Lecture 4: Hamilton-Jacobi-Bellman Equations, Stochastic Differential Equations

ECO 521: Advanced Macroeconomics I

Benjamin Moll

Princeton University

Fall 2012

Outline

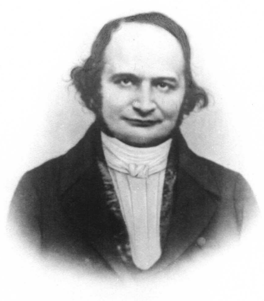
(1) Hamilton-Jacobi-Bellman equations in deterministic settings (with derivation)

(2) Numerical solution: finite difference method

(3) Stochastic differential equations

Hamilton-Jacobi-Bellman Equation: Some "History"

William Hamilton



Carl Jacobi

Richard Bellman

- Aside: why called "dynamic programming"?
- Bellman: *"Try thinking of some combination that will possibly give it a pejorative meaning. It's impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities."* http://www.ingre.unimore.it/or/corsi/vecchi_corsi/complementiro/materialedidattico/originidp.pdf

Hamilton-Jacobi-Bellman Equations

- Recall the generic deterministic optimal control problem from Lecture 1:

$$V(x_0) = \max_{u(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h(x(t), u(t))\, dt$$

subject to the law of motion for the state

$$\dot{x}(t) = g(x(t), u(t)) \text{ and } u(t) \in U$$

for $t \geq 0$, $x(0) = x_0$ given.

- $\rho \geq 0$: discount rate
- $x \in X \subseteq \mathbb{R}^m$: state vector
- $u \in U \subseteq \mathbb{R}^n$: control vector
- $h : X \times U \to \mathbb{R}$: instantaneous return function

Example: Neoclassical Growth Model

$$V(k_0) = \max_{c(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} U(c(t)) dt$$

subject to

$$\dot{k}(t) = F(k(t)) - \delta k(t) - c(t)$$

for $t \geq 0$, $k(0) = k_0$ given.

- Here the state is $x = k$ and the control $u = c$
- $h(x, u) = U(u)$
- $g(x, u) = F(x) - \delta x - u$

Generic HJB Equation

- The value function of the generic optimal control problem satisfies the Hamilton-Jacobi-Bellman equation

$$\rho V(x) = \max_{u \in U} \; h(x,u) + V'(x) \cdot g(x,u)$$

- In the case with more than one state variable $m > 1$, $V'(x) \in \mathbb{R}^m$ is the gradient of the value function.

Example: Neoclassical Growth Model

- "cookbook" implies:

$$\rho V(k) = \max_c \; U(c) + V'(k)[F(k) - \delta k - c]$$

- Proceed by taking first-order conditions etc

$$U'(c) = V'(k)$$

Derivation from Discrete-time Bellman

- Here: derivation for neoclassical growth model.
- Extra class notes: generic derivation.
- Time periods of length Δ
- discount factor

$$\beta(\Delta) = e^{-\rho\Delta}$$

- Note that $\lim_{\Delta \to 0} \beta(\Delta) = 1$ and $\lim_{\Delta \to \infty} \beta(\Delta) = 0$.
- Discrete-time Bellman equation:

$$V(k_t) = \max_{c_t} \; \Delta U(c_t) + e^{-\rho\Delta} V(k_{t+\Delta}) \quad \text{s.t.}$$

$$k_{t+\Delta} = \Delta[F(k_t) - \delta k_t - c_t] + k_t$$

Derivation from Discrete-time Bellman

- For small Δ (will take $\Delta \to 0$), $e^{-\rho\Delta} = 1 - \rho\Delta$

$$V(k_t) = \max_{c_t} \ \Delta U(c_t) + (1 - \rho\Delta) V(k_{t+\Delta})$$

- Subtract $(1 - \rho\Delta)V(k_t)$ from both sides

$$\rho\Delta V(k_t) = \max_{c_t} \ \Delta U(c_t) + (1 - \Delta\rho)[V(k_{t+\Delta}) - V(k_t)]$$

- Divide by Δ and manipulate last term

$$\rho V(k_t) = \max_{c_t} \ U(c_t) + (1 - \Delta\rho) \frac{V(k_{t+\Delta}) - V(k_t)}{k_{t+\Delta} - k_t} \frac{k_{t+\Delta} - k_t}{\Delta}$$

Take $\Delta \to 0$

$$\rho V(k_t) = \max_{c_t} \ U(c_t) + V'(k_t)\dot{k}_t$$

Connection Between HJB Equation and Hamiltonian

- Hamiltonian

$$\mathcal{H}(x, u, \lambda) = h(x, u) + \lambda g(x, u)$$

- Bellman

$$\rho V(x) = \max_{u \in U} \; h(x, u) + V'(x) g(x, u)$$

- Connection: $\lambda(t) = V'(x(t))$, i.e. co-state = shadow value
- Bellman can be written as

$$\rho V(x) = \max_{u \in U} \; \mathcal{H}(x, u, V'(x))$$

- Hence the "Hamilton" in Hamilton-Jacobi-Bellman
- Can show: playing around with FOC and envelope condition gives conditions for optimum from Lecture 1.

