Lecture 2:

New Keynesian Model in Continuous Time

ECO 521: Advanced Macroeconomics I

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New Keynesian Model

- New Keynesian model = RBC model with sticky prices
- References:
  - Gali (2008): most accessible intro
  - Woodford (2003): New Keynesian bible
  - Clarida, Gali and Gertler (1999): most influential article
  - Gali and Monacelli (2005): small open economy version

Why Should You Care?

- Simple framework to think about relationship between monetary policy, inflation and the business cycle
- RBC model: cannot even think about these issues! Real variables are completely separate from nominal variables ("monetary neutrality", "classical dichotomy")
- Corollary: monetary policy has no effect on any real variables
- Sticky prices break "monetary neutrality"
- Workhorse model at central banks (see Fed presentation `https://www.dropbox.com/s/74x17k3pgq1h5g2/MacroModelsAtTheFed.pdf?dl=0`)
- Makes some sense of newspaper statements like: "a boom leads the economy to overheat and creates inflationary pressure"
- Some reason to believe that "demand shocks" (e.g. consumer confidence, animal spirits) may drive business cycle. Sticky prices = one way to get this story off the ground.

Outline

(1) Model with flexible prices

(2) Model with sticky prices

Setup: Flexible Prices

- Households maximize
$$\int_0^\infty e^{-\rho t}\left\{\log C(t) - \frac{N(t)^{1+\varphi}}{1+\varphi}\right\} dt$$
subject to
$$PC + \dot{B} = iB + WN$$

- C: consumption
- N: labor
- P: price level
- B: bonds
- i: nominal interest rate
- W: nominal wage
- Note: no capital

Households

- Hamiltonian

$$\mathcal{H}(B, C, N, \lambda) = \log C - \frac{N^{1+\varphi}}{1+\varphi} + \lambda[iB + WN - PC]$$

- Conditions for optimum

$$\dot{\lambda} = \rho\lambda - \lambda i$$

$$\frac{1}{C} = \lambda P \quad \Rightarrow \quad \frac{\dot{C}}{C} = -\frac{\dot{\lambda}}{\lambda} - \frac{\dot{P}}{P}$$

$$N^{\varphi} = \lambda W$$

- Defining the inflation rate $\pi = \dot{P}/P$

$$\frac{\dot{C}}{C} = i - \pi - \rho$$

$$CN^{\varphi} = \frac{W}{P}$$

Firms – Final Goods Producer

- A competitive final goods producer aggregates a continuum of intermediate inputs

$$Y = \left(\int_0^1 y_j^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

- Cost minimization $\Rightarrow$ demand for intermediate good j

$$y_j(p_j) = \left(\frac{p_j}{P} \right)^{-\varepsilon} Y$$

where

$$P = \left(\int_0^1 p_j^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}$$

- For a derivation see the Technical Appendix of `http://www.crei.cat/people/gali/pdf_files/monograph/slides-ch3.pdf`

Firms – Intermediate Goods Producers

- Continuum of monopolistically competitive intermediate goods producers $j \in [0, 1]$.
- Production uses labor only

$$y_j(t) = A(t)n_j(t).$$

- Solve (drop j subscripts for simplicity)

$$\max_p \; p\left(\frac{p}{P(t)}\right)^{-\varepsilon} Y(t) - \frac{W(t)}{A(t)}\left(\frac{p}{P(t)}\right)^{-\varepsilon} Y(t)$$

- Solution

$$p(t) = P(t) = \frac{\varepsilon}{\varepsilon - 1}\frac{W(t)}{A(t)}$$

where $P = p_j$ follows because all producers are identical.

