# Mean Field Games in Economics Part II

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# Plan

## Lecture 1

1. A benchmark MFG for macroeconomics: the Aiyagari-Bewley-Huggett (ABH) heterogeneous agent model
2. The ABH model with common noise (“Krusell-Smith”)
3. If time: some interesting extensions of the ABH model
    - the “wealthy hand-to-mouth” and marginal propensities to consume (MPCs)
    - present bias and self-control (economics meets psychology)

## Lecture 2

1. Numerical solution of MFGs with common noise based on “When Inequality Matters for Macro...”
2. Other stuff...

# Recall Stationary MFG, Aiyagari's Variant

Functions $v$ and $g$ on $(\underline{a}, \infty) \times (\underline{y}, \bar{y})$ and scalar $r$ satisfy

$$\rho v = H(\partial_a v) + (wy + ra)\partial_a v + \mu(y)\partial_y v + \frac{\sigma^2(y)}{2}\partial_{yy} v \qquad \text{(HJB)}$$

$$\text{where } H(p) := \max_{c \geq 0} \{u(c) - pc\}, \quad \text{with state constraint } a \geq \underline{a}$$

$$\text{and } 0 = \partial_y v(a, \underline{y}) = \partial_y v(a, \bar{y}) \text{ all } a$$

$$0 = -\partial_a((wy + ra + H'(\partial_a v))g) - \partial_y(\mu(y)g) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g) \qquad \text{(FP)}$$

$$1 = \int_0^\infty \int_{\underline{a}}^\infty g \mathrm{d}a\mathrm{d}y, \quad g \geq 0$$

$$r = e^Z \partial_K F(K, L) = \frac{1}{2}e^Z\sqrt{L/K}, \quad w = e^Z \partial_L F(K, L) = \frac{1}{2}e^Z\sqrt{K/L},$$

$$K = \int_0^\infty \int_{\underline{a}}^\infty ag \mathrm{d}a\mathrm{d}y, \quad L = \int_0^\infty \int_{\underline{a}}^\infty yg \mathrm{d}a\mathrm{d}y \qquad \text{(EQ)}$$

- Coupling through scalars $r$ and $w$ (prices) determined by (EQ)
- Algorithm: guess $(r, w)$, solve (HJB), solve (FP), check (EQ)

# Macroeconomic MFGs with Common Noise

- **This is where the money is!**
- Can fit 90% of macroeconomics into this apparatus so any progress would be extremely valuable
- To understand setup consider Aiyagari (1994) with stochastic aggregate productivity, $Z$, common to all firms
- First studied by
  - Per Krusell and Tony Smith (1998), ”Income and Wealth Heterogeneity in the Macroeconomy”, J of Political Economy
  - Wouter Den Haan (1996), “Heterogeneity, Aggregate Uncertainty, and the Short-Term Interest Rate”, Journal of Business and Economic Statistics
- Language: instead of “common noise” economists say “aggregate shocks” or “aggregate uncertainty”

# Macroeconomic MFGs with Common Noise

- Households:

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \qquad \text{s.t.}$$

$$da_t = (w_t y_t + r_t a_t - c_t) dt$$

$$dy_t = \mu(y_t) dt + \sigma(y_t) dW_t$$

$$a_t \geq \underline{a}$$

- Firms:

$$\max_{K_t, L_t} \left\{ e^{Z_t} F(K_t, L_t) - r_t K_t - w_t L_t \right\}$$

$$dZ_t = -\theta Z_t dt + \eta dB_t, \quad \text{common } B_t \text{ for all firms}$$

$$\Rightarrow r_t = e^{Z_t} \partial_K F(K_t, L_t), \quad w_t = e^{Z_t} \partial_L F(K_t, L_t)$$

- Equilibrium:

$$L_t = \int_0^\infty \int_{\underline{a}}^\infty y g(a, y, t) \mathrm{d}a \mathrm{d}y, \quad K_t = \int_0^\infty \int_{\underline{a}}^\infty a g(a, y, t) \mathrm{d}a \mathrm{d}y$$

# Macroeconomic MFGs with Common Noise

- Households:

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \qquad \text{s.t.}$$

$$da_t = (w_t y_t + r_t a_t - c_t) dt$$

$$dy_t = \mu(y_t) dt + \sigma(y_t) dW_t$$

$$a_t \geq \underline{a}$$

- Firms:

$$\max_{K_t, L_t} \left\{ e^{Z_t} F(K_t, L_t) - r_t K_t - w_t L_t \right\}$$

$$dZ_t = -\theta Z_t dt + \eta dB_t, \quad \text{common } B_t \text{ for all firms}$$

$$\Rightarrow r_t = e^{Z_t} \partial_K F(K_t, L_t), \quad w_t = e^{Z_t} \partial_L F(K_t, L_t)$$

- Equilibrium if restrict to stationary $y$-process with 1st moment = 1:

$$L_t = 1, \quad K_t = \int_0^\infty \int_{\underline{a}}^\infty a g(a, y, t) \mathrm{d}a \mathrm{d}y$$

