# Heterogeneous Agent Models in Continuous Time
# Part II

Benjamin Moll
Princeton

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# Outline

## Lecture 1

1. Refresher: HJB equations
2. Textbook heterogeneous agent model
3. Numerical solution of HJB equations
4. Models with non-convexities (Skiba)

## Lecture 2

1. Analysis and numerical solution of heterogeneous agent model
2. Transition dynamics/MIT shocks
3. Stopping time problems
4. Models with multiple assets (HANK)

## "When Inequality Matters for Macro and Macro Matters for Inequality"

1. Aggregate shocks via perturbation (Reiter)
2. Application to consumption dynamics

# Analysis and Numerical Solution of Heterogeneous Agent Model

# Recall Textbook Heterogeneous Agent Model

$$\rho v_j(a) = \max_c \; u(c) + v_j'(a)(y_j + ra - c) + \lambda_j(v_{-j}(a) - v_j(a)) \qquad \text{(HJB)}$$

$$0 = -\frac{d}{da}[s_j(a)g_j(a)] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a), \qquad \text{(KF)}$$

$$s_j(a) = y_j + ra - c_j(a) = \text{saving policy function from (HJB)},$$

$$\int_{\underline{a}}^{\infty} (g_1(a) + g_2(a))da = 1, \quad g_1, g_2 \geq 0$$

$$S(r) := \int_{\underline{a}}^{\infty} ag_1(a)da + \int_{\underline{a}}^{\infty} ag_2(a)da = B, \qquad B \geq 0 \qquad \text{(EQ)}$$

- The two PDEs (HJB) and (KF) together with (EQ) fully characterize stationary equilibrium ▸ Derivation of (HJB) ▸ (KF)

# Borrowing Constraints?

- Q: where is borrowing constraint $a \geq \underline{a}$ in (HJB)?
- A: "in" boundary condition
- Result: $v_j$ must satisfy

$$v_j'(\underline{a}) \geq u'(y_j + r\underline{a}), \quad j = 1, 2 \tag{BC}$$

- Derivation:
  - the FOC still holds at the borrowing constraint

$$u'(c_j(\underline{a})) = v_j'(\underline{a}) \tag{FOC}$$

  - for borrowing constraint not to be violated, need

$$s_j(\underline{a}) = y_j + r\underline{a} - c_j(\underline{a}) \geq 0 \tag{$*$}$$

  - (FOC) and ($*$) $\Rightarrow$ (BC).

- See slides on viscosity solutions for more rigorous discussion

`http://www.princeton.edu/~moll/viscosity_slides.pdf`

# Plan

- New theoretical results:

   1. analytics: consumption, saving, MPCs of the poor

   2. closed-form for wealth distribution with 2 income types

   3. unique stationary equilibrium if IES $\geq 1$ (sufficient condition)

   Note: for 1. and 2. analyze partial equilibrium with $r < \rho$

- Computational algorithm:

   - problems with non-convexities

   - transition dynamics

# Result 1: Consumption, Saving Behavior of the Poor

Behavior near borrowing constraint depends on two factors

1. tightness of constraint
2. properties of $u$ as $c \to 0$

Assumption 1:
*As $a \to \underline{a}$, coefficient of absolute risk aversion $R(c) = -u''(c)/u'(c)$ remains finite*

$$\underline{R} := -\lim_{a \to \underline{a}} \frac{u''(y_1 + ra)}{u'(y_1 + ra)} < \infty$$

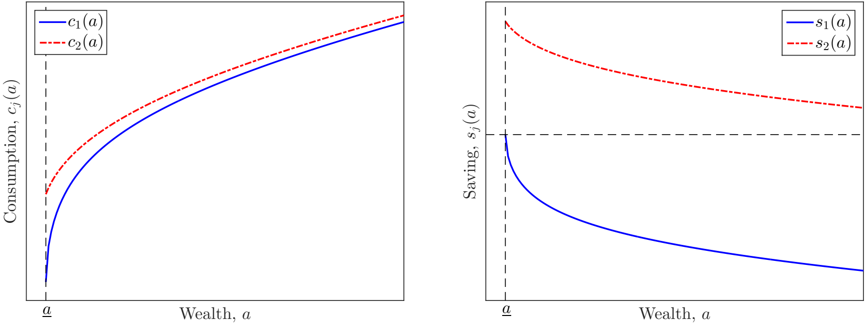
- sufficient condition for A1: borrowing constraint is tighter than "natural borrowing constraint" $\underline{a} > -y_1/r$
- e.g. with CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \Rightarrow \quad \underline{R} = \frac{\gamma}{y_1 + r\underline{a}}$$

- but weaker: e.g. A1 satisfied with $\underline{a} = -y_1/r$ and $u(c) = -e^{-\theta c}/\theta$