Equilibrium with Flexible Prices

- Market clearing:

$$C = AN$$

- Combining with household FOC $CN^{\varphi} = W/P$ and $P = \frac{\varepsilon}{\varepsilon-1}W/A$

$$C = Y = A\left(\frac{\varepsilon}{\varepsilon - 1}\right)^{\frac{-1}{1+\varphi}}$$

- Note: distortion from monopolistic competition

- Back out real interest rate from

$$r = i - \pi = \rho - \frac{\dot{C}}{C} = \rho + \frac{\dot{A}}{A} = \rho + g$$

Some Notable Features

- Like an RBC model, this model features "monetary neutrality"
  `http://lmgtfy.com/?q=monetary+neutrality`
- Equivalently: there is a "classical dichotomy"
  `http://lmgtfy.com/?q=classical+dichotomy`
- Real variables $(C(t), Y(t), N(t), W(t)/P(t), r(t))$ are determined completely separately from nominal variables $(P(t), W(t), \pi(t), i(t))$
- In fact, $P(t)$ and $\pi(t)$ are not even determined in the absence of a description of a determination of the economy's money stock (e.g. through monetary policy). But this doesn't matter for real variables
- As a corollary, monetary policy has no effect on real variables

Sticky Prices

- Everything same except intermediate goods producers
- Per period profits are still

$$\Pi_t(p) = p\left(\frac{p}{P(t)}\right)^{-\varepsilon} Y(t) - \frac{W(t)}{A(t)}\left(\frac{p}{P(t)}\right)^{-\varepsilon} Y(t)$$

- But now have to pay quadratic price adjustment cost

$$\Theta_t\left(\frac{\dot{p}}{p}\right) = \frac{\theta}{2}\left(\frac{\dot{p}}{p}\right)^2 P(t)Y(t)$$

- Optimal control problem:

$$V_0(p_0) = \max_{p(t),t\geq 0} \int_0^\infty e^{-\int_0^t i(s)ds}\left\{\Pi_t(p(t)) - \Theta_t\left(\frac{\dot{p}(t)}{p(t)}\right)\right\} dt$$

- θ: degree of price stickiness

Comparison to Literature

- Note: my formulation uses quadratic price adjustment costs as in Rotemberg (1982)
- Different from standard Calvo (1983) pricing formulation: allowed to change price at Poisson rate α
- I like Rotemberg better because pricing is state dependent as opposed time dependent ("Calvo fairy")
- Closer to "menu cost" models
- Many other papers, e.g. Schmitt-Grohe and Uribe (2004), Fernandez-Villaverde et al. (2011) also use Rotemberg
- I also assume that adjustment costs are paid as a transfer to consumers, $T = \Theta_t(\pi) = (\theta/2)\pi^2 PY$. Just a trick to eliminate real resource costs of inflation ($\Theta_t(\pi) \approx 0$ anyway)

Optimal Price Setting

- Hamiltonian (state: p, control: $\dot{p}$, co-state: η):

$$\mathcal{H}(p, \dot{p}, \eta) = p\left(\frac{p}{P}\right)^{-\varepsilon} Y - \frac{W}{A}\left(\frac{p}{P}\right)^{-\varepsilon} Y - \frac{\theta}{2}\left(\frac{\dot{p}}{p}\right)^{2} PY + \eta\dot{p}$$

- Conditions for optimum

$$\theta\frac{\dot{p}}{p}\frac{P}{p}Y = \eta$$

$$\dot{\eta} = i\eta - \left[(1-\varepsilon)\left(\frac{p}{P}\right)^{-\varepsilon} Y + \varepsilon\frac{W}{p}\frac{1}{A}\left(\frac{p}{P}\right)^{-\varepsilon} Y + \theta\left(\frac{\dot{p}}{p}\right)^{2}\frac{P}{p}Y\right].$$

- Symmetric equilibrium: $p = P$

$$\theta\pi Y = \eta$$

$$\dot{\eta} = i\eta - \left[(1-\varepsilon)Y + \varepsilon\frac{W}{P}\frac{1}{A}Y + \theta\pi^{2}Y\right].$$

Optimal Price Setting

- Recall the FOC: $\theta \pi Y = \eta$. Differentiate with respect to time

$$\theta \dot{\pi} Y + \theta \pi \dot{Y} = \dot{\eta}$$

- Substitute into equation for co-state and rearrange

Lemma

The price setting of firms implies that the inflation rate $\pi = \dot{P}/P$ is determined by

$$\left(i - \pi - \frac{\dot{Y}}{Y}\right)\pi = \frac{\varepsilon - 1}{\theta}\left(\frac{\varepsilon}{\varepsilon - 1}\frac{W}{P}\frac{1}{A} - 1\right) + \dot{\pi}$$