# MFG System with Common Noise

- both $g_t$ and $v_t$ are now random variables
- dynamic programming notation w.r.t. individual states only
- $\mathbb{E}_t$ is conditional expectation w.r.t. future $(g_t, Z_t)$

$$\rho v_t(a,y) = H(\partial_a v_t(a,y)) + \partial_a v_t(a,y)(w_t y + r_t a) \qquad \text{(HJB)}$$
$$+ \mu(y)\partial_y v_t(a,y) + \frac{\sigma^2(y)}{2}\partial_{yy} v_t(a,y) + \frac{1}{dt}\mathbb{E}_t\left[dv_t(a,y)\right],$$

$$\partial_t g_t(a,y) = -\partial_a[(w_t y + r_t a + H'(\partial_a v_t(a,y)))g_t(a,y)] \qquad \text{(KF)}$$
$$-\partial_y(\mu(y) g_t(a,y)) + \frac{1}{2}\partial_{yy}(\sigma^2(y) g_t(a,y)),$$

$$w_t = \frac{1}{2}e^{Z_t}\sqrt{1/K_t}, \quad r_t = \frac{1}{2}e^{Z_t}\sqrt{K_t}, \quad K_t = \int a g_t(a,y)\,\mathrm{d}a\mathrm{d}y$$

$$dZ_t = -\theta Z_t dt + \eta dB_t$$

Note: $\frac{1}{dt}\mathbb{E}_t\left[dv_t\right]$ means $\lim_{s\downarrow 0}\mathbb{E}_t[v_{t+s} - v_t]/s$ – sorry if weird notation

# Analogous System for Textbook MFG

- See Cardialaguet-Delarue-Lasry-Lions
  `https://arxiv.org/abs/1509.02505`
- Standard MFG with common noise $W_t$

$$dX_{i,t} = \ldots + \sqrt{2}dB_{i,t} + \sqrt{2\beta}dW_t$$

- See their equation (8) for MFG system with common noise:

$$\begin{cases} d_t u_t = \big\{-(1+\beta)\Delta u_t + H(x, Du_t) - F(x, m_t) - \sqrt{2\beta}\mathrm{div}(v_t)\big\}dt + v_t \cdot \sqrt{2\beta}dW_t \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times \mathbb{T}^d, \\ d_t m_t = \big[(1+\beta)\Delta m_t + \mathrm{div}\big(m_t D_p H(m_t, Du_t)\big)\big]dt - \mathrm{div}(m_t \sqrt{2\beta}dW_t\big), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times \mathbb{T}^d, \\ u_T(x) = G(x, m_T),\ m_0 = m_{(0)}, \qquad \text{in } \mathbb{T}^d \end{cases}$$

- "where the map $v_t$ is a random vector field that forces the solution $u_t$ of the backward equation to be adapted to the filtration generated by $(W_t)_{t\in[0,T]}$"
- Previous slide is my sloppy version of this for my particular model

# Today

- A computational method for MFGs with common noise, based on "When Inequality Matters for Macro..."
- Idea: linearize MFG with common noise $Z_t$ around MFG without common noise $Z_t = 0$
- Works beautifully in practice and in many different applications
- But we have no idea about the underlying mathematics!
- $\Rightarrow$ Great problem for mathematicians
- Today: will do in terms of our specific example (Krusell-Smith)
- Good exercise for you: work this out for equation (8) in Cardialaguet-Delarue-Lasry-Lions

# Warm-Up: Linearizing Economic Models

- Economists often solve dynamic economic models using linearization methods
- Explain in context of particularly basic macroeconomic model: "neoclassical growth model"
  - for the moment: no heterogeneity, "representative agent"

$$\max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.} \quad \dot{k}_t = f(k_t) - c_t, \quad k_t \geq 0, \quad c_t \geq 0$$

  - $c_t$: consumption
  - $u$: utility function, $u' > 0$, $u'' < 0$
  - $\rho$: discount rate
  - $k_t$: capital stock, $k_0 = \bar{k}_0$ given
  - $f$: production function, $f' > 0$, $f'' < 0$, $f'(\infty) < \rho < f'(0)$
- Interpretation: a fictitious "social planner" decides how to allocate production $f(k_t)$ between consumption $c_t$ and investment $\dot{k}_t$

# Warm-Up: Linearizing Economic Models

- You can obviously solve this problem numerically from the HJB equation: value function $v$ satisfies

$$\rho v(k) = \max_{c \geq 0} \; u(c) + v'(k)(f(k) - c) \quad \text{on} \quad (0, \infty)$$

- But suppose you don't want to do this for some reason
  - e.g. don't know finite difference methods
  - or want to know more about optimal $k_t$
- Can proceed as follows: differentiate HJB equation w.r.t. $k$

$$v''(k)(f(k) - c(k)) = (\rho - f'(k))v'(k)$$

- Define $\nu_t = v'(k_t)$, evaluate along characteristic $\dot{k}_t = f(k_t) - c_t$

$$\dot{\nu}_t = (\rho - f'(k_t))\nu_t$$
$$\dot{k}_t = f(k_t) - (u')^{-1}(\nu_t)$$

- $(\nu_t, k_t)$ satisfy two ODEs with initial condition $k_0 = \bar{k}_0$, and can also derive terminal condition: $\lim_{t\to\infty} e^{-\rho t}\nu_t k_t = 0$

# Warm-Up: Linearizing Economic Models

- Recall $(\nu_t, k_t)$ satisfy two ODEs

$$\begin{aligned} \dot{\nu}_t &= (\rho - f'(k_t))\nu_t \\ \dot{k}_t &= f(k_t) - (u')^{-1}(\nu_t) \end{aligned} \qquad \text{(ODEs)}$$

$$\text{with } k_0 = \bar{k}_0, \quad \lim_{t\to\infty} e^{-\rho t}\nu_t k_t = 0 \qquad \text{(BOUNDARY)}$$

- Unique stationary $(\nu^*, k^*)$ satisfying $f'(k^*) = \rho$, $\nu^* = u'(f(k^*))$
- To understand dynamics: first-order expansion around $(\nu^*, k^*)$

$$\begin{bmatrix} \dot{\widehat{\nu}}_t \\ \dot{\widehat{k}}_t \end{bmatrix} \approx \underbrace{\begin{bmatrix} 0 & -f''(k^*)\nu^* \\ -\frac{1}{u''(c^*)} & \rho \end{bmatrix}}_{\text{B}} \begin{bmatrix} \widehat{\nu}_t \\ \widehat{k}_t \end{bmatrix}, \quad \begin{bmatrix} \widehat{\nu}_t \\ \widehat{k}_t \end{bmatrix} := \begin{bmatrix} \nu_t - \nu^* \\ k_t - k^* \end{bmatrix}$$

