# Lecture 7: Stopping Time Problems
# Firm Size Distribution

ECO 521: Advanced Macroeconomics I

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# Outline

(1) Stopping time problems

(2) Luttmer (2007)

(3) Other related literature

# Stopping Time Problems

- In lots of problems in economics, agents have to choose an optimal **stopping time**.
- Quite often these problems entail some form of **non-convexity**
- Examples:
  - how long should a low productivity firm wait before it exits an industry?
  - how long should a firm wait before it resets its prices?
  - when should you exercise an option?
  - etc... Stokey's book is all about these kind of problems
- These problems are very awkward in discrete time because you run into integer problems.
- **Big payoff** from working in continuous time.

# Stopping Time Problems

- Based on Stokey (2008), chapter 6 "Exercising an Option" (Chapter 5 in draft in Dropbox course folder).
- First deterministic problem
- Then stochastic problem, use HJB approach in 6.3.
- Also see Dixit and Pindyck (1994), chapter 1.G which you may find more intuitive

# Deterministic Problem

- Plant has profits

$$\pi(X(t))$$

- $X(t)$: state variable = stand in for demand, plant capacity etc

$$X(t) = x_0 + \mu t \quad \Leftrightarrow \quad dX(t) = \mu dt$$

- Can shut down plant at any time, get scrap value $S$, but cannot reopen.
- Problem: choose **stopping time** $T$ to solve

$$V(x_0) = \max_{T \geq 0} \left[ \int_0^T e^{-rt} \pi(X(t)) dt + e^{-rT} S \right]$$

- Assumptions to make sure $T^* < \infty$:

$$\pi'(x) > 0, \quad \mu < 0, \quad \lim_{x \to -\infty} \pi(x) < rS < \lim_{x \to +\infty} \pi(x)$$

# Deterministic Problem

- FOC

$$e^{-rT^*}[\pi(X(T^*)) - rS] \leq 0, \quad \text{with equality if } T^* > 0$$

- Can write this in terms of cutoff $b^* = X(T^*)$

$$\pi(b^*) = rS$$

- Optimal stopping time is

$$T^* = \begin{cases} 0, & \text{if } x < b^*, \\ (b^* - x)/\mu, & \text{if } x \geq b^* \end{cases}$$

# Deterministic Problem: HJB Approach

**Claim** (Stokey, Proposition 6.2): The value function, $V$, and optimal threshold, $b^*$, have the following properties:

(i) $V$ satisfies the **HJB equation**

$$
\begin{aligned}
rV(x) &= \pi(x) + V'(x)\mu, && x \geq b^* \\
V(x) &= S, && x \leq b^*
\end{aligned}
$$

(ii) $V$ is continuous at $b^*$ (**value matching**)

$$\lim_{x \downarrow b^*} V(x) = S$$

(iii) $V'$ is continuous at $b^*$ (**smooth pasting**)

$$\lim_{x \downarrow b^*} V'(x) = 0$$

# Intuitive Derivation

- Periods of length $\Delta t$,
- Value of a firm with $x_0 = x$:

$$V(x) = \max\{\tilde{V}(x), S\}$$

- $S$: value of exiting
- $\tilde{V}(x)$: value of staying in industry satisfying

$$\tilde{V}(x) = \pi(x)\Delta t + (1 - r\Delta t)V(x + \mu\Delta t)$$

# Derivation: Value Matching $\lim_{x \downarrow b} V(x) = S$

- Consider some (not necessarily optimal) threshold $b$
- By definition of $b$:

$$V(x) = \begin{cases} \tilde{V}(x), & x > b \\ S, & x \leq b \end{cases}$$

(Note: could write $x \geq b$ and $x < b$, would need to slightly change argument below; just definition of $b$ in any case.)