# Result 1: Consumption, Saving Behavior of the Poor

Rough version of Proposition: under A1 policy functions look like this



# Result 1: Consumption, Saving Behavior of the Poor

**Proposition:** Assume $r < \rho$, $y_1 < y_2$ and that A1 holds. The solution to (HJB) has following properties:

1. $s_1(\underline{a}) = 0$ but $s_1(a) < 0$ all $a > \underline{a}$: only households exactly at the borrowing constraint are constrained
2. Saving and consumption policy functions close to $a = \underline{a}$ satisfy

$$s_1(a) \sim -\sqrt{2\nu_1}\sqrt{a - \underline{a}}$$

$$c_1(a) \sim y_1 + ra + \sqrt{2\nu_1}\sqrt{a - \underline{a}}$$

$$c_1'(a) \sim r + \frac{1}{2}\sqrt{\frac{\nu_1}{2(a - \underline{a})}}$$

$$\nu_1 = \frac{(\rho - r)u'(\underline{c}_1) + \lambda_1(u'(\underline{c}_1) - u'(\underline{c}_2))}{-u''(\underline{c}_1)}$$

Note: "$f(a) \sim g(a)$" means $\lim_{a \to \underline{a}} f(a)/g(a) = 1$, "$f$ behaves like $g$ close to $\underline{a}$"

# Result 1: Consumption, Saving Behavior of the Poor

**Corollary:** The wealth of worker who keeps $y_1$ converges to borrowing constraint in finite time at speed governed by $\nu_1$:

$$a(t) - \underline{a} \sim \frac{\nu_1}{2}(T - t)^2, \quad 0 \leq t \leq T, \quad \text{where}$$

$$T := \sqrt{\frac{2(a_0 - \underline{a})}{\nu_1}} = \text{"hitting time"}$$

Proof: integrate $\dot{a}(t) = -\sqrt{2\nu_1}\sqrt{a(t) - \underline{a}}$

And have analytic solution for speed

$$\nu_1 = \frac{(\rho - r)u'(\underline{c}_1) + \lambda_1(u'(\underline{c}_1) - u'(\underline{c}_2))}{-u''(\underline{c}_1)}$$

$$\approx (\rho - r)\text{IES}(\underline{c}_1)\underline{c}_1 + \lambda_1(\underline{c}_2 - \underline{c}_1)$$

# Result 2: Stationary Wealth Distribution

- Recall equation for stationary distribution

$$0 = -\frac{d}{da}[s_j(a)g_j(a)] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a) \qquad \text{(KF)}$$

- **Lemma:** the solution to (KF) is

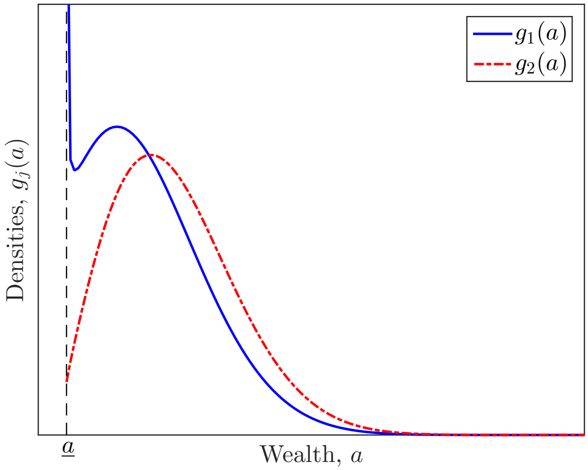
$$g_i(a) = \frac{\kappa_j}{s_j(a)} \exp\left(-\int_{\underline{a}}^{a} \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)} dx\right)\right)$$

  with $\kappa_1, \kappa_2$ pinned down by $g_j$'s integrating to one

- Features of wealth distribution:
  - Dirac point mass of type $y_1$ individuals at constraint $G_1(\underline{a}) > 0$
  - thin right tail: $g(a) \sim \xi(a_{\max} - a)^{\lambda_2/\zeta_2 - 1}$, i.e. not Pareto
  - see paper for more

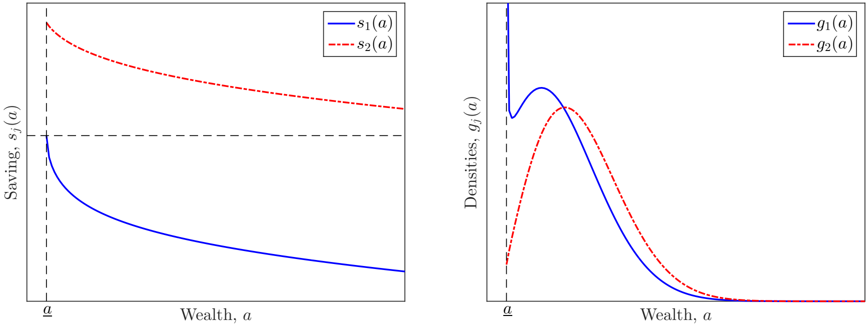
- Later in paper: extension with Pareto tail (Benhabib-Bisin-Zhu)

