Lecture 3: Growth Model, Dynamic Optimization in Continuous Time (Hamiltonians)

ECO 503: Macroeconomic Theory I

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Plan of Lecture

Growth model in continuous time

- Hamiltonians: system of differential equations
- Phase diagrams
- Finite difference methods and shooting algorithm

Growth Model in Continous Time

- **Preferences:** representative household with utility function

$$\int_0^\infty e^{-\rho t} u(c(t)) dt$$

$\rho \geq 0 =$ discount *rate* (as opposed to $\beta =$ discount *factor*)

- **Technology:**

$$y(t) = f(k(t)), \quad c(t) + i(t) = y(t)$$
$$\dot{k}(t) = i(t) - \delta k(t), \quad c(t) \geq 0, \quad k(t) \geq 0$$

- **Endowments:** $\hat{k}_0$ of capital at $t = 0$

- Pareto optimal allocation solves

$$V(\hat{k}_0) = \max_{c(t)_{t=0}^\infty} \int_0^\infty e^{-\rho t} u(c(t)) dt \quad \text{s.t.}$$
$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t), \quad k(0) = \hat{k}_0$$

Hamiltonians

- Pretty much all deterministic optimal control problems in continuous time can be written as

$$V(\hat{x}_0) = \max_{z(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h(x(t), z(t))\, dt$$

subject to the law of motion for the state

$$\dot{x}(t) = g(x(t), z(t)) \text{ and } z(t) \in Z$$

for $t \geq 0$, $x(0) = \hat{x}_0$ given.

- $\rho \geq 0$: discount rate
- $x \in X \subseteq \mathbb{R}^m$: state vector
- $z \in Z \subseteq \mathbb{R}^k$: control vector
- $h : X \times Z \to \mathbb{R}$: instantaneous return function

Example: Growth Model

$$V\left(\hat{k}_0\right) = \max_{c(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u(c(t))dt \quad \text{s.t.}$$

$$\dot{k}\left(t\right) = f(k(t)) - \delta k(t) - c(t), \quad k(0) = \hat{k}_0$$

- Here the state is $x = k$ and the control $z = c$
- $h(x, z) = u(z)$
- $g(x, z) = f(x) - \delta x - z$

Hamiltonian: General Formulation

- Consider the general optimal control problem two slides back.
- Can obtain necessary and sufficient conditions for an optimum using the following procedure ("cookbook")
- Current-value Hamiltonian

$$\mathcal{H}(x, z, \lambda) = h(x, z) + \lambda g(x, z).$$

- $\lambda \in \mathbb{R}^m$: "co-state"

Hamiltonian: General Formulation

- Necessary and sufficient conditions:

$$H_z\left(x\left(t\right),z\left(t\right),\lambda\left(t\right)\right)=0$$
$$\dot{\lambda}\left(t\right)=\rho\lambda\left(t\right)-H_x\left(x\left(t\right),z\left(t\right),\lambda\left(t\right)\right)$$
$$\dot{x}\left(t\right)=g\left(x\left(t\right),z\left(t\right)\right)$$

  for all $t\geq 0$.

- Initial value for state variable(s): $x(0)=\hat{x}_0$.
- Boundary condition for co-state variable(s) $\lambda\left(t\right)$, called "Transversality condition"

$$\lim_{T\to\infty} e^{-\rho T}\lambda\left(T\right)x\left(T\right)=0.$$

- Note: initial value of the co-state variable $\lambda\left(0\right)$ not predetermined.

Example: Neoclassical Growth Model

- Recall: $h(x, z) = u(z)$ and $g(x, z) = f(x) - \delta x - z$
- Using the "cookbook"

$$\mathcal{H}(k, c, \lambda) = u(c) + \lambda[f(k) - \delta k - c]$$

- We have

$$\mathcal{H}_c(k, c, \lambda) = u'(c) - \lambda$$

$$\mathcal{H}_k(k, c, \lambda) = \lambda(f'(k) - \delta)$$

- Therefore conditions for optimum are:

$$\dot{\lambda} = \lambda(\rho + \delta - f'(k))$$

$$\dot{k} = f(k) - \delta k - c \qquad \text{(ODE)}$$

$$u'(c) = \lambda$$

with $k(0) = \hat{k}_0$ and $\lim_{T\to\infty} e^{-\rho T}\lambda(T)k(T) = 0$.