- Subtract $(1 - r\Delta t)\tilde{V}(x)$ from both sides and divide by $\Delta t$

$$r\tilde{V}(x) = \pi(x) + (1 - r\Delta t)\frac{V(x + \mu\Delta t) - \tilde{V}(x)}{\Delta t}$$

## Derivation: Value Matching $\lim_{x \downarrow b} V(x) = S$

- Evaluate $\tilde{V}$ at $x = b - \mu \Delta t$, i.e. at an $x$ just above the threshold (recall $\mu < 0$).

$$r\tilde{V}(b - \mu\Delta t) = \pi(b - \mu\Delta t) + (1 - r\Delta t)\frac{S - \tilde{V}(b - \mu\Delta t)}{\Delta t}$$

- Want to take $\Delta t \to 0$. Note:

$$\lim_{\Delta t \to 0} \tilde{V}(b - \mu\Delta t) = \lim_{x \downarrow b} \tilde{V}(x)$$

- **Proof** by contradiction. Suppose $\lim_{x \downarrow b} \tilde{V}(x) < S$.
  - Then $\frac{S - \tilde{V}(b - \mu\Delta t)}{\Delta t} \to \infty$ and hence $r\tilde{V}(b - \mu\Delta t) \to \infty$.
  - But $\lim_{x \downarrow b} \tilde{V}(x) = \infty$ contradicts $\lim_{x \downarrow b} \tilde{V}(x) < S$.
- Symmetric argument for $\lim_{x \downarrow b} \tilde{V}(x) > S$
- Since $V(x) = \tilde{V}(x)$ for $x > b$, also $\lim_{x \downarrow b} V(x) = S$
- Note: this has to hold for any threshold $b$, also suboptimal ones. Continuous problems have continuous value functions.

# Derivation: Smooth Pasting $\lim_{x \downarrow b^*} V'(x) = 0$

- Now consider the optimal threshold choice.
- The value of staying, $\tilde{V}$, satisfies the Bellman equation

$$\tilde{V}(x) = \pi(x)\Delta t + (1 - r\Delta t) \max \left\{ \tilde{V}(x + \mu\Delta t), S \right\}$$

where the max is over stay, exit

- Subtract $(1 - r\Delta t)\tilde{V}(x)$ from both sides and divide by $\Delta t$

$$r\tilde{V}(x) = \pi(x) + (1 - r\Delta t) \max \left\{ \frac{\tilde{V}(x + \mu\Delta t) - \tilde{V}(x)}{\Delta t}, \frac{S - \tilde{V}(x)}{\Delta t} \right\}$$

- Evaluate at $x = b - \mu\Delta t$

$$r\tilde{V}(b - \mu\Delta t) = \pi(b - \mu\Delta t) + (1 - r\Delta t) \max \left\{ \frac{\tilde{V}(b) - \tilde{V}(b - \mu\Delta t)}{\Delta t}, \frac{S - \tilde{V}(b - \mu\Delta t)}{\Delta t} \right\}$$

## Derivation: Smooth Pasting $\lim_{x\downarrow b^*} V'(x) = 0$

- Given value matching, $\lim_{x\downarrow b} \tilde{V}(x) = S$, can take $\Delta t \to 0$

$$r\tilde{V}(b) = \pi(b) + \max\left\{\lim_{x\downarrow b} \tilde{V}'(x)\mu, 0\right\}$$

- At the **optimal** $b^*$ need to be indifferent so

$$\lim_{x\downarrow b^*} \tilde{V}'(x) = 0$$

  This is **smooth pasting**.

  - If $\lim_{x\downarrow b} \tilde{V}'(x)\mu > 0$: should wait longer, i.e. decrease $b$
  - If $\lim_{x\downarrow b} \tilde{V}'(x)\mu < 0$: waited too long, i.e. increase $b$

- Since $V(x) = \tilde{V}(x)$ for $x > b^*$, we also have

$$\lim_{x\downarrow b^*} V'(x) = 0$$

- Note: at $b^*$ also obtain the FOC from before

$$rS = \pi(b^*)$$

# Deterministic Problem: Extensions

- Suppose the scrap value is $S(x)$ rather than $S$.
- And further that drift is $\mu(x)$ rather than $\mu$
- Can use the same approach as above to show that
  - **Value Matching**:

$$\lim_{x \downarrow b^*} V(x) = S(b^*)$$

  - **Smooth Pasting**:

$$\lim_{x \downarrow b^*} V'(x) = S'(b^*)$$

# Stochastic Problem

- Assume $X$ is a Brownian motion:

$$dX(t) = \mu dt + \sigma dW(t)$$

- Problem

$$v(x) = \max_{b \leq x} \mathbb{E}_x \left[ \int_0^{T(b)} e^{-rt} \pi(X(t)) dt + e^{-rT(b)} S \right]$$