# Result 2: Stationary Wealth Distribution

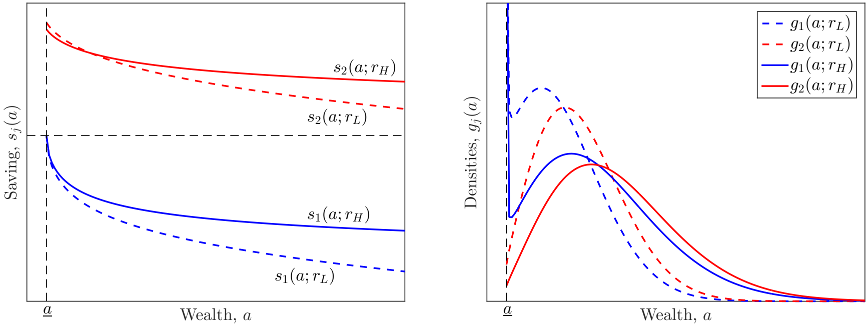



Note: in numerical solution, Dirac mass = finite spike in density

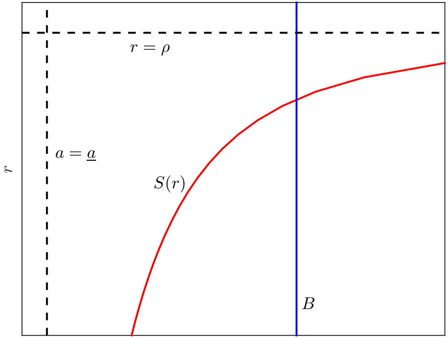
# General Equilibrium: Existence and Uniqueness



# Increase in $r$ from $r_L$ to $r_H > r_L$



# Stationary Equilibrium




Asset Supply $\quad S(r) = \int_{\underline{a}}^{\infty} a g_1(a; r) da + \int_{\underline{a}}^{\infty} a g_2(a; r) da$

- **Proposition:** a stationary equilibrium exists
- **Proposition:** if IES$(c) \geq 1$ for all $c$ and no borrowing $a \geq 0$, stationary equilibrium is unique

# Computations for Heterogeneous Agent Model

# Computations for Heterogeneous Agent Model

- Hard part: HJB equation. But already know how to do that.
- Easy part: KF equation. Once you solved HJB equation, get KF equation "for free"
- System to be solved

$$\rho v_1(a) = \max_c \; u(c) + v_1'(a)(y_1 + ra - c) + \lambda_1(v_2(a) - v_1(a))$$

$$\rho v_2(a) = \max_c \; u(c) + v_2'(a)(y_2 + ra - c) + \lambda_2(v_1(a) - v_2(a))$$

$$0 = -\frac{d}{da}[s_1(a)g_1(a)] - \lambda_1 g_1(a) + \lambda_2 g_2(a)$$

$$0 = -\frac{d}{da}[s_2(a)g_2(a)] - \lambda_2 g_2(a) + \lambda_1 g_1(a)$$

$$1 = \int_{\underline{a}}^{\infty} g_1(a)da + \int_{\underline{a}}^{\infty} g_2(a)da$$

$$0 = \int_{\underline{a}}^{\infty} ag_1(a)da + \int_{\underline{a}}^{\infty} ag_2(a)da \equiv S(r)$$

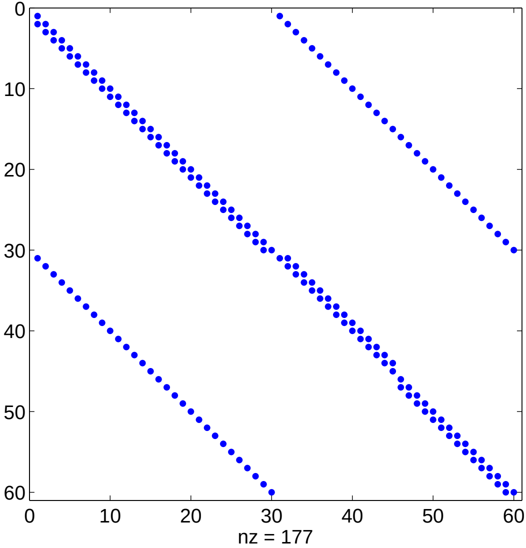
# Computations for Heterogeneous Agent Model

- As before, discretized HJB equation is

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v})\mathbf{v} \tag{HJBd}$$

- $\mathbf{A}$ is $N \times N$ transition matrix
  - here $N = 2 \times I$, $I$=number of wealth grid points
  - $\mathbf{A}$ depends on $\mathbf{v}$ (nonlinear problem)
  - solve using implicit scheme