- Easy to show: eigenvalues $(\lambda_1, \lambda_2)$ of **B** are real, $\lambda_1 < 0 < \lambda_2$

$$\Rightarrow \quad \begin{bmatrix} \widehat{\nu}_t \\ \widehat{k}_t \end{bmatrix} \approx c_1 e^{\lambda_1 t}\phi_1 + c_2 e^{\lambda_2 t}\phi_2, \quad \phi_j \in \mathbb{R}^2 = \text{eigenvectors}$$

- constants $(c_1, c_2)$ pinned down from (BOUNDARY) $\Rightarrow$ need $c_2 = 0$

# Warm-Up: Linearizing Economic Models

- Linearization strategy also works with common noise. Consider

$$\max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.}$$

$$\dot{k}_t = e^{Z_t} f(k_t) - c_t, \qquad dZ_t = -\theta Z_t dt + \eta dB_t = \text{common noise}$$

- Value function $v(k, Z)$. Differentiate with respect to $k$:

$$(\rho - e^Z f'(k))\partial_k v = (e^Z f(k) - c(k, Z))\partial_{kk} v - \theta Z \partial_{kZ} v + \frac{\eta^2}{2}\partial_{kZZ} v$$

- Define $\nu_t := \partial_k v(k_t, Z_t)$. Then Ito's formula yields:

$$d\nu_t = b(k_t, Z_t)dt + \eta \partial_{kZ} v(k_t, Z_t) dB_t$$

$$b(k_t, Z_t) := (e^{Z_t} f(k_t) - c_t)\partial_{kk} v(k_t, Z_t) - \theta Z_t \partial_{kZ} v(k_t, Z_t) + \frac{\eta^2}{2}\partial_{kZZ} v(k_t, Z_t)$$

$$\Rightarrow \quad \nu_{t+s} - \nu_t = \int_t^{t+s} b(k_u, Z_u) du + \eta \int_t^{t+s} \partial_{kZ} v(k_u, Z_u) dB_u$$

expanding right-hand side terms $\quad \Rightarrow \quad \lim_{s \downarrow 0} \frac{1}{s} \mathbb{E}_t[\nu_{t+s} - \nu_t] = b(k_t, Z_t)$

# Warm-Up: Linearizing Economic Models

- Recall
$(\rho - e^Z f'(k))\partial_k v = (e^Z f(k) - c(k,Z))\partial_{kk} v - \theta Z \partial_{kZ} v + \frac{\eta^2}{2}\partial_{kZZ} v$
- Evaluate along characteristic $(k_t, Z_t)$ using previous slide

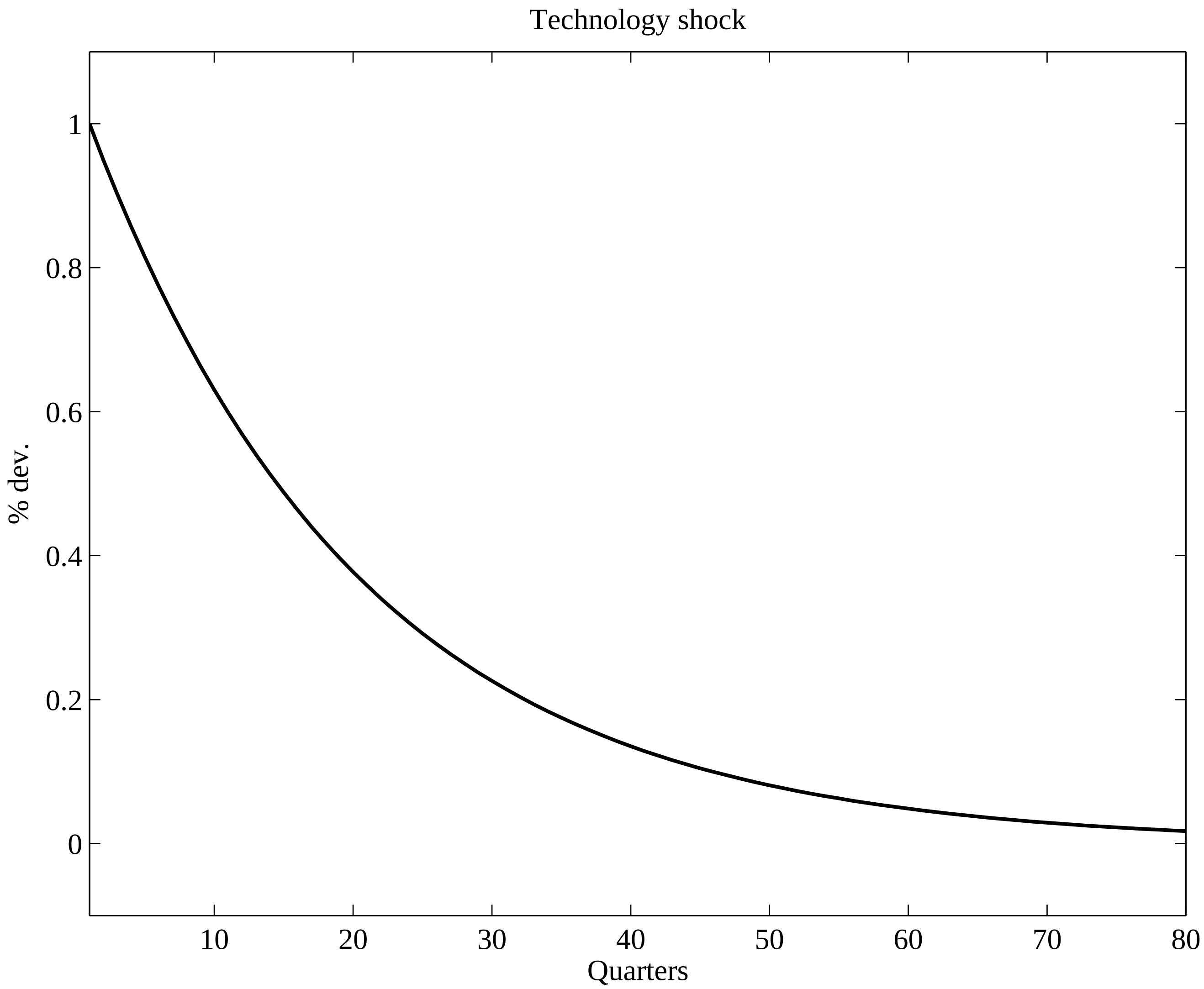
$$\begin{aligned} \mathbb{E}_t[d\nu_t] &= (\rho - e^{Z_t} f'(k_t))dt \\ dk_t &= e^{Z_t} f(k_t) - (u')^{-1}(\nu_t) \qquad (*) \\ dZ_t &= -\theta Z_t dt + \eta dB_t \end{aligned}$$