- Can also attack problem with a direct approach, but big mess (see Stokey, chapter 6.2)
- HJB approach much more convenient and general

# Stochastic Problem: HJB Approach

**Claim** (Stokey, Proposition 6.4): The value function, $V$, and optimal threshold, $b^*$, have the following properties:

(i) $v$ satisfies the **HJB equation**

$$rv(x) = \pi(x) + v'(x)\mu + \frac{1}{2}v''(x)\sigma^2, \qquad x \geq b^*$$
$$v(x) = S, \qquad x \leq b^*$$

(ii) $v$ is continuous at $b^*$ (**value matching**)

$$\lim_{x \downarrow b^*} v(x) = S$$

(iii) $v'$ is continuous at $b^*$ (**smooth pasting**)

$$\lim_{x \downarrow b^*} v'(x) = 0$$

(iv) $v$ has the limiting property (**no bubble**)

$$\lim_{x \to \infty} [v(x) - v_P(x)] = 0, \quad v_P(x) \equiv \mathbb{E}_0 \int_0^\infty e^{-rt}\pi(X(t))dt$$

# Intuitive Derivation

- Use random walk approximation to Brownian motion described in Stokey.
- First derive **HJB equation** above threshold (mainly to get you accustomed to using approximation)
- Then derive **smooth pasting**
- Do not derive **value matching** here because intuitive
- Random walk approximation to Brownian motion:
  - Divide time into discrete periods of length $\Delta t$; start at some $x$; with probability $p$ the process moves up some distance $\Delta x$ and with probability $q = 1 - p$ it moves down.
  - Step size and probabilities

$$\Delta x = \sigma\sqrt{\Delta t}, \quad p = \frac{1}{2}\left[1 + \frac{\mu\sqrt{\Delta t}}{\sigma}\right]$$

# HJB Equation Above Threshold

- Suppose $x$ far enough above threshold that don't want to exit.

$$v(x) = \pi(x)\Delta t + (1 - r\Delta t)\left[pv(x + \sigma\sqrt{\Delta t}) + qv(x - \sigma\sqrt{\Delta t})\right]$$

- Use Taylor series approximations

$$v(x - \sigma\sqrt{\Delta t}) = v(x) - v'(x)\sigma\sqrt{\Delta t} + \frac{1}{2}v''(x)\sigma^2\Delta t + o(\Delta t)$$

and similar for $v(x + \sigma\sqrt{\Delta t})$.

- Subtract $(1 - r\Delta t)v(x)$ from both sides

$$r\Delta t v(x) = \pi(x)\Delta t + (1 - r\Delta t)\left[(2p-1)v'(x)\sigma\sqrt{\Delta t} + \frac{1}{2}v''(x)\sigma^2\Delta t + o(\Delta t)\right]$$

- Divide by $\Delta t$ and use that $(2p-1)/\sqrt{\Delta t} = \mu/\sigma$

$$rv(x) = \pi(x) + (1 - r\Delta t)\left[v'(x)\mu + \frac{1}{2}v''(x)\sigma^2 + \frac{o(\Delta t)}{\Delta t}\right]$$

- Take $\Delta t \to 0$. Done.

# Smooth Pasting

- Now consider $x$ closer to threshold
- Value of exiting: $S$
- Value of staying in industry:

$$\tilde{v}(x) = \pi(x)\Delta t + (1 - r\Delta t) \times \left[ p \max\{\tilde{v}(x + \sigma\sqrt{\Delta t}), S\} + q \max\{\tilde{v}(x - \sigma\sqrt{\Delta t}), S\} \right]$$

where the max is over stay, exit

- Do usual manip., evaluate at $b$ (value matching $\tilde{v}(b) = S$)

$$r\Delta t S = \pi(b)\Delta t + (1 - r\Delta t)[p \max\left\{ \tilde{v}'(x)\sigma\sqrt{\Delta t} + \frac{1}{2}\tilde{v}''(x)\sigma^2\Delta t + o(\Delta t), 0 \right\} + q \max\left\{ -\tilde{v}'(x)\sigma\sqrt{\Delta t} + \frac{1}{2}\tilde{v}''(x)\sigma^2\Delta t + o(\Delta t), 0 \right\}]$$