# Visualization of **A** (output of `spy(A)` in Matlab)



# Computing the FK Equation

- Equations to be solved

$$0 = -\frac{d}{da}[s_1(a)g_1(a)] - \lambda_1 g_1(a) + \lambda_2 g_2(a)$$
$$0 = -\frac{d}{da}[s_2(a)g_2(a)] - \lambda_2 g_2(a) + \lambda_1 g_1(a)$$

  with $1 = \int_{\underline{a}}^{\infty} g_1(a)da + \int_{\underline{a}}^{\infty} g_2(a)da$
- Actually, super easy: discretized version is simply

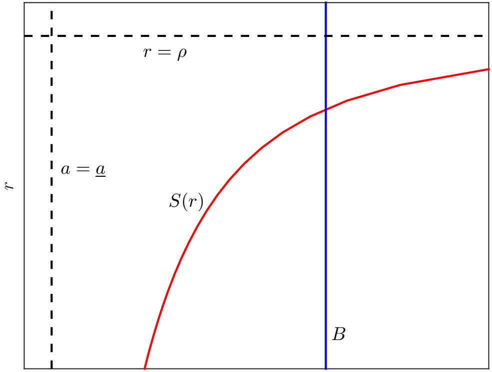
$$0 = \mathbf{A}(\mathbf{v})^{\mathrm{T}}\mathbf{g} \qquad \text{(KFd)}$$

  - eigenvalue problem
  - get KF for free, one more reason for using implicit scheme
- Why transpose?
  - operator in (HJB) is "adjoint" of operator in (KF)
  - "adjoint" = infinite-dimensional analogue of matrix transpose
- In principle, can use similar strategy in discrete time

# Finding the Equilibrium Interest Rate

Use bisection method

- increase $r$ whenever $S(r) < B$
- decrease $r$ whenever $S(r) > B$



# A Model with a Continuum of Income Types

- Assume idiosyncratic income follows diffusion process

$$dy_t = \mu(y_t)dt + \sigma(y_t)dW_t$$

- Reflecting barriers at $\underline{y}$ and $\bar{y}$

$$\rho v(a,y) = \max_c \; u(c) + \partial_a v(a,y)(y + ra - c) + \mu(y)\partial_y v(a,y) + \frac{\sigma^2(y)}{2}\partial_{yy}v(a,y)$$

$$0 = -\partial_a[s(a,y)g(a,y)] - \partial_y[\mu(y)g(a,y)] + \frac{1}{2}\partial_{yy}[\sigma^2(y)g(a,y)]$$

$$1 = \int_0^\infty \int_{\underline{a}}^\infty g(a,y)dady$$

$$0 = \int_0^\infty \int_{\underline{a}}^\infty ag(a,y)dady =: S(r)$$

- Borrowing constraint: $\partial_a v(\underline{a},y) \geq u'(y + r\underline{a})$, all $y$
- reflecting barriers (see e.g. Dixit “Art of Smooth Pasting”)

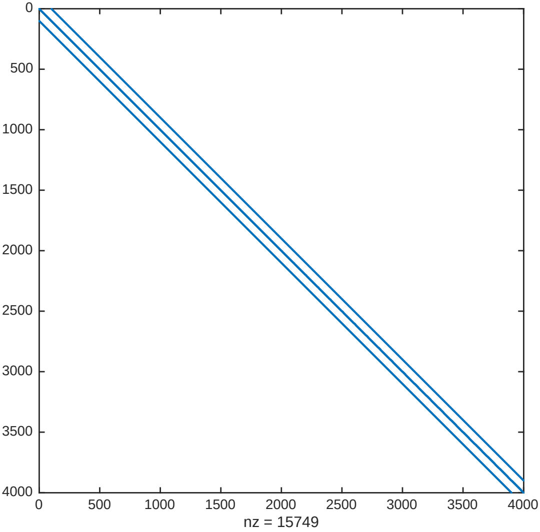
$$0 = \partial_y v(a,\underline{y}) = \partial_y v(a,\bar{y})$$

It doesn't matter whether you solve ODEs or PDEs

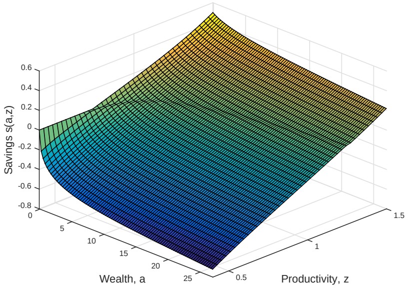
$\Rightarrow$ everything generalizes

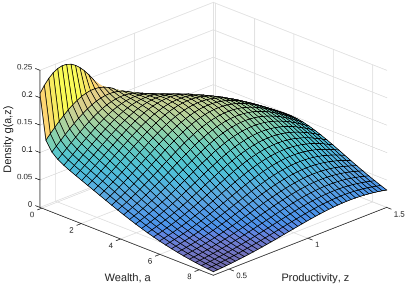
`http://www.princeton.edu/~moll/HACTproject/huggett_diffusion_partialeq.m`