Numerical Solution: Finite Difference Method

- Example: Neoclassical Growth Model

$$\rho V(k) = \max_c \; U(c) + V'(k)[F(k) - \delta k - c]$$

- Functional forms

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad F(k) = k^\alpha$$

- **See material at**

  `http://www.princeton.edu/~moll/HACTproject.htm`

  particularly

  - `http://www.princeton.edu/~moll/HACTproject/HACT_Additional_Codes.pdf`
  - Code 1: `http://www.princeton.edu/~moll/HACTproject/HJB_NGM.m`
  - Code 2: `http://www.princeton.edu/~moll/HACTproject/HJB_NGM_implicit.m`

Diffusion Processes

- A diffusion is simply a continuous-time Markov process (with continuous sample paths, i.e. no jumps)
- Simplest possible diffusion: standard Brownian motion (sometimes also called "Wiener process")
- **Definition:** a standard Brownian motion is a stochastic process W which satisfies

$$W(t+\Delta t) - W(t) = \varepsilon_t \sqrt{\Delta t}, \quad \varepsilon_t \sim N(0,1), \quad W(0) = 0$$

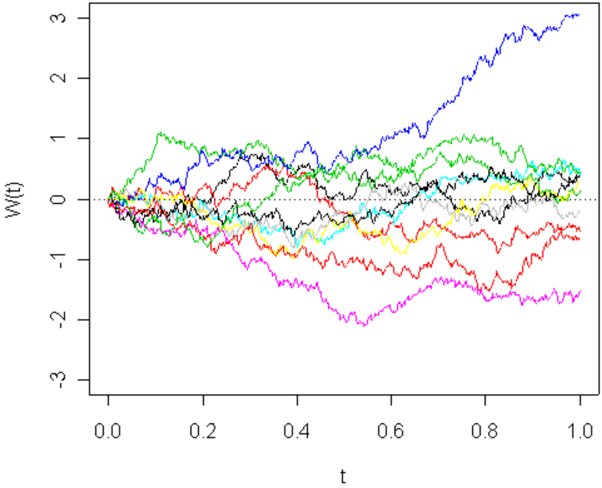
- Not hard to see

$$W(t) \sim N(0,t)$$

- Continuous time analogue of a discrete time random walk:

$$W_{t+1} = W_t + \varepsilon_t, \quad \varepsilon_t \sim N(0,1)$$

Standard Brownian Motion



- Note: mean zero, $\mathbb{E}(W(t)) = 0$...
- ... but blows up $Var(W(t)) = t$.

Brownian Motion

- Can be generalized

$$x(t) = x(0) + \mu t + \sigma W(t)$$

- Since $\mathbb{E}(W(t)) = 0$ and $Var(W(t)) = t$

$$\mathbb{E}[x(t) - x(0)] = \mu t, \quad Var[x(t) - x(0)] = \sigma^2 t$$

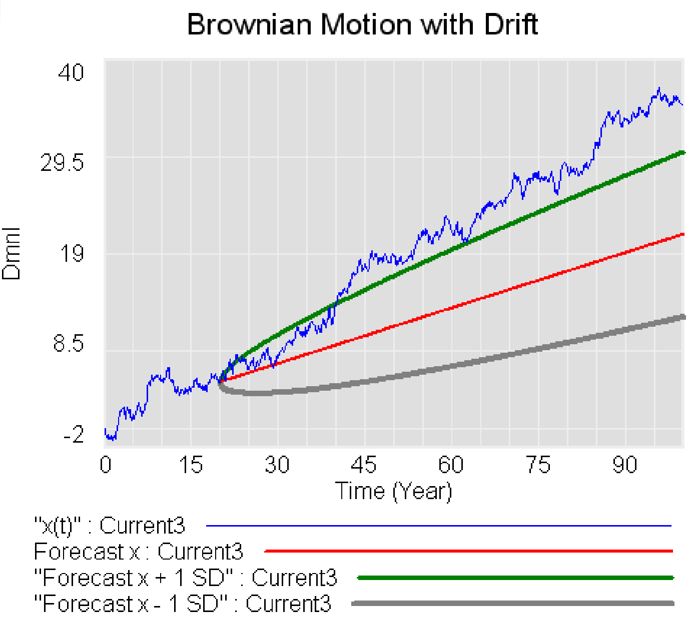
- This is called a Brownian motion with drift μ and variance σ^2
- Can write this in differential form as

$$dx(t) = \mu dt + \sigma dW(t)$$

where $dW(t) \equiv \lim_{\Delta t \to 0} \varepsilon_t \sqrt{\Delta t}$, with $\varepsilon_t \sim N(0, 1)$