# Smooth Pasting

- Divide by $\sqrt{\Delta t}$ (not $\Delta t$!)

$$r\sqrt{\Delta t}S = \pi(b)\sqrt{\Delta t} + (1 - r\Delta t)[p \max\left\{\tilde{v}'(x)\sigma + \frac{1}{2}\tilde{v}''(x)\sigma^2\sqrt{\Delta t} + \frac{o(\Delta t)}{\sqrt{\Delta t}}, 0\right\}$$
$$+q \max\left\{-\tilde{v}'(x)\sigma + \frac{1}{2}\tilde{v}''(x)\sigma^2\sqrt{\Delta t} + \frac{o(\Delta t)}{\sqrt{\Delta t}}, 0\right\}]$$

- Take $\sqrt{\Delta t} \to 0$ using $p \to 1/2$

$$0 = \frac{1}{2}\max\left\{\tilde{v}'(b)\sigma, 0\right\} + \frac{1}{2}\max\left\{-\tilde{v}'(b)\sigma, 0\right\}$$

- So need **smooth pasting**

$$\tilde{v}'(b^*) = 0$$

- If $\tilde{v}'(b) > 0$ should **wait and see** rather than exit at $b$

# Intuitive Derivation: Extensions

- Suppose again that scrap value depends on $x$, $S(x)$.
- and that $X$ is any general diffusion

$$dX = \mu(X)dt + \sigma(X)dW$$

- Argument can again be generalized. Simply use discrete approximation with

$$\Delta x = \sigma\sqrt{\Delta t}, \quad p = \frac{1}{2}\left[1 + \frac{\mu(x)\sqrt{\Delta t}}{\sigma(x)}\right]$$

- Generalized conditions:
  - **Value Matching**:

$$\lim_{x \downarrow b^*} v(x) = S(b^*)$$

  - **Smooth Pasting**:

$$\lim_{x \downarrow b^*} v'(x) = S'(b^*)$$

# Particular Examples

- For some functional forms for $\pi(x)$, can solve the ODE in closed form
- Example 1: $\pi(x) = x$
- Example 2: $\pi(x) = ae^{\eta x}$
- Do at end of lecture if I have time.

# Luttmer (2007)

- Firms are monopolistic competitors
- Permanent shocks to preferences and technologies associated with firms
- Low productivity firms exit, new firms imitate and attempt to enter
  - Selection produces Pareto right tail rather than log-normal.
  - Population productivity grows faster than mean of incumbents.
  - Thickness of right tail depends on the difference.
  - Zipf tail when entry costs are high or imitation is difficult.

# Size Distribution

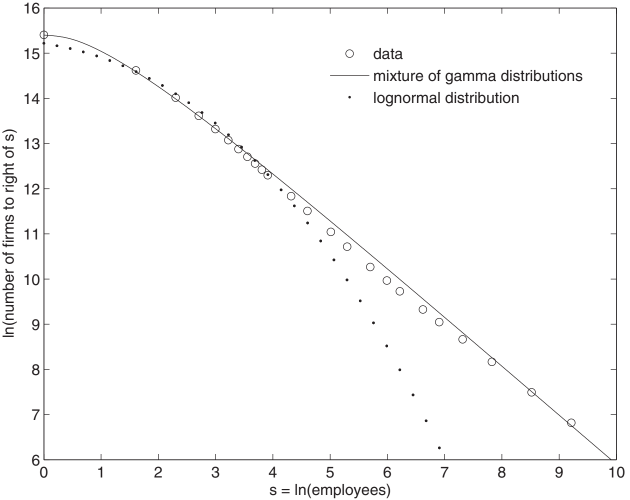



FIGURE I
Size Distribution of U. S. Firms in 2002

# Luttmer (2007)

- Preferences:
  - differentiated commodities with permanent taste shocks
- Technologies:
  - at a cost, entrants draw technologies from some distribution
  - fixed overhead labor, asymptotic constant returns to scale
  - random productivity, quality growth.