# Visualization of **A** (output of `spy(A)` in Matlab)



# Saving Policy Function and Stationary Distribution



# Summary: Stationary Equilibrium

- Can always write as

$$\begin{aligned}\rho \mathbf{v} &= \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}, \mathbf{p})\mathbf{v} \\ 0 &= \mathbf{A}(\mathbf{v}, \mathbf{p})^{\top}\mathbf{g} \\ 0 &= \mathbf{F}(\mathbf{p}, \mathbf{g})\end{aligned}$$

  where **p** is a vector of prices.

# Accuracy of Finite Difference Method

# Accuracy of Finite Difference Method?

Two experiments:

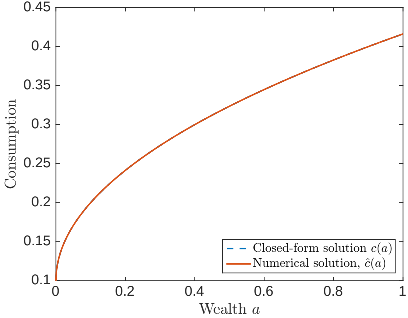
1. special case: comparison with closed-form solution
2. general case: comparison with numerical solution computed using very fine grid

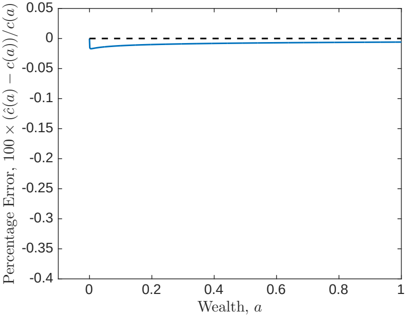
# Accuracy of Finite Difference Method, Experiment 1

- see `http://www.princeton.edu/~moll/HACTproject/HJB_accuracy1.m`
- Achdou et al. (2017) get closed-form solution if
  - exponential utility $u'(c) = c^{-\theta c}$
  - no income risk and $r = 0$ so that $\dot{a} = y - c$ (and $a \geq 0$)

$$\Rightarrow \qquad s(a) = -\sqrt{2\nu a}, \qquad c(a) = y + \sqrt{2\nu a}, \qquad \nu := \frac{\rho}{\theta}$$

- Accuracy with $I = 1000$ grid points ($\widehat{c}(a)$ = numerical solution)



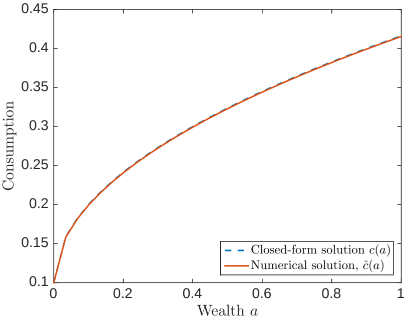



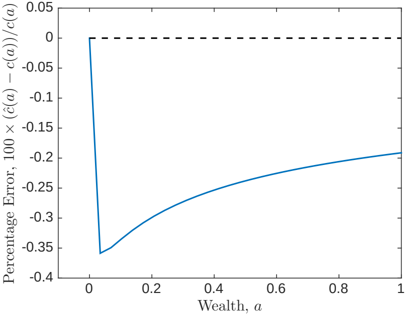
# Accuracy of Finite Difference Method, Experiment 1

- see `http://www.princeton.edu/~moll/HACTproject/HJB_accuracy1.m`
- Achdou et al. (2017) get closed-form solution if
  - exponential utility $u'(c) = c^{-\theta c}$
  - no income risk and $r = 0$ so that $\dot{a} = y - c$ (and $a \geq 0$)

$$\Rightarrow \qquad s(a) = -\sqrt{2\nu a}, \qquad c(a) = y + \sqrt{2\nu a}, \qquad \nu := \frac{\rho}{\theta}$$

- Accuracy with $I = 30$ grid points ($\widehat{c}(a)$ = numerical solution)



# Accuracy of Finite Difference Method, Experiment 2

- see `http://www.princeton.edu/~moll/HACTproject/HJB_accuracy2.m`
- Consider HJB equation with continuum of income types

$$\rho v(a,y) = \max_c \; u(c) + \partial_a v(a,y)(y + ra - c) + \mu(y)\partial_y v(a,y) + \frac{\sigma^2(y)}{2}\partial_{yy} v(a,y)$$