- This is called a **stochastic differential equation**
- Analogue of stochastic difference equation:

$$x_{t+1} = \mu t + x_t + \sigma \varepsilon_t, \quad \varepsilon_t \sim N(0, 1)$$



Brownian Motion with Drift
40
29.5
19
8.5
-2
Dmnl
0
15
30
45
60
75
90
Time (Year)
"x(t)" : Current3
Forecast x : Current3
"Forecast x + 1 SD" : Current3
"Forecast x - 1 SD" : Current3

Further Generalizations: Diffusion Processes

- Can be generalized further (suppressing dependence of x and W on t)

  $$dx = \mu(x)dt + \sigma(x)dW$$

  where μ and σ are any non-linear etc etc functions.
- This is called a "diffusion process"
- $\mu(\cdot)$ is called the drift and $\sigma(\cdot)$ the diffusion.
- all results can be extended to the case where they depend on t, $\mu(x,t), \sigma(x,t)$ but abstract from this for now.
- The amazing thing about diffusion processes: **by choosing functions μ and σ, you can get pretty much any stochastic process you want** (except jumps)

Example 1: Ornstein-Uhlenbeck Process

- Brownian motion $dx = \mu dt + \sigma dW$ is not stationary (random walk). But the following process is

$$dx = \theta(\bar{x} - x)dt + \sigma dW$$

- Analogue of AR(1) process, autocorrelation $e^{-\theta} \approx 1 - \theta$

$$x_{t+1} = \theta\bar{x} + (1 - \theta)x_t + \sigma\varepsilon_t$$

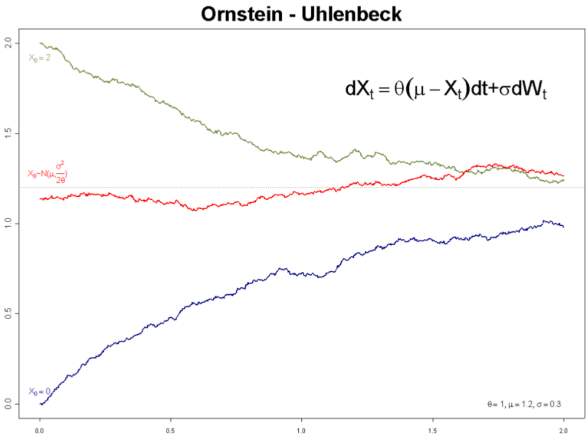
- That is, we just choose

$$\mu(x) = \theta(\bar{x} - x)$$

  and we get a nice stationary process!

- This is called an "Ornstein-Uhlenbeck process"

Ornstein-Uhlenbeck Process



- Can show: stationary distribution is $N(\bar{x}, \sigma^2/(2\theta))$

Example 2: “Moll Process”

- Design a process that stays in the interval $[0, 1]$ and mean-reverts around $1/2$

$$\mu(x) = \theta\left(1/2 - x\right), \quad \sigma(x) = \sigma x(1 - x)$$

$$dx = \theta\left(1/2 - x\right) dt + \sigma x(1 - x) dW$$

- Note: diffusion goes to zero at boundaries $\sigma(0) = \sigma(1) = 0$ & mean-reverts $\Rightarrow$ always stay in $[0, 1]$

Other Examples

- Geometric Brownian motion:

$$dx = \mu x dt + \sigma x dW$$

$x \in [0, \infty)$, no stationary distribution:

$$\log x(t) \sim N((\mu - \sigma^2/2)t, \sigma^2 t).$$

- Feller square root process (finance: "Cox-Ingersoll-Ross")

$$dx = \theta(\bar{x} - x)dt + \sigma\sqrt{x}dW$$

$x \in [0, \infty)$, stationary distribution is $Gamma(\gamma, 1/\beta)$, i.e.

$$f_\infty(x) \propto e^{-\beta x} x^{\gamma - 1}, \quad \beta = 2\theta\bar{x}/\sigma^2, \quad \gamma = 2\theta\bar{x}/\sigma^2$$

- Other processes in Wong (1964), "The Construction of a Class of Stationary Markoff Processes."

Next Time

(1) Hamilton-Jacobi-Bellman equations in stochastic settings (without derivation)

(2) Ito's Lemma

(3) Kolmogorov Forward Equations

(4) Application: Power laws (Gabaix, 2009)