# Consumers

- A population $He^{\eta t}$ with preferences over per-capita consumption $C_t e^{-\eta t}$:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \frac{(C_t e^{-\eta t})^{1-\gamma}}{1-\gamma} dt$$

- where

$$C_t = \left[ \int u^{1-\beta} c_t^\beta(u) dM_t(u) \right]^{1/\beta}$$

- Elasticity of substitution is $\sigma = 1/(1-\beta)$
- Demands

$$c_t(u,p) = \left( \frac{p}{P_t} \right)^{-1/(1-\beta)} uC_t$$

where

$$P_t = \left( \int u p^{-\beta/(1-\beta)} dM_t(u) \right)^{-(1-\beta)/\beta}$$

# Firms

- Firms indexed by age $a$ and date of birth $t$.
- Calendar time $= t + a$
- Production function

$$y_{t,a} = z_{t,a} L_{t,a}$$

- Revenues

$$R_{t,a} = C_{t+a}^{1-\beta} (Z_{t,a} L_{t,a})^{\beta}, \quad Z_{t,a} \equiv (u_{t,a}^{1-\beta} z_{t,a}^{\beta})^{1/\beta}$$

- $Z_{t,a}$: combined quality and technology shock

# Firms

- $Z_{t,a}$: combined quality and technology shock ("productivity") evolves according to

$$Z_{t,a} = Z \exp(\theta_E t + \theta_I a + \sigma_Z dW_{t,a})$$

- That is, $Z_{t,a}$ is a geometric Brownian motion

$$\frac{dZ_{t,a}}{Z_{t,a}} = \theta_E dt + \theta_I da + \sigma_Z dW_{t,a}, \quad Z_{0,0} = Z$$

- $\theta_E$: growth of productivity of new firms
- $\theta_I$: growth of productivity of incumbent firms
- $\theta_I - \theta_E$ is key parameter.

# Firms

- Continuation requires $\lambda_F$ units of labor per unit of time.
- Value of a firm:

$$V_t(Z) = \max_{L,\tau} \mathbb{E}_t \int_0^\tau e^{-ra}(R_{t,a} - w_{t+a}[L_{t,a} + \lambda_F])da$$

- $\tau$: stopping time

# Balanced Growth Path

- Will look for equilibria where a bunch of things are growing at a constant growth rate $\kappa$
- Aggregate labor supply: $H_t = He^{\eta t}$
- Number of firms: $M_t = Me^{\eta t}$
- Initial productivity $Z_{t,0} = Ze^{\theta_E t}$
- Total consumption $C_t = Ce^{\kappa t}$. Per capita $C_t e^{-\eta t} = Ce^{(\kappa - \eta)t}$.
- Revenues $R_{t,a} = C_{t+a}^{1-\beta}(Z_{t,a}L_{t,a})^{\beta}$ also grow at $\kappa$.
- Growth rate

$$\kappa = \theta_E + \left(\frac{1-\beta}{\beta}\right)\eta$$

# Production Decisions along BGP

- Firms maximize variable profits $R_{t,a} - w_{t+a}L_{t,a}$. Solution:

$$R_{t,a} - w_{t+a}L_{t,a} = (1-\beta)\left(\frac{\beta Z_{t,a}}{w_{t+a}}\right)^{\beta/(1-\beta)} C_{t+a}$$

- Therefore total profits can be written as

$$R_{t,a} - w_{t+a}L_{t,a} - w_{t+a}\lambda_F = w_{t+a}\lambda_F(e^{s_a} - 1)$$

$$\text{where} \quad s_a \equiv S(Z) + \frac{\beta}{1-\beta}\left[\ln\left(\frac{Z_{t,a}}{Z_{t,0}} - \theta_E a\right)\right]$$

$$\text{and} \quad e^{S(Z)} \equiv \frac{1-\beta}{\lambda_F}\frac{C}{w}\left(\frac{\beta Z}{w}\right)^{\beta/(1-\beta)}$$