with $k_0 = \bar{k}_0$, $Z_0 = \bar{Z}_0$ and a terminal condition for $\nu_t$ (in expect.)
- Expansion around stationary point w/o common noise $(\nu^*, k^*, 0)$:

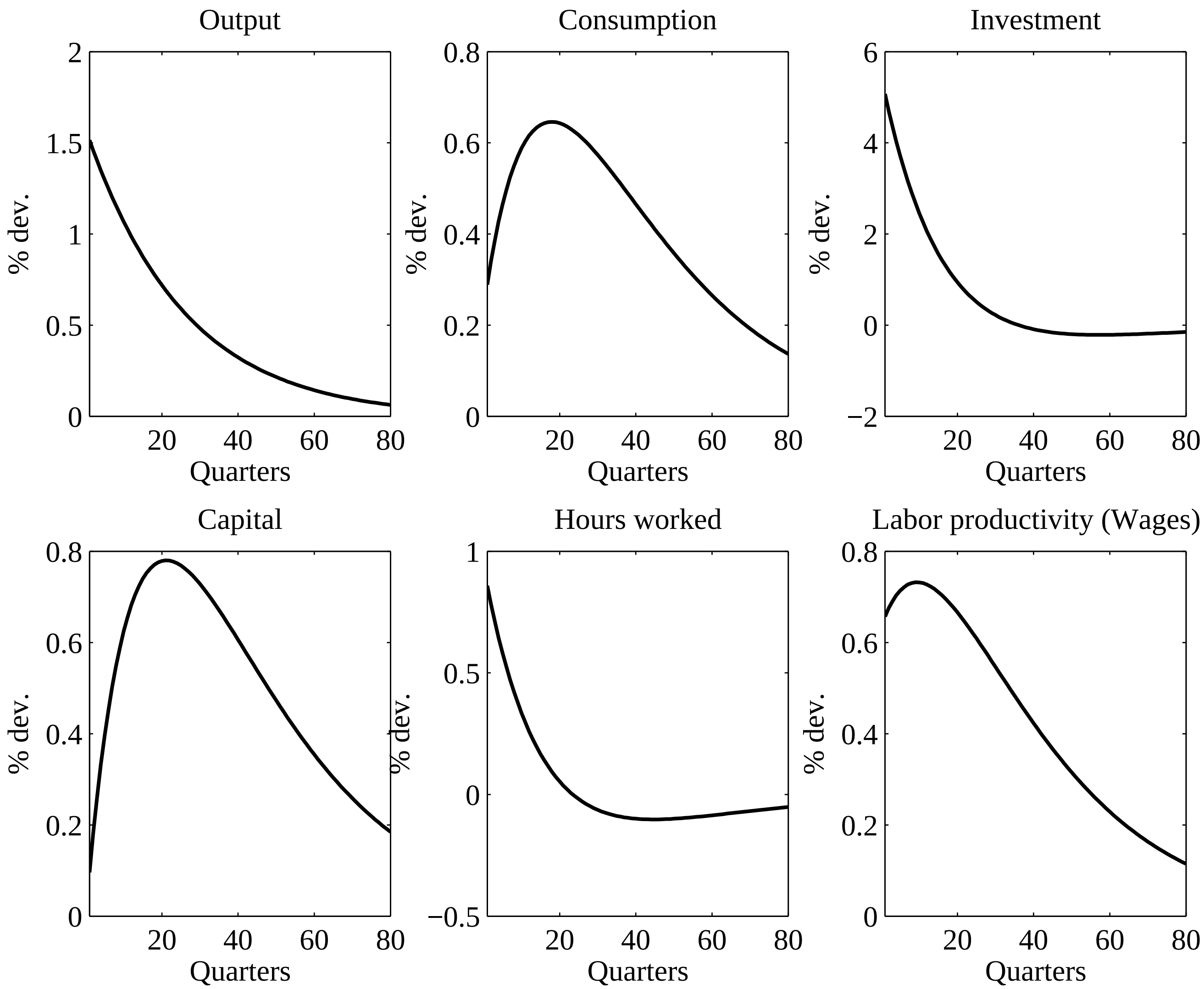
$$\begin{bmatrix} \mathbb{E}_t[d\widehat{\nu}_t] \\ d\widehat{k}_t \\ dZ_t \end{bmatrix} \approx \mathbf{B} \begin{bmatrix} \widehat{\nu}_t \\ \widehat{k}_t \\ Z_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ \eta \end{bmatrix} dB_t, \qquad \begin{bmatrix} \widehat{\nu}_t \\ \widehat{k}_t \\ Z_t \end{bmatrix} = \begin{bmatrix} \nu_t - \nu^* \\ k_t - k^* \\ Z_t - 0 \end{bmatrix}$$