- Compute twice:
  1. with very fine grid: $I = 3000$ wealth grid points
  2. with coarse grid: $I = 300$ wealth grid points

  then examine speed-accuracy tradeoff (accuracy = error in agg $C$)

| | Speed (in secs) | Aggregate $C$ |
|---|---|---|
| $I = 3000$ | 0.916 | 1.1541 |
| $I = 300$ | 0.076 | 1.1606 |
| row 2/row 1 | 0.0876 | 1.005629 |

- i.e. going from $I = 3000$ to $I = 300$ yields $> 10\times$ speed gain and 0.5% reduction in accuracy (but note: even $I = 3000$ very fast)
- Other comparisons? Feel free to play around with `HJB_accuracy2.m`

# Transition Dynamics/MIT Shocks

# Transition Dynamics

Do Aiyagari version of the model

$$r(t) = F_K(K(t),1) - \delta, \qquad w(t) = F_L(K(t),1) \tag{P}$$

$$K(t) = \int a g_1(a,t)da + \int a g_2(a,t)da \tag{K}$$

$$\begin{aligned}\rho v_j(a,t) = & \max_c \; u(c) + \partial_a v_j(a,t)(w(t)z_j + r(t)a - c) \\ & + \lambda_j(v_{-j}(a,t) - v_j(a,t)) + \partial_t v_j(a,t),\end{aligned} \tag{HJB}$$

$$\partial_t g_j(a,t) = -\partial_a[s_j(a,t)g_j(a,t)] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t), \tag{KF}$$

$$s_j(a,t) = w(t)z_j + r(t)a - c_j(a,t), \quad c_j(a,t) = (u')^{-1}(\partial_a v_j(a,t))$$

- Given initial condition $g_{j,0}(a)$, the two PDEs (HJB) and (KF) together with (P) and (K) fully characterize equilibrium.

# Transition Dynamics

- Recall discretized equations for stationary equilibrium

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v})\mathbf{v}$$
$$0 = \mathbf{A}(\mathbf{v})^{\top}\mathbf{g}$$

- Transition dynamics
    - denote $v_{i,j}^n = v_j(a_i, t^n)$ and stack into $\mathbf{v}^n$
    - denote $g_{i,j}^n = g_j(a_i, t^n)$ and stack into $\mathbf{g}^n$

$$\rho \mathbf{v}^n = \mathbf{u}(\mathbf{v}^{n+1}) + \mathbf{A}(\mathbf{v}^{n+1})\mathbf{v}^n + \frac{1}{\Delta t}(\mathbf{v}^{n+1} - \mathbf{v}^n)$$
$$\frac{\mathbf{g}^{n+1} - \mathbf{g}^n}{\Delta t} = \mathbf{A}(\mathbf{v}^n)^{\top}\mathbf{g}^{n+1}$$

- Terminal condition for $\mathbf{v}$: $\mathbf{v}^N = \mathbf{v}_\infty$ (steady state)
- Initial condition for $\mathbf{g}$: $\mathbf{g}^1 = \mathbf{g}_0$.

# Transition Dynamics

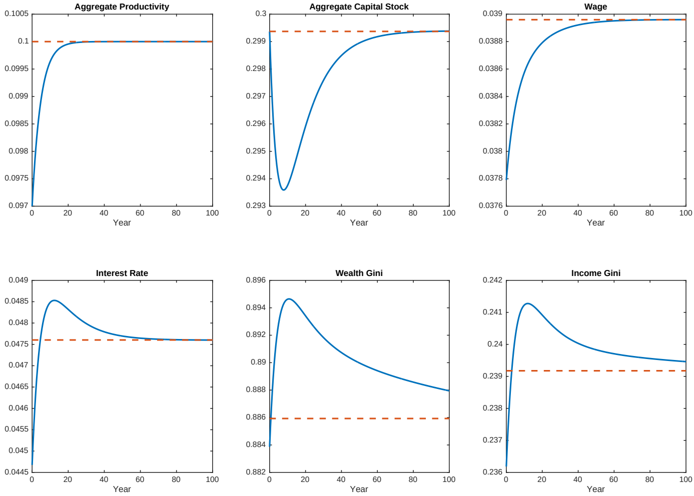
- (HJB) looks forward, runs backwards in time
- (KF) looks backward, runs forward in time
- Algorithm: Guess $K^0(t)$ and then for $\ell = 0, 1, 2, \ldots$

1. find prices $r^\ell(t)$ and $w^\ell(t)$
2. solve (HJB) backwards in time given terminal cond'n $v_{j,\infty}(a)$
3. solve (KF) forward in time given given initial condition $g_{j,0}(a)$
4. Compute $S^\ell(t) = \int a g_1^\ell(a,t) da + \int a g_2^\ell(a,t) da$
5. Update $K^{\ell+1}(t) = (1-\xi)K^\ell(t) + \xi S^\ell(t)$ where $\xi \in (0,1]$ is a relaxation parameter