- $s_a$: firm size relative to fixed costs. This is a Brownian motion

$$ds_a = \mu da + \sigma dW_{t,a}$$

$$\text{where} \quad \mu \equiv \frac{\beta}{1-\beta}(\theta_I - \theta_E), \quad \sigma = \frac{\beta}{1-\beta}\sigma_Z$$

# Exit Decision: Stopping Time Problem

- Value of a firm is

$$V_t(Z) = w_t \lambda_F V(S(Z))$$

where

$$V(s) = \max_{\tau} \ \mathbb{E}\left[\int_0^{\tau} e^{-(r-\kappa)a}(e^{s_a} - 1)\right]$$

- Stopping time problem $\Rightarrow$ threshold policy: shut down when $s$ falls below $b$.
- For $s > b$, the **HJB equation** holds

$$(r-\kappa)V(s) = e^s - 1 + V'(s)\mu + \frac{1}{2}V''(s)\sigma^2$$

- $b$ determined by **value matching** and **smooth pasting**

$$V(b) = 0, \quad V'(b) = 0$$

# Exit Decision: Stopping Time Problem

- Can show: exit barrier determined by

$$e^b = \left(\frac{\xi}{1+\xi}\right)\left(1 - \frac{\mu + \sigma^2/2}{r-\kappa}\right)$$

$$\text{where} \quad \xi \equiv \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{r-\kappa}{\sigma^2/2}}$$

and the HJB equation has solution

$$V(s) = \frac{1}{r-\kappa}\left(\frac{\xi}{1+\xi}\right)\left(e^{s-b} - 1 - \frac{1-e^{-\xi(s-b)}}{\xi}\right), \quad s \geq b$$

- Faster aggregate productivity growth $\theta_E \uparrow \Rightarrow \mu \propto \theta_I - \theta_E \downarrow$ $\Rightarrow b \uparrow$, i.e. incumbents more likely to exit.

# Entry

- Labor cost of an arrival rate of $\ell_t$ entry opportunities per unit of time:

$$L_{E,t} = \lambda_E \ell_t$$

- An entry opportunity yields a draw $Z$ from a distribution $J$
- Zero profit condition

$$\lambda_E = \lambda_F \int V(S(Z))dJ(Z)$$

- For now: $J$ exogenous

# Kolmogorov Forward Equation

- Density of measure of firms of age $a$ and size $s$ at time $t$

$$f(a, s, t) = m(a, s) I e^{\eta t}$$

- The KFE is

$$\frac{\partial f(a,s,t)}{\partial t} = -\frac{\partial}{\partial a} f(a,s,t) - \frac{\partial}{\partial s}[\mu f(a,s,t)] + \frac{1}{2}\frac{\partial^2}{\partial s^2}[\sigma^2 f(a,s,t)]$$

- Note: unit drift of age $da = dt$
- Substituting in $f(a, s, t) = m(a, s) I e^{\eta t}$ yields

$$\frac{\partial m(a,s)}{\partial a} = -\eta m(a,s) - \frac{\partial}{\partial s}[\mu m(a,s)] + \frac{1}{2}\frac{\partial^2}{\partial s^2}[\sigma^2 m(a,s)]$$

# Boundary Conditions

- Denote size distribution of entering firms by $G(s)$, derived from $J(Z) = G(S(Z))$
- First boundary condition: at age zero

