Lecture 6: Competitive Equilibrium in the Growth Model (II)

ECO 503: Macroeconomic Theory I

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Plan of Lecture

1. Sequence of markets CE
2. The growth model and the data

Sequence of Markets CE

- **Arrow-Debreu CE**
  - period 0: markets for **everything**
- **Sequence of Markets CE**: particular markets at particular points in time

Period 0	Period 1	Period 2	...
market for period 0 capital,	...	...	...
period 0 labor,	...	...	...
period 0 output,	...	...	...
period 0 labor,	...	...	...
1 period ahead borrowing/leding	...	...	...

- Individ. formulates plan at $t = 0$, but executes it in real time
  - in contrast, in ADCE everything happens in period 0
- SOMCE features explicit **borrowing & lending**
  - riskless one-period bond that pays real interest rate r_t

Sequence of Market CE

- **Definition**: A SOMCE for the growth model are sequences $\{c_t, h_t, k_t, a_t, w_t, R_t, r_t\}_{t=0}^{\infty}$ s.t.
  1. (HH max) Taking $\{w_t, R_t, r_t\}$ as given, $\{c_t, h_t, k_t, a_t\}$ solves
  $$\max_{\{c_t, h_t, k_t, a_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$
  $$c_t + k_{t+1} - (1-\delta)k_t + a_{t+1} \leq R_t k_t + w_t h_t + (1 + r_t)a_t$$
  $$c_t \geq 0, \quad 0 \leq h_t \leq 1, \quad k_{t+1} \geq 0, \quad k_0 = \bar{k}_0, \quad a_0 = 0$$
  $$\lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0 \qquad (*)$$
  2. (Firm max) Taking $\{w_t, R_t, r_t\}$ as given, $\{k_t, h_t\}$ solves
  $$\max_{k_t, h_t} F(k_t, h_t) - w_t h_t - R_t k_t \quad k_t \geq 0, \quad h_t \geq 0 \quad \forall t.$$
  3. (Market clearing) For each t:
  $$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, h_t)$$
  $$a_{t+1} = 0 \qquad (**)$$

Comments

- a_t = HH bond holdings
  - $a_t > 0$: HH saves, $a_t < 0$: HH borrows
  - period-t price of bond that pays off at $t+1$: $q_t = 1/(1+r_t)$
  - some people like to write

$$c_t + k_{t+1} - (1-\delta)k_t + q_t b_{t+1} \leq R_t k_t + w_t h_t + b_t$$

  - this is equivalent with $b_t = (1+r_t)a_t$ and $q_t = 1/(1+r_t)$
- Interpretation of bond market clearing condition $(**)$
  - bonds are in **zero net supply**
  - more generally, in economy with individuals $i = 1, ..., N$

$$\sum_{i=1}^{N} a_{i,t+1} = 0$$

  - for every dollar borrowed, someone else saves a dollar
  - here only one type, so $a_{t+1} = 0$.
  - Q: since $a_t = 0$, why not eliminate? A: need to know eq. r_t

Comments

- $(*)$ is a so-called "no-Ponzi condition"
  - with period budget constraints only, individuals could choose time paths with $a_t \to -\infty$
  - no-Ponzi condition $(*)$ rules out such time paths: a_t cannot become too negative
  - implies that sequence of budget constraints can be written as present-value (or time-zero) budget constraint
  - return to this momentarily

- Could have written firm's problem as

$$\max_{\{k_t, h_t\}} \sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \frac{1}{1 + r_s} \right) (F(k_t, h_t) - w_t h_t - R_t k_t) \quad k_t \geq 0, \quad h_t \geq 0$$

but this is a sequence of static problems so can split them up

Sequence BC + no-Ponzi $\Rightarrow$ PVBC

- **Result:** If $\{c_t, i_t, h_t\}$ satisfy the sequence budget constraint

$$c_t + i_t + a_{t+1} = R_t k_t + w_t h_t + (1 + r_t) a_t$$

and if the no-Ponzi condition $(*)$ holds with equality, then $\{c_t, i_t, h_t\}$ satisfy the present value budget constraint

$$\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \frac{1}{1 + r_s} \right) (c_t + i_t) = \sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \frac{1}{1 + r_s} \right) (R_t k_t + w_t h_t)$$

- **Proof:** next slide

Proof

- Write period t budget constraint as

$$\frac{1}{1+r_t} a_{t+1} = \frac{1}{1+r_t} \left(R_t k_t + w_t h_t - c_t - i_t \right) + a_t$$

- At $t = 0, t = 1, ...$

$$\begin{aligned} \frac{1}{1+r_0} a_1 &= \frac{1}{1+r_0} \left(R_0 k_0 + w_0 h_0 - c_0 - i_0 \right) + a_0 \\ \frac{1}{1+r_0} \frac{1}{1+r_1} a_2 &= \frac{1}{1+r_0} \frac{1}{1+r_1} \left(R_1 k_1 + w_1 h_1 - c_1 - i_1 \right) \\ &\quad + \frac{1}{1+r_0} \left(R_0 k_0 + w_0 h_0 - c_0 - i_0 \right) + a_0 \end{aligned}$$

- By induction/repeated substitution

$$\left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = \sum_{t=0}^{T} \left(\prod_{s=0}^{t} \frac{1}{1+r_s} \right) \left(R_t k_t + w_t h_t - i_t - c_t \right)$$

- Result follows from taking $T \to \infty$ and imposing $(*)$

Why no-Ponzi Condition?

