\xi)t}$$

with moment-specific speed of convergence

$$\lambda(\xi) = \mu\xi - \frac{\sigma^2}{2}\xi^2 + \delta$$

- Hence, for $\xi < 0$, the higher the moment $-\xi$, the slower the convergence (for high enough $|\xi| < \zeta$)

- Key step: Laplace transform transforms PDE $(*)$ into ODE

$$\frac{\partial \widehat{p}(\xi, t)}{\partial t} = -\xi\mu\widehat{p}(\xi, t) + \xi^2\frac{\sigma^2}{2}\widehat{p}(\xi, t) - \delta\widehat{p}(\xi, t) + \delta\widehat{\psi}(\xi)$$

Can the model explain the fast rise in inequality?

- Recall process for log wages

$$d \log w_{it} = \mu dt + \sigma dZ_{it} \qquad + \text{death at rate } \delta$$

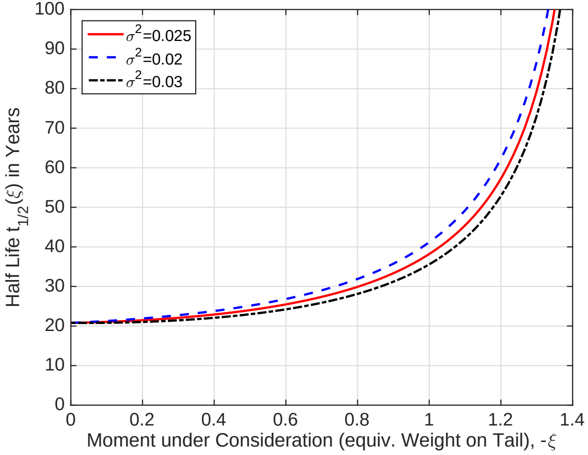
- σ^2 = Var(permanent earnings)

- Literature: σ has increased over last forty years
  - Kopczuk, Saez and Song (2010), DeBacker et al. (2013), Heathcote, Perri and Violante (2010) using PSID
  - but Guvenen, Ozkan and Song (2014): σ flat/decreasing in SSA data

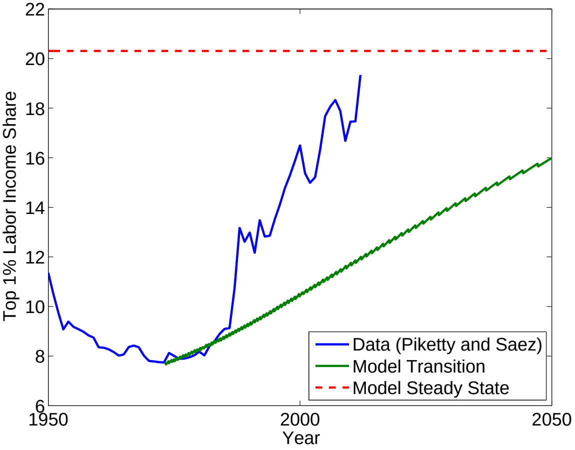
- Can increase in σ explain increase in top income inequality?
  - experiment: $\sigma^2 \uparrow$ from 0.01 in 1973 to 0.025 in 2014 (Heathcote-Perri-Violante)

Putting the Theory to Work

- Recall formula $\lambda(\xi) = \mu\xi - \frac{\sigma^2}{2}\xi^2 + \delta$
- Compute half-life $t_{1/2}(\xi) = \log 2/\lambda(\xi)$



Transition following Increase in σ^2 from 0.01 to 0.025



OK, so what drives top inequality then?

Two candidates:

1. "type dependence": heterogeneity in mean growth rates

2. "scale dependence": "superstar shocks" to skill prices

Both are violations of Gibrat's law

Type Dependence

- Casual evidence: very rapid income growth rates since 1980s (Bill Gates, Mark Zuckerberg)
- Two regimes: H and L with $\mu_H > \mu_L$

$$dx_{it} = \mu_H dt + \sigma_H dZ_{it}$$
$$dx_{it} = \mu_L dt + \sigma_L dZ_{it}$$

- Assumptions
  - drop from H to L at rate ψ
  - retire at rate δ
- See Luttmer (2011) for similar model of firm dynamics
- Proposition: Speed of transition determined by

$$\lambda_H(\xi) := \xi\mu_H - \xi^2 \frac{\sigma_H^2}{2} + \psi + \delta \gg \lambda_L(\xi)$$

Scale Dependence

- Second candidate for fast transitions: $x_{it} = \log w_{it}$ satisfies

$$\begin{aligned} x_{it} &= \chi_t y_{it} \\ dy_{it} &= \mu dt + \sigma dZ_{it} \end{aligned} \qquad (*)$$

  i.e. $w_{it} = (e^{y_{it}})^{\chi_t}$ and χ_t = stochastic process $\neq 1$

- Note: implies deviations from Gibrat's law

$$dx_{it} = \mu dt + x_{it} dS_t + \sigma dZ_{it}, \qquad S_t := \log \chi_t \neq 0$$

- Call χ_t (equiv. S_t) "superstar shocks"

- Proposition: The process $(*)$ has an infinitely fast speed of adjustment: $\lambda = \infty$. Indeed

$$\zeta_t^x = \zeta^y / \chi_t \qquad \text{or} \qquad \eta_t^x = \chi_t \eta^y$$

  where ζ_t^x, ζ^y are the PL exponents of incomes x_{it} and y_{it}.