# An MIT Shock

- Modification: $Y_t = F_t(K, L) = A_t K^\alpha L^{1-\alpha}$, $dA_t = \nu(\bar{A} - A_t)dt$

  `http://www.princeton.edu/~moll/HACTproject/aiyagari_poisson_MITshock.m`



# Stopping Time Problems

# Stopping Time Problems

- In lots of problems in economics, agents have to choose an optimal stopping time
- Quite often these problems entail some form of non-convexity
- Examples:
  - how long should a low productivity firm wait before it exits an industry?
  - how long should a firm wait before it resets its prices?
  - when should you exercise an option?
  - etc... Stokey's book is all about these kind of problems
- These problems are very awkward in discrete time because you run into integer problems
- Big payoff from working in continuous time
- Next: flexible algorithm for solving such problems, also works if don't have simple threshold rules and with states $> 1$

# Exercising an Option (Stokey, Ch. 6)

- Plant has profits

$$\pi(x_t)$$

- $x_t$: state variable = stand in for demand, plant capacity etc

$$dx_t = \mu(x_t)dt + \sigma(x_t)dW_t$$

  where $dW_t := \lim_{\Delta t \to 0} \varepsilon\sqrt{\Delta t}, \varepsilon \sim \mathcal{N}(0,1)$

- Can shut down plant at any time, get scrap value $S(x_t)$, but cannot reopen
- Problem: choose stopping time $\tau$ to solve

$$v(x_0) = \max_{\tau \geq 0} \left\{ \mathbb{E}_0 \int_0^\tau e^{-\rho t}\pi(x_t)dt + e^{-\rho\tau}S(x_\tau) \right\}$$

- Assumptions to make sure $\tau^* < \infty$:

  $\pi'(x) > 0,\ \mu(x) < 0,\ \lim_{x \downarrow -\infty} \left( \frac{\pi(x)}{\rho} - S(x) \right) < 0,\ \lim_{x \uparrow +\infty} \left( \frac{\pi(x)}{\rho} - S(x) \right) >$

- Analytic solution if $\mu(x) = \bar{\mu}, \sigma(x) = \bar{\sigma}, S(x) = \bar{S}$, but not in general

# Exercising an Option: Standard Approach

- Assume scrap value is independent of $x$: $S(x) = \bar{S}$
- Optimal policy = threshold rule: exit if $x_t$ falls below $\underline{x}$
- Standard approach (see e.g. Stokey, Ch.6):

$$\rho v(x) = \pi(x) + \mu(x)v'(x) + \frac{\sigma^2(x)}{2}v''(x), \qquad x > \underline{x}$$

with "value matching" and "smooth pasting" at $\underline{x}$:

$$v(\underline{x}) = \bar{S}, \qquad v'(\underline{x}) = 0$$

- but things more complicated if $S$ depends on $x$ or if dimension $> 1$
- $\Rightarrow$ can't use threshold property
- want algorithm that works also in those cases

# Exercising an Option: HJBVI Approach

- Denote $X$ = set of $x$ such that don't exit:

$$x \in X: \quad v(x) \geq S(x), \quad \rho v(x) = \pi(x) + \mu(x)v'(x) + \frac{\sigma^2(x)}{2}v''(x)$$
$$x \notin X: \quad v(x) = S(x), \quad \rho v(x) \geq \pi(x) + \mu(x)v'(x) + \frac{\sigma^2(x)}{2}v''(x)$$

- Can write compactly as:

$$\min\left\{\rho v(x) - \pi(x) - \mu(x)v'(x) - \frac{\sigma^2(x)}{2}v''(x), v(x) - S(x)\right\} = 0 \quad (*)$$

- Note: have used that following two statements are equivalent
  1. for all $x$, either $f(x) \geq 0, g(x) = 0$ or $f(x) = 0, g(x) \geq 0$
  2. $\min\{f(x), g(x)\} = 0$ for all $x$
- $(*)$ is called "HJB variational inequality" (HJBVI)
- Important: did not impose smooth pasting
  - instead, it's a result: if $\bar{S}$, can prove that $(*)$ implies $v'(\underline{x}) = 0$
  - see e.g. Oksendal `http://th.if.uj.edu.pl/~gudowska/dydaktyka/Oksendal.pdf` (who calls "smooth pasting" "high contact (or smooth fit) principle")

# Finite Difference Scheme for solving HJBVI

- Codes
  http://www.princeton.edu/~moll/HACTproject/option_simple_LCP.m,
  http://www.mathworks.com/matlabcentral/fileexchange/20952

- Main insight: discretized HJBVI = Linear Complementarity Problem (LCP) https://en.wikipedia.org/wiki/Linear_complementarity_problem