Example: Neoclassical Growth Model

- Interpretation: continuous time Euler equation
- In discrete time

$$
\begin{aligned}
\lambda_t &= \beta\lambda_{t+1}(f'(k_{t+1}) + 1 - \delta) \\
k_{t+1} &= f(k_t) + (1-\delta)k_t - c_t \\
u'(c_t) &= \lambda_t
\end{aligned}
$$

- (ODE) is continous-time analogue

Phase Diagrams

- How analyze (ODE)? In one-dimensional case (scalar x): use phase-diagram
- Two possible phase-diagrams:
  (i) in (λ, k)-space: more general strategy.
  (ii) in (c, k)-space: nicer in terms of the economics.
- For (i), use $u'(c) = \lambda$ or $c = (u')^{-1}(\lambda)$ to write (ODE) as

$$\begin{aligned} \dot{\lambda} &= \lambda(\rho + \delta - f'(k)) \\ \dot{k} &= f(k) - \delta k - (u')^{-1}(\lambda) \end{aligned} \qquad \text{(ODE')}$$

  with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T}\lambda(T)k(T) = 0$.
- Exercise: draw phase-diagram in (λ, k)-space.

Phase Diagrams

- For (ii), assume CRRA utility

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

- Not necessary but makes algebra easier.

$$c^{-\sigma} = \lambda \quad \Rightarrow \quad -\sigma \log c(t) = \log \lambda(t) \quad \Rightarrow \quad -\sigma \frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda}$$

- Therefore write (ODE) as

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\sigma}(f'(k) - \rho - \delta) \\ \dot{k} &= f(k) - \delta k - c \end{aligned} \qquad \text{(ODE")}$$

with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T} c(T)^{-\sigma} k(T) = 0$.

Steady State

- In steady state $\dot{k} = \dot{c} = 0$. Therefore

$$f'(k^*) = \rho + \delta$$

$$c^* = f(k^*) - \delta k^*$$

- Same as in discrete time with $\beta = 1/(1+\rho)$.
- For example, if $f(k) = Ak^\alpha, \alpha < 1$. Then

$$k^* = \left(\frac{\alpha A}{\rho + \delta}\right)^{\frac{1}{1-\alpha}}$$

Phase Diagram

- See graph that I drew in lecture by hand or Figure 8.1 in Acemoglu's textbook.
- Obtain saddle path.
- Prove stability of steady state.
- Important: saddle path is **not** a "knife edge" case in the sense that the system only converges to steady state if $(c(0), k(0))$ happens to lie on the saddle path and diverges for all other initial conditions.
- In contrast to the state variable $k(t)$, $c(t)$ is a "jump variable." That is, $c(0)$ is free and **always** adjusts so as to lie on the saddle path.

Violations of Transversality Condition

- **Question:** how do you know that trajectories with $c(0)$ off the saddle path violate the transversality condition?
- See Acemoglu, chapter 8 "The Neoclassical Growth Model" section 5 "Transitional Dynamics"
  - if $c(0)$ below saddle path, $k(t) \to k_{\text{max}}$ and $c(t) \to 0$
  - if $c(0)$ above saddle path, $k(t) \to 0$ in finite time while $c(t) > 0$. Violates feasibility.
  - local analysis/linearization gives same answer. Next lecture.
  - notes that most rigorous and straightforward way is to use that concave problems have unique solution (his Theorem 7.14)

Numerical Solution: Finite-Diff. Methods

- By far the simplest and most transparent method for numerically solving differential equations.
- Approximate $k(t)$ and $c(t)$ at N discrete points in the time dimension, $t^n, n = 1, ..., N$. Denote distance between grid points by Δt.
- Use short-hand notation $k^n = k(t^n)$.
- Approximate derivatives

$$\dot{k}(t^n) \approx \frac{k^{n+1} - k^n}{\Delta t}$$

- Approximate (ODE") as

$$\frac{c^{n+1} - c^n}{\Delta t} \frac{1}{c^n} = \frac{1}{\sigma}(f'(k^n) - \rho - \delta)$$
$$\frac{k^{n+1} - k^n}{\Delta t} = f(k^n) - \delta k^n - c^n$$

Finite-Diff. Methods/Shooting Algorithm

- Or

$$\begin{aligned} c^{n+1} &= \Delta t c^n \frac{1}{\sigma}(f'(k^n) - \rho - \delta) + c^n \\ k^{n+1} &= \Delta t(f(k^n) - \delta k^n - c^n) + k^n \end{aligned} \qquad \text{(FD)}$$

  with $k^0 = k_0$ given.

- Exercise: draw phase diagram/saddle path in MATLAB.
- Assume $f(k) = Ak^\alpha$, $A = 1$, $\alpha = 0.3$, $\sigma = 2$, $\rho = \delta = 0.05$, $k_0 = \frac{1}{2}k^*$, $\Delta t = 0.1$, $N = 700$.
- Algorithm:
  - (i) guess c^0
  - (ii) obtain $(c^n, k^n), n = 1, ..., N$ by running (FD) forward in time.
  - (iii) If the sequence converges to (c^*, k^*), then you have obtained the correct saddle path. If not, back to (i) and try different c^0.
- This is called a "shooting algorithm"