- Can show: $\mathbf{B} \in \mathbb{R}^{3\times 3}$ has real eigenvalues $\lambda_1 \leq \lambda_2 < 0 < \lambda_3 \Rightarrow$ system of SDEs has unique sol'n satisfying boundary conditions
- Impulse response functions (IRFs): $(\widehat{\nu}_t, \widehat{k}_t, Z_t)$, $t \geq 0$ after $dB_0 = 1$

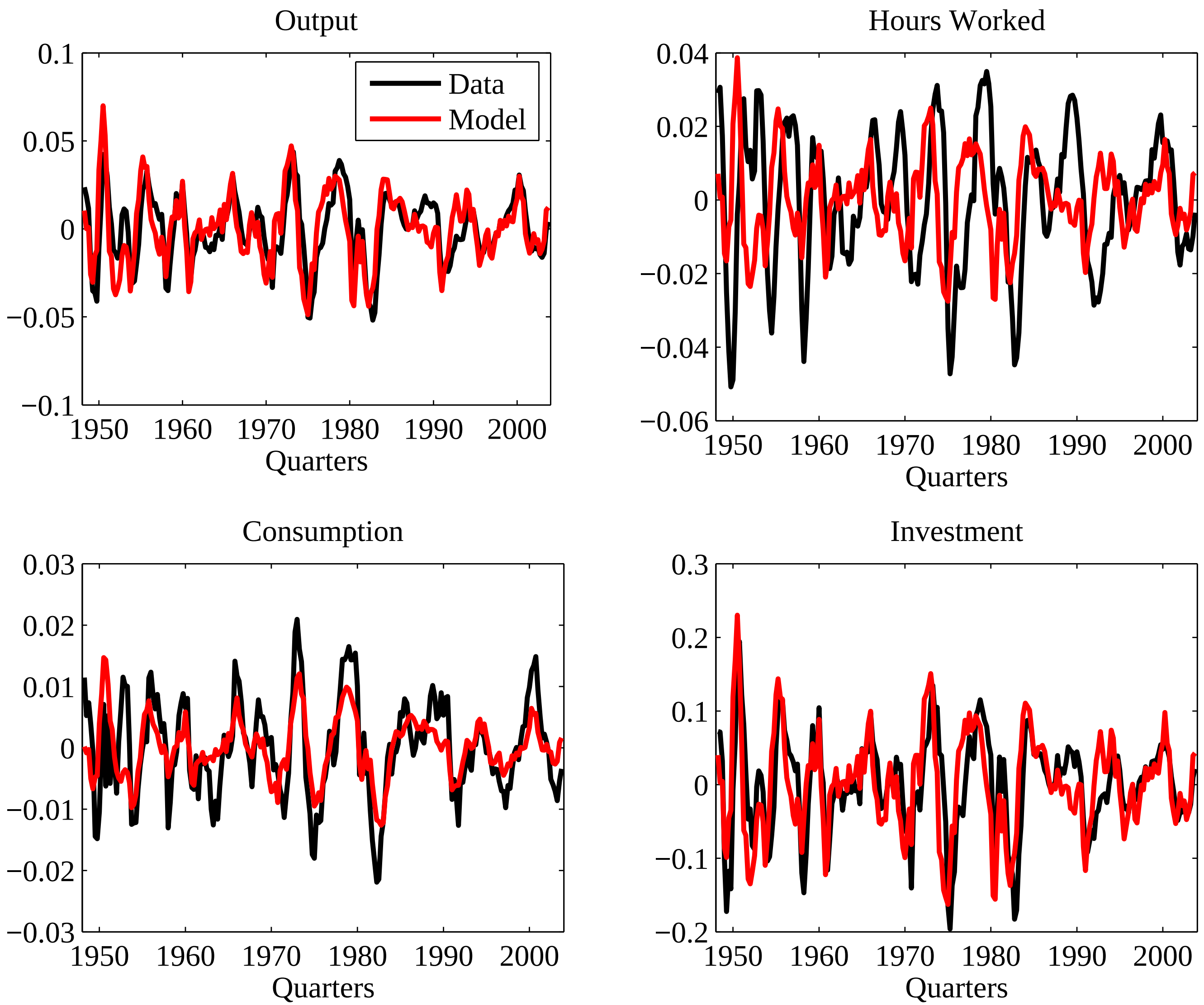
# IRF to A Technological Shock



# IRF to A Technological Shock



# A good fit with estimated shocks



# Real Business Cycle (RBC) Model

- Aside: this model (neoclassical growth model + common noise in productivity $Z_t$) with addition of hours worked choice is called the "Real Business Cycle" (RBC) model
  - fits aggregate data surprisingly well
  - Finn Kydland and Ed Prescott got a Nobel prize for it
  - what's a negative "technology shock"? Do we suddenly forget how to produce stuff?
  - one example is oil price shock, but technology shocks probably a bit of a stretch