Optimal Price Setting in Equilibrium

- In equilibrium $C = Y$ and Euler equation

$$\frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = i - \pi - \rho$$

- Substitute into expression on previous slide $\Rightarrow$ Inflation determined by

$$\rho\pi = \frac{\varepsilon - 1}{\theta}\left(\frac{\varepsilon}{\varepsilon - 1}\frac{W}{P}\frac{1}{A} - 1\right) + \dot{\pi}. \qquad (*)$$

- In integral form (check that differentiating gives back above)

$$\pi(t) = \frac{\varepsilon - 1}{\theta}\int_t^\infty e^{-\rho(s-t)}\left(\frac{\varepsilon}{\varepsilon - 1}\frac{W(s)}{P(s)}\frac{1}{A(s)} - 1\right) ds$$

- Compare with equation (16) in Chapter 3.3. of Gali's book and expression just below.

Optimal Price Setting in Equilibrium

- Inflation determined by

$$\pi(t) = \frac{\varepsilon - 1}{\theta} \int_t^\infty e^{-\rho(s-t)} \left(\frac{\varepsilon}{\varepsilon - 1} \frac{W(s)}{P(s)} \frac{1}{A(s)} - 1 \right) ds$$

- Intuition: term in brackets = marginal payoff to a firm from increasing its price

$$\Pi_t'(P(t)) = (\varepsilon - 1) Y(t) \left(\frac{\varepsilon}{\varepsilon - 1} \frac{W(t)}{P(t)} \frac{1}{A(t)} - 1 \right)$$

- Positive whenever P less than optimal markup $\frac{\varepsilon}{\varepsilon-1}$ over marginal cost W/A
- With flexible prices, $\theta = 0$: $\Pi_t'(P(t)) = 0$ for all t, $P = \frac{\varepsilon}{\varepsilon-1}\frac{W}{A}$
- With sticky prices, $\theta > 0$: π = PDV of all future $\Pi_t'(P(t))$
- Adjustment cost is convex. So if expect reason to adjust in the future – e.g. $W(t)/A(t) \uparrow$ – already adjust now

IS Curve and Phillips Curve

- Call outcomes under flexible prices, $\theta = 0$, "natural" output Y^n and "natural" real interest rate. Recall

$$Y^n = A\left(\frac{\varepsilon}{\varepsilon - 1}\right)^{\frac{-1}{1+\varphi}}, \quad \frac{\dot{Y}^n}{Y^n} = r - \rho, \quad r = \rho + \frac{\dot{A}}{A}$$

- Define output gap: $X = Y/Y^n$. Recall Euler equation under sticky prices

$$\frac{\dot{Y}}{Y} = i - \pi - \rho$$

- Euler equation in terms of output gap $\dot{X}/X = \dot{Y}/Y - \dot{Y}^n/Y^n$

$$\frac{\dot{X}}{X} = i - \pi - r$$

- This is basically an IS curve

IS Curve and Phillips Curve

- Can obtain "Phillips Curve" in similar way. Recall

$$P^n = \frac{\varepsilon}{\varepsilon - 1}\frac{W^n}{A} \quad \Rightarrow \quad \frac{W}{P}\frac{1}{A} = \frac{W/P}{W^n/P^n}$$

- Equation for inflation $(*)$ becomes

$$\rho\pi = \frac{\varepsilon - 1}{\theta}\frac{W/P - W^n/P^n}{W^n/P^n} + \dot{\pi}.$$

- From FOC $CN^{\varphi} = \frac{W}{P}$, and mkt clearing $C = Y$, $N = Y/A$

$$\frac{W/P}{W^n/P^n} = \left(\frac{Y}{Y^n}\right)^{1+\varphi} = X^{1+\varphi}.$$

IS Curve and Phillips Curve

- Relation between inflation and output gap: "New Keynesian Phillips Curve"

$$\rho\pi = \frac{\varepsilon - 1}{\theta}\left(X^{1+\varphi} - 1\right) + \dot{\pi}$$

- In integral form

$$\pi(t) = \frac{\varepsilon - 1}{\theta}\int_t^\infty e^{-\rho(s-t)}\left(X(s)^{1+\varphi} - 1\right) ds$$