$$\int_b^s m(0,x)dx = G(s) - G(b) \quad \text{all } s > b$$

or more intuitively in terms of the density $g(s) = G'(s)$

$$m(0,s) = g(s), \quad \text{all } s > b$$

- Second boundary condition: at the exit threshold

$$m(a,b) = 0, \quad \text{all } a > 0$$

# Boundary Conditions

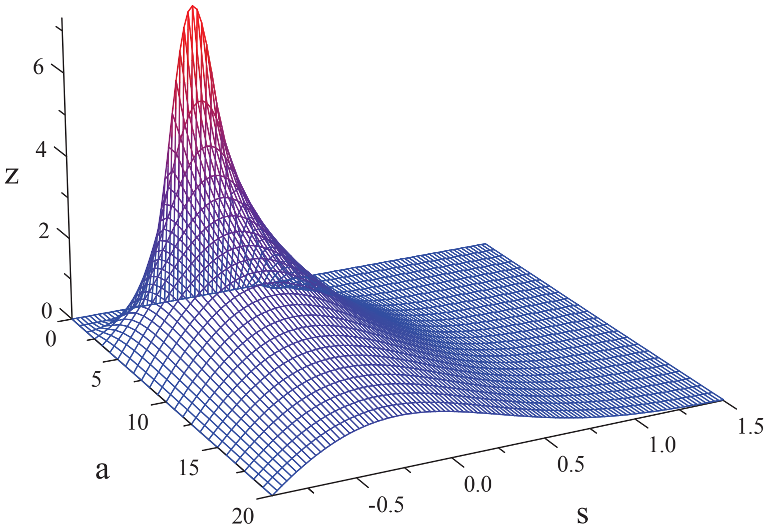
- **Lemma 1** the solution to the KFE subject to the boundary conditions is

$$m(a,s) = \int_b^\infty e^{-\eta a}\psi(a,s|x)dG(x)$$

$$\psi(a,s|x) = \frac{1}{\sigma\sqrt{a}}\left[\phi\left(\frac{s-x-\mu a}{\sigma\sqrt{a}}\right) - e^{-\mu(x-b)/(\sigma^2/2)}\phi\left(\frac{s+x-2b-\mu a}{\sigma\sqrt{a}}\right)\right]$$

- where $\phi$ is the standard normal probability density.
- $\psi(a,s|x)$ is the density of survivors at age a with size $s$ of the cohort that entered with the same initial size $x$ (not a p.d.f.)

# Life of a Cohort: evolution of $m(a, s)$



# Aside: Practical Advice

- Question: how to find solutions for these kinds of ODEs/PDEs?
- Answer: there is a collection of **known solutions** to a big number of ODEs/PDEs. This one apparently from Harrison (1985, p.46)
- if you ever encounter an ODE or PDE that you need to solve, plug into **Mathematica** (function DSolve). Knows **all known solutions**.

# Size Distribution

- Want to obtain size distribution. Almost there.
- Denote by $\pi(a, s|x)$ the probability density of survivors at age $a$ with size $s$ of the cohort that entered with the same initial size $x$ (proportional to $\psi(a, s|x)$)

$$\pi(a, s|x) = \left( \frac{1 - e^{-\alpha_*(x-b)}}{\eta} \right)^{-1} e^{-\eta a} \psi(a, s|x)$$

- Integrate this over all ages, $a$, to get density conditional on initial size

$$\pi(s|x) \propto e^{-\alpha(s-b)} \min \left\{ e^{(\alpha+\alpha_*)(s-b)} - 1, e^{(\alpha+\alpha_*)(x-b)} - 1 \right\}$$

- Density of $e^s$ is our friend the double Pareto distribution. Can write in a better way.
- From fact: if $s$ has an exponential distribution, then $e^s$ has a Pareto distribution.

# Special Case: $\eta = 0$

- when $\eta = 0$, then the tail exponents are $\alpha_* = 0$ and

$$\alpha = -\frac{\mu}{\sigma^2/2} = \frac{\theta_E - \theta_I}{\left(\frac{\beta}{1-\beta}\sigma_Z^2/2\right)}$$

# Stationary Size Distribution

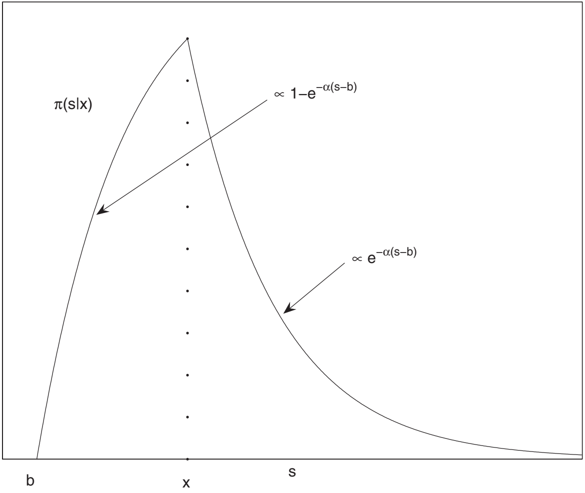



FIGURE II
Size Density Conditional on Initial Size