# Summary of Linearization Method

1. Compute stationary point without common noise

2. Compute first-order Taylor expansion around stationary point without common noise

3. Solve linear stochastic differential equations

# Key idea: same strategy in MFG with common noise

1. Compute stationary MFG without common noise

2. Compute first-order Taylor expansion around stationary MFG without common noise

3. Solve linear stochastic differential equations

# MFG System with Common Noise

Recall MFG System with Common Noise

$$
\begin{aligned}
\rho v_t(a,y) =& H(\partial_a v_t(a,y)) + \partial_a v_t(a,y)(w_t y + r_t a) \\
&+ \mu(y)\partial_y v_t(a,y) + \frac{\sigma^2(y)}{2}\partial_{yy} v_t(a,y) + \frac{1}{dt}\mathbb{E}_t\left[dv_t(a,y)\right],
\end{aligned} \qquad \text{(HJB)}
$$

$$
\begin{aligned}
\partial_t g_t(a,y) =& -\partial_a[(w_t y + r_t a + H'(\partial_a v_t(a,y)))g_t(a,y)] \\
&- \partial_y(\mu(y)g_t(a,y)) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g_t(a,y)),
\end{aligned} \qquad \text{(KF)}
$$

$$
w_t = \frac{1}{2}e^{Z_t}\sqrt{1/K_t}, \quad r_t = \frac{1}{2}e^{Z_t}\sqrt{K_t}, \quad K_t = \int a g_t(a,y)\mathrm{d}a\mathrm{d}y
$$

$$
dZ_t = -\theta Z_t dt + \eta dB_t
$$

# Linearization and Discretization: Which Order?

- Numerical solution method has two components
  - linearization (first-order Taylor expansion) around MFG without common noise
  - discretization of $(v, g)$ via finite difference method

- What we do:
  1. discretization
  2. linearization

  Reason: don't understand linearized infinite-dimensional system

- What one probably should do:
  1. linearization
  2. discretization

  i.e. analyze linearized infinite-dimensional system before discretizing and putting on computer

# Interesting Exercise

- Start with equation (8) in Cardialaguet-Delarue-Lasry-Lions
  `https://arxiv.org/abs/1509.02505`

$$\begin{cases} d_t u_t = \left\{-(1+\beta)\Delta u_t + H(x, Du_t) - F(x, m_t) - \sqrt{2\beta}\operatorname{div}(v_t)\right\}dt + v_t \cdot \sqrt{2\beta}dW_t \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T]\times\mathbb{T}^d, \\ d_t m_t = \left[(1+\beta)\Delta m_t + \operatorname{div}\big(m_t D_p H(m_t, Du_t)\big)\right]dt - \operatorname{div}\big(m_t\sqrt{2\beta}dW_t\big), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T]\times\mathbb{T}^d, \\ u_T(x) = G(x, m_T),\ m_0 = m_{(0)}, \qquad \text{in } \mathbb{T}^d \end{cases}$$

- Linearize this system around stationary MFG with $\beta = 0$

$$\begin{cases} 0 = -\Delta u + H(x, Du) & \text{in } \mathbb{T}^d \\ 0 = -\Delta m + \operatorname{div}(mD_pH(x, Du)) & \text{in } \mathbb{T}^d \end{cases}$$

# Linearization: Three Steps

1. Compute stationary MFG without common noise

2. Compute first-order Taylor expansion around stationary MFG without common noise

3. Solve linear stochastic differential equations

# Step 1: Compute stationary MFG w/o common noise

$$\rho v = H(\partial_a v) + (wy + ra)\partial_a v + \mu(y)\partial_y v + \frac{\sigma^2(y)}{2}\partial_{yy} v \qquad \text{(HJB*)}$$

$$0 = -\partial_a((wy + ra + H'(\partial_a v))g) - \partial_y(\mu(y)g) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g) \qquad \text{(FP*)}$$

$$r = \frac{1}{2}\sqrt{1/K}, \quad w = \frac{1}{2}\sqrt{K}, \quad K = \int_0^\infty \int_{\underline{a}}^\infty ag\,\mathrm{d}a\,\mathrm{d}y \qquad \text{(EQ*)}$$

# Step 1: Compute stationary MFG w/o common noise

Compute using finite difference method, notation: $\partial_a v(a_i, y_j) \approx \partial_a v_{i,j}$

$$\rho v_{i,j} = H(\partial_a v_{i,j}) + (w y_j + r a_i)\partial_a v_{i,j} + \mu(y_j)\partial_y v_{i,j} + \frac{\sigma^2(y_j)}{2}\partial_{yy} v_{i,j} \text{ (HJB}^*\text{)}$$

$$0 = -\partial_a((wy + ra + H'(\partial_a v))g) - \partial_y(\mu(y)g) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g) \text{ (FP}^*\text{)}$$

$$r = \frac{1}{2}\sqrt{1/K}, \quad w = \frac{1}{2}\sqrt{K}, \quad K = \int_0^\infty \int_{\underline{a}}^\infty a g \, da \, dy \qquad \text{(EQ}^*\text{)}$$

# Step 1: Compute stationary MFG w/o common noise

Compute using finite difference method, notation: $\mathbf{v} = (v_{1,1}, \ldots, v_{I,J})^{\mathsf{T}}$

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; \mathbf{p})\, \mathbf{v}, \qquad \mathbf{p} := (r, w) \tag{HJB*}$$

$$0 = -\partial_a((wy + ra + H'(\partial_a v))g) - \partial_y(\mu(y)g) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g) \tag{FP*}$$

$$r = \frac{1}{2}\sqrt{1/K}, \quad w = \frac{1}{2}\sqrt{K}, \quad K = \int_0^\infty \int_{\underline{a}}^\infty a g \,\mathrm{d}a \,\mathrm{d}y \tag{EQ*}$$

# Step 1: Compute stationary MFG w/o common noise

Compute using finite difference method, notation: $\mathbf{g} = (g_{1,1}, ..., g_{I,J})^{\top}$