- Prototypical LCP: given matrix **B** and vector $q$, find $z$ such that

$$
\begin{aligned}
z'(\mathbf{B}z + q) &= 0 \\
z &\geq 0 \\
\mathbf{B}z + q &\geq 0
\end{aligned}
$$

- There are many good LCP solvers in Matlab and other languages

- Best one I've found if **B** large but sparse (Newton-based):
  http://www.mathworks.com/matlabcentral/fileexchange/20952

# Finite Difference Scheme for solving HJBVI

- Recall HJBVI

$$\min\left\{\rho v(x) - \pi(x) - \mu(x)v'(x) - \frac{\sigma^2(x)}{2}v''(x), v(x) - S(x)\right\} = 0$$

- Without exit, discretize as

$$\rho v_i = \pi_i + \mu_i(v_i)' + \frac{\sigma_i^2}{2}(v_i)'' \qquad \Leftrightarrow \qquad \rho v = \pi + \mathbf{A}v$$

- With exit:

$$\min\{\rho v - \pi - \mathbf{A}v, v - S\} = 0$$

- Equivalently:

$$(v - S)'(\rho v - \pi - \mathbf{A}v) = 0$$
$$v \geq S$$
$$\rho v - \pi - \mathbf{A}v \geq 0$$

- But this is just an LCP with $z = v - S$, $\mathbf{B} = \rho\mathbf{I} - \mathbf{A}$, $q = -\pi + \mathbf{B}$!!

# Generalization: Menu Cost Model

- Work in progress: menu cost model (Golosov-Lucas) via HJBVI
  - HANK + menu cost model + aggregate shocks

# Multiple Assets

# Solution Method in Deterministic Version

$$\max_{\{c_t, d_t\}_{t \geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.}$$
$$\dot{b}_t = y + r^b b_t - d_t - \chi(d_t, a_t) - c_t$$
$$\dot{a}_t = r^a a_t + d_t$$
$$a_t \geq \underline{a}, \quad b_t \geq \underline{b}$$

- $a_t$: illiquid assets
- $b_t$: liquid assets
- $c_t$: consumption
- $y$: individual income
- $d_t$: deposits into illiquid account
- $\chi$: transaction cost function $\chi(d, a) = \chi_0 |d| + \frac{\chi_1}{2} \left(\frac{d}{a}\right)^2 a$

No uncertainty, but easily extended to $y$=Markov process

# How to "upwind" with two endogenous states

- HJB equation

$$\rho v(a,b) = \max_c \; u(c) + \partial_b v(a,b)(y + r^b b - d - \chi(d,a) - c) + \partial_a v(a,b)(d + r^a a)$$

- FOC for $d$: $(1 + \chi_d(d,a))\partial_b v = \partial_a v$

$$\Rightarrow \quad d = \left(\frac{\partial_a v}{\partial_b v} - 1 + \chi_0\right)^{-} \frac{a}{\chi_1} + \left(\frac{\partial_a v}{\partial_b v} - 1 - \chi_0\right)^{+} \frac{a}{\chi_1}$$

- Applying standard upwind scheme

$$\rho v_{i,j} = u(c_i) + \frac{v_{i+1,j} - v_{i,j}}{\Delta b}(s^b_{i,j})^+ + \frac{v_{i,j} - v_{i-1,j}}{\Delta b}(s^b_{i,j})^+ + \frac{v_{i,j+1} - v_{i,j}}{\Delta a}(s^a_{i,j})^+ + \frac{v_{i,j} - v_{i,j-1}}{\Delta a}(s^a_{i,j})^-$$

where e.g. $s^b_{i,j} = y + r^b b_i - d_{i,j} - \chi(d_{i,j}, a_j) - c_{i,j}$

- Hard: $d_{i,j}$ depends on forward/backward choice for $\partial_b v_{i,j}, \partial_a v_{i,j}$

# How to "upwind" with two endogenous states

- Convenient trick: "splitting the drift"

$$
\begin{aligned}
\rho v(a,b) = \max_c \; u(c) &+ \partial_b v(a,b)(y + r^b b - c) \\
&+ \partial_b v(a,b)(-d - \chi(d,a)) \\
&+ \partial_a v(a,b) d \\
&+ \partial_a v(a,b) r^a a
\end{aligned}
$$

  and upwind each term separately

- Can check this satisfies Barles-Souganidis monotonicity condition
- For an application, see

  `http://www.princeton.edu/~moll/HACTproject/two_asset_kinked.pdf`

  `http://www.princeton.edu/~moll/HACTproject/two_asset_kinked.m`

  Subroutines

  `http://www.princeton.edu/~moll/HACTproject/two_asset_kinked_cost.m`

  `http://www.princeton.edu/~moll/HACTproject/two_asset_kinked_FOC.m`