- Expression also provides some intuition for no-Ponzi condition

$$\left(\prod_{t=0}^{T} \frac{1}{1+r_t}\right) a_{T+1} = \sum_{t=0}^{T} \left(\prod_{s=0}^{t} \frac{1}{1+r_s}\right) (R_t k_t + w_t h_t - i_t - c_t)$$

- Suppose for the moment this were a finite horizon economy
  - would impose: die without debt, i.e.

$$a_{T+1} \geq 0$$

  - in fact, HH's would always choose $a_{T+1} = 0$
- Right analogue for infinite horizon economy

$$\lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t}\right) a_{T+1} \geq 0$$

and HH's choose $\{a_t\}$ so that this holds with equality

- no-Ponzi condition **not needed** for **physical capital** because natural constraint $k_t \geq 0$.

Characterizing SOMCE

- Necessary conditions for **consumer problem** ($h_t = 1$ wlog)

$$c_t : \quad \beta^t u'(c_t) = \lambda_t = \text{multiplier on period } t \text{ b.c.} \tag{1}$$

$$k_{t+1} : \quad \lambda_t = \lambda_{t+1}(R_{t+1} + 1 - \delta) \tag{2}$$

$$a_{t+1} : \quad \lambda_t = \lambda_{t+1}(1 + r_{t+1}) \tag{3}$$

$$c_t + k_{t+1} - (1-\delta)k_t + a_{t+1} = R_t k_t + w_t h_t + (1 + r_t)a_t \tag{4}$$

$$\text{no-Ponzi: } \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0 \tag{5}$$

$$\text{TVC on } k : \quad \lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0 \tag{6}$$

$$\text{TVC on } a : \quad \lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0 \tag{7}$$

$$\text{initial} : \quad k_0 = \bar{k}_0, \quad a_0 = 0 \tag{8}$$

Characterizing SOMCE

- Necessary conditions for **firm problem**

$$F_k(k_t, h_t) = R_t, \quad F_h(k_t, h_t) = w_t \tag{9}$$

- **Market clearing**

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, h_t), \quad a_{t+1} = 0 \tag{10}$$

Characterizing SOMCE

- (1), (3) and (5)

$$\beta^T u'(c_T) = \lambda_T = \prod_{t=0}^{T} \frac{1}{1+r_t}$$

$$\Rightarrow \lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} \geq 0$$

- No-Ponzi condition looks very similar to TVC on $\{a_t\}$
- But no-Ponzi and TVC are **different conditions**
- Kamihigashi (2008) "A no-Ponzi-game condition is a constraint that prevents overaccumulation of debt, while a typical transversality condition is an optimality condition that rules out overaccumulation of wealth. They place opposite restrictions, and should not be confused."

Characterizing SOMCE

- (2) and (3)

$$1 + r_{t+1} = R_{t+1} + 1 - \delta$$

i.e. rate of return on bonds = rate of return on capital

  - arbitrage condition
  - if this holds, HH is indifferent between a and k

- (1), (2) and (9) $\Rightarrow$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta \tag{11}$$

- (11) + TVC (6) + initial condition (8) + market clearing (10) = same set of equations as for SP problem
- Hence: SOMCE allocation is same as social planner's allocation
  - this is actually somewhat surprising, see next slide

Why is SOMCE allocation =SP's alloc.?

- Relative to ADCE, we closed down many markets
- Q: Why do we still get SP solution even though we closed down many markets?
- A: We only closed down markets that didn't matter
- In fact, ADCE and SOMCE are equivalent

Equivalence of SOMCE and ADCE

- Recall HH's problem in ADCE (last lecture):

$$\max_{\{c_t, h_t, k_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} p_t(R_t k_t + w_t h_t)$$

- Have shown earlier: HH's problem in SOMCE is same with present-value budget constraint

$$\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \frac{1}{1+r_s} \right) (c_t + k_{t+1} - (1-\delta)k_t) = \sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \frac{1}{1+r_s} \right) (R_t k_t + w_t h_t)$$

- Clearly these are equivalent
    - ADCE is SOMCE with $p_t = \prod_{s=0}^{t} \frac{1}{1+r_s}$
    - SOMCE is ADCE with $1 + r_{t+1} = p_t / p_{t+1}$
- Firm's problems are also equivalent.

Why is SOMCE allocation =SP's alloc.?

- riskless one-period bond is surprisingly powerful
- one period ahead borrowing and lending $\Rightarrow$ arbitrary period ahead borrowing and lending
- When is SOMCE allocation with one-period bonds $\neq$ SP's allocation? That is, when do the welfare theorems fail?
  - risk (idiosyncratic or aggregate)
    - welfare theorems may hold if sufficiently rich insurance markets
  - "financial frictions." Examples:
    - interest rate $= r_t(a_t)$ with $r_t' \neq 0$.
    - in more general environments: borrowing constraint $-a_t \leq 0$ or collateral constraints (need to back debt with collateral)

$$-a_{t+1} \leq \theta k_{t+1}$$

  - ...