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; \mathbf{p})\, \mathbf{v} \tag{HJB*}$$

$$\mathbf{0} = \mathbf{A}(\mathbf{v}; \mathbf{p})^{\top} \mathbf{g} \tag{FP*}$$

$$r = \frac{1}{2}\sqrt{1/K}, \quad w = \frac{1}{2}\sqrt{K}, \quad K = \int_0^{\infty}\int_{\underline{a}}^{\infty} a g \,\mathrm{d}a \,\mathrm{d}y \tag{EQ*}$$

# Step 1: Compute stationary MFG w/o common noise

Compute using finite difference method

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v}; \mathbf{p})\,\mathbf{v} \qquad \text{(HJB*)}$$

$$\mathbf{0} = \mathbf{A}(\mathbf{v}; \mathbf{p})^{\top}\,\mathbf{g} \qquad \text{(FP*)}$$

$$\mathbf{p} = \mathbf{F}(\mathbf{g}) \qquad \text{(EQ*)}$$

# Linearization: Three steps

1. Compute stationary MFG without common noise
    - Yves' finite difference method
    - stationary MFG reduces to sparse matrix equations
2. **Compute first-order Taylor expansion around stationary MFG without common noise**
    - use automatic differentiation routine
3. Solve linear stochastic differential equation

# Step 2: Linearize discretized system w common noise

- Discretized system with common noise

$$\rho \mathbf{v}_t = \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)\, \mathbf{v}_t + \frac{1}{dt}\mathbb{E}_t[d\mathbf{v}_t]$$

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)^\top \mathbf{g}_t$$

$$\mathbf{p}_t = \mathbf{F}(\mathbf{g}_\mathrm{t}; Z_t)$$

$$dZ_t = -\theta Z_t dt + \eta dB_t$$

# Step 2: Linearize discretized system w common noise

- Discretized system with common noise

$$\rho \mathbf{v}_t = \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)\mathbf{v}_t + \frac{1}{dt}\mathbb{E}_t[d\mathbf{v}_t]$$

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)^{\top}\mathbf{g}_t$$

$$\mathbf{p}_t = \mathbf{F}(\mathbf{g}_t; Z_t)$$

$$dZ_t = -\theta Z_t dt + \eta dB_t$$

- Structure basically the same as

$$\mathbb{E}_t[d\nu_t] = (\rho - e^{Z_t} f'(k_t))dt$$

$$dk_t = e^{Z_t} f(k_t) - (u')^{-1}(\nu_t)$$

$$dZ_t = -\theta Z_t dt + \eta dB_t$$

from warm-up exercise

# Step 2: Linearize discretized system w common noise

- Discretized system with common noise

$$\rho \mathbf{v}_t = \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)\mathbf{v}_t + \frac{1}{dt}\mathbb{E}_t[d\mathbf{v}_t]$$

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)^\top \mathbf{g}_t$$

$$\mathbf{p}_t = \mathbf{F}(\mathbf{g}_\mathrm{t}; Z_t)$$

$$dZ_t = -\theta Z_t dt + \eta dB_t$$

- ... which we linearized as

$$\begin{bmatrix} \mathbb{E}_t[d\widehat{\nu}_t] \\ d\widehat{k}_t \\ dZ_t \end{bmatrix} \approx \mathbf{B} \begin{bmatrix} \widehat{\nu}_t \\ \widehat{k}_t \\ Z_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ \eta \end{bmatrix} dB_t, \qquad \begin{bmatrix} \widehat{\nu}_t \\ \widehat{k}_t \\ Z_t \end{bmatrix} = \begin{bmatrix} \nu_t - \nu^* \\ k_t - k^* \\ Z_t - 0 \end{bmatrix}$$

# Step 2: Linearize discretized system w common noise

- Discretized system with common noise

$$\rho \mathbf{v}_t = \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)\,\mathbf{v}_t + \frac{1}{dt}\mathbb{E}_t[d\mathbf{v}_t]$$

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)^\top \mathbf{g}_t$$

$$\mathbf{p}_t = \mathbf{F}(\mathbf{g}_t; Z_t)$$

$$dZ_t = -\theta Z_t dt + \eta dB_t$$

- $\Rightarrow$ Linearize in analogous fashion (using automatic differentiation)

$$\begin{bmatrix} \mathbb{E}_t[d\widehat{\mathbf{v}}_t] \\ d\widehat{\mathbf{g}}_t \\ \mathbf{0} \\ dZ_t \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{B}_{vv} & \mathbf{0} & \mathbf{B}_{vp} & \mathbf{0} \\ \mathbf{B}_{gv} & \mathbf{B}_{gg} & \mathbf{B}_{gp} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{pg} & -\mathbf{I} & \mathbf{B}_{pZ} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\theta \end{bmatrix}}_{\mathrm{B}} \begin{bmatrix} \widehat{\mathbf{v}}_t \\ \widehat{\mathbf{g}}_t \\ \widehat{\mathbf{p}}_t \\ Z_t \end{bmatrix} dt + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \eta \end{bmatrix} dB_t$$

# Step 2: Linearize discretized system w common noise

- Discretized system with common noise

$$\rho \mathbf{v}_t = \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)\mathbf{v}_t + \frac{1}{dt}\mathbb{E}_t[d\mathbf{v}_t]$$