- Inflation high when future output gaps are high, i.e. when economy "overheats"

Three Equation Model

- Recall: IS curve and Phillips curve

$$\frac{\dot{X}}{X} = i - \pi - r \tag{IS}$$

$$\rho\pi = \frac{\varepsilon - 1}{\theta}\left(X^{1+\varphi} - 1\right) + \dot{\pi} \tag{PC}$$

- To close model: Taylor rule

$$i = i^* + \phi\pi + \phi_x \log X \tag{TR}$$

- "Three equation model," see modern undergraduate textbooks (e.g. Carlin and Soskice)
- Substitute (TR) into (IS) $\Rightarrow$ system of two ODEs in (π, X), analyze with phase diagram

Three Equation Model in Literature

- Literature uses log-linearization all over the place
- Obtain exact analogues by defining

$$x := \log X = \log Y - \log Y^n$$

- Using that for small x (Taylor-series)

$$X^{1+\varphi} - 1 = e^{(1+\varphi)x} - 1 \approx (1+\varphi)x$$

- and defining $\kappa := (\varepsilon - 1)(1+\varphi)/\theta$

$$\dot{x} = i - \pi - r \tag{IS'}$$

$$\rho\pi = \kappa x + \dot{\pi} \tag{PC'}$$

$$i = i^* + \phi\pi + \phi_x x \tag{TR'}$$

- Exact continuous time analogues of (21), (22), (25) in Chapter 3 of Gali's book, same as in Werning (2012)

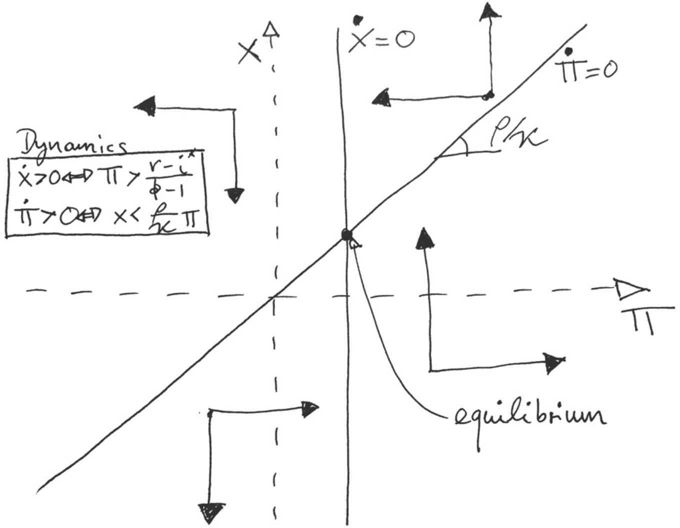
Phase Diagrams

- For simplicity, assume $\phi_x = 0$. Makes some math easier.
- Also ignore ZLB, $i \geq 0$ (see Werning paper on reading list).
- Substitute (TR') into (IS')

$$\begin{aligned} \dot{x} &= i^* - r + (\phi - 1)\pi \\ \dot{\pi} &= \rho\pi - \kappa x \end{aligned} \tag{ODE}$$

- See phase diagrams on next slide.
- Important: both π and x are jump-variables. No state variables.
- Two cases:
  - $\phi > 1$: unique equilibrium. "Taylor principle": i increases more than one-for-one with π so that also real rates increase.
  - $\phi < 1$: equilibrium indeterminacy
- From now assume $\phi > 1$

Phase Diagram with $\phi > 1$



Phase Diagram with $\phi < 1$

Homework 3

Rigorous Analysis of Uniqueness/Determinacy

- Examine eigenvalues of system (ODE). For intro see here:
  `http://www.princeton.edu/~moll/ECO503Web/Lecture4_ECO503.pdf`
- Consider case $i^* = r \Rightarrow$ st. st. $= (\pi^*, x^*) = (0, 0)$. Write (ODE) as

$$\begin{bmatrix} \dot{x} \\ \dot{\pi} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ \pi \end{bmatrix}, \qquad \mathbf{A} := \begin{bmatrix} 0 & \phi - 1 \\ -\kappa & \rho \end{bmatrix}$$

- Find eigenvalues