- Intuition: if power χ_t jumps up, top inequality jumps up

A Microfoundation for "Superstar Shocks"

- χ_t term can be microfounded with changing skill prices in assignment models (Sattinger, 1979; Rosen, 1981)
- Here adopt Gabaix and Landier (2008)
  - continuum of firms of different size $S \sim$ Pareto$(1/\alpha_t)$.
  - continuum of managers with different talent T, distribution

    $$T(n) = T_{\max} - \frac{B}{\beta} n^{\beta_t}$$

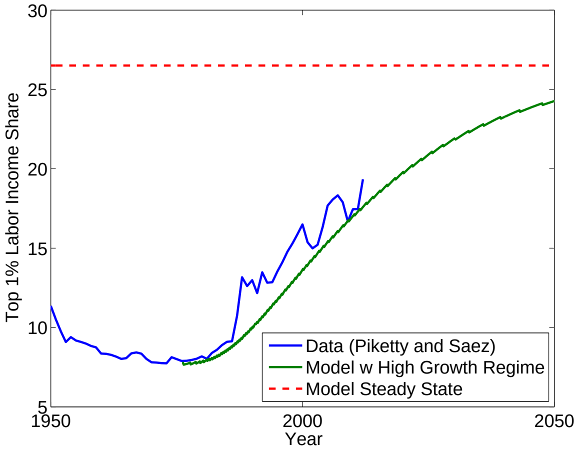
    where n:= rank/quantile of manager talent
  - Match generates firm value: constant $\times T S^{\gamma_t}$
- Can show: $w(n) = e^{a_t} n^{-\chi_t}$ $(= e^{a_t + \chi_t y_{it}}, y_{it} = -\log n_{it})$

  $$\chi_t = \alpha_t \gamma_t - \beta_t$$

- Increase in χ_t due to
  - β_t, γ_t: (perceived) importance of talent in production, e.g. due to ICT (Garicano & Rossi-Hansberg, 2006)
- Other assignment models (e.g. with rent-seeking, inefficiencies)

Revisiting the Rise in Income Inequality

- Jones and Kim (2015): in IRS/SSA data, $\mu_H \uparrow$ since 1970s
- Experiment: in 1973 $\mu_H \uparrow$ by 8%



Conclusion

- Transition dynamics of standard random growth models too slow relative to those observed in the data
- Two parsimonious deviations that generate fast transitions
  1. heterogeneity in mean growth rates
  2. “superstar shocks” to skill prices
- Rise in top income inequality due to
  - ~~simple tax stories, stories about Var(permanent earnings)~~
  - rise in superstar growth (and churn) in two-regime world
  - “superstar shocks” to skill prices
- See paper for wealth inequality results

  `https://benjaminmoll.com/dynamics_wealth/`

Tools Summary

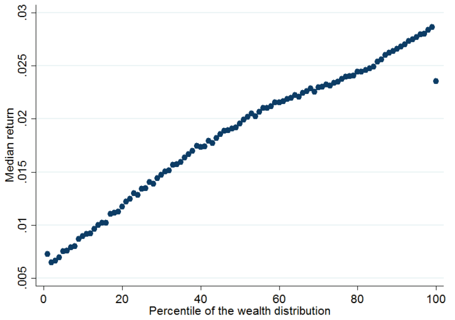
- Differential operators as transition matrices
- At fundamental level, everything same whether discrete/continuous time/space
  - nothing special about continuous t
  - nothing special about continuous x
  - all results from discrete time/space carry over to infinite-dimensional (i.e. continuous) case
  - but computational advantages (e.g. sparsity) – next lecture
- Analogies
  - function $p \Leftrightarrow$ vector $\mathbf{p}$
  - (linear) operator $\mathcal{A} \Leftrightarrow$ matrix $\mathbf{A}$
  - adjoint $\mathcal{A}^* \Leftrightarrow$ transpose $\mathbf{A}^\top$

Open Questions

- "What fraction" of top inequality is efficient in the sense of people getting paid marginal product? What fraction due to rent-seeking?
- What are the underlying economic forces that drove the increase in top inequality?
  - technical change?
  - globalization?
  - superstars?
  - rent-seeking?
  - particular sectors/occupations?
- Evidence for scale- and type-dependence?
  - for wealth: Fagereng, Guiso, Malacrino and Pistaferri (2016), "Heterogeneity and Persistence in Returns to Wealth"
  - what about income?

Fagereng-Guiso-Malacrino-Pistaferri

- Using Norwegian administrative data (Norway has wealth tax), document massive heterogeneity in returns to wealth
  - range of over 500 basis points between 10th and 90th pctile
  - returns positively correlated with wealth



- Interesting open question: can a process for returns to wealth like the one documented by FGMP quantitatively generate fast dynamics in top wealth inequality?