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)^\top \mathbf{g}_t$$

$$\mathbf{p}_t = \mathbf{F}(\mathbf{g}_{\mathrm{t}}; Z_t)$$

$$dZ_t = -\theta Z_t dt + \eta dB_t$$

- Can simplify further by eliminating $\widehat{\mathbf{p}}_t$

$$\begin{bmatrix} \mathbb{E}_t[d\widehat{\mathbf{v}}_t] \\ d\widehat{\mathbf{g}}_t \\ dZ_t \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{vv} & \mathbf{B}_{vp}\mathbf{B}_{pg} & \mathbf{B}_{vp}\mathbf{B}_{pZ} \\ \mathbf{B}_{gv} & \mathbf{B}_{gg} + \mathbf{B}_{gp}\mathbf{B}_{pg} & \mathbf{B}_{gp}\mathbf{B}_{pZ} \\ \mathbf{0} & \mathbf{0} & -\theta \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{v}}_t \\ \widehat{\mathbf{g}}_t \\ Z_t \end{bmatrix} dt + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \eta \end{bmatrix} dB_t$$

Only difference to $(\widehat{\nu}_t, \widehat{k}_t, Z_t)$ system: dimensionality

- rep agent model: dimension 3
- MFG: $2 \times N + 1$, $N = I \times J$, e.g. $= 2001$ if $I = 50$, $J = 20$

# Linearization: Three steps

1. Compute stationary MFG without common noise
   - Yves' finite difference method
   - stationary MFG reduces to sparse matrix equations
2. **Compute first-order Taylor expansion around stationary MFG without common noise**
   - use automatic differentiation routine
3. Solve linear stochastic differential equation
   - moderately-sized systems $\Rightarrow$ can diagonalize system, compute eigenvalues (typically $N + 1$ are $< 0$)
   - large systems, e.g. two-asset model from Lecture 1 $\Longrightarrow$ dimensionality reduction

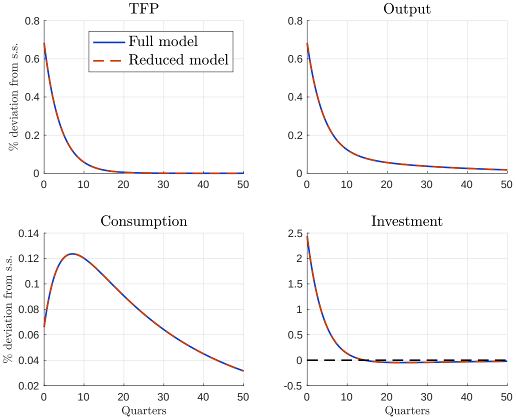
# Dimensionality Reduction in Step 3

- Use tools from engineering literature: “Model reduction”
  - Antoulas (2005), “Approximation of Large-Scale Dynamical Systems”, available at
    `http://epubs.siam.org/doi/book/10.1137/1.9780898718713`
  - Amsallem and Farhat (2011), Lecture Notes for Stanford CME345 “Model Reduction”, available at
    `https://web.stanford.edu/group/frg/course_work/CME345/`
- Approximate *N*-dimensional distribution by projecting onto *k*-dimensional subspace of $\mathbb{R}^N$ with $k << N$

$$\mathbf{g}_t \approx \gamma_{1t}\mathbf{x}_1 + \ldots + \gamma_{kt}\mathbf{x}_k$$

- Adapt to problems with forward-looking decisions
- For details, see “When Inequality Matters for Macro...”

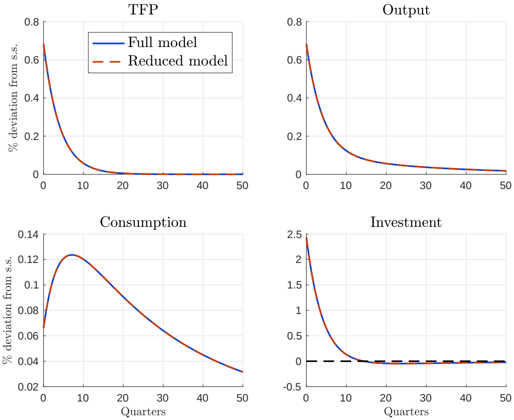
# IRFs in Krusell & Smith Model




- Comparison of full distribution vs. $k = 1$ approximation

  $\Longrightarrow$ recovers Krusell & Smith's result: ok to work with 1D object

# IRFs in Krusell & Smith Model




- Instead two-asset model from Lecture 1 requires $k = 300$
  $\Longrightarrow$ not ok to work with 1D object

# Our Method Is Fast, Accurate in Krusell & Smith Model

## Our method is fast

| | **w/o Reduction** | **w/ Reduction** |
|---|---|---|
| Steady State | 0.082 sec | 0.082 sec |
| Linearize | 0.021 sec | 0.021 sec |
| Reduction | $\times$ | 0.007 sec |
| Solve | 0.14 sec | 0.002 sec |
| **Total** | 0.243 sec | 0.112 sec |

- JEDC comparison project (2010): fastest alternative $\approx$ 7 minutes

## Our method is accurate

| | | | | | |
|---|---|---|---|---|---|
| Common noise $\eta$ | 0.01% | 0.1% | 0.7% | 1% | 5% |
| Den Haan Error | 0.000% | 0.002% | 0.053% | 0.135% | 3.347% |

- JEDC comparison project: most accurate alternative $\approx$ 0.16%

# Linearizing MFGs with Common Noise: Summary

- Method works beautifully in practice ...
- ... and in many applications
- But we don’t understand underlying mathematics
- Great problem for mathematicians!
- Again, from economists’ point of view, MFGs with common noise is where the money is
- Probably want to switch oder:
    1. linearize ...
    2. ... then discretize and put on computer

# Conclusion

- Mean field games extremely useful in economics...
- ... lots of exciting questions involve mean field type interactions...
- ... but mathematics often pretty challenging, at least for the average economist
- Potentially high payoff from mathematicians working on this!
- Questions? Come talk to me or shoot me an email `moll@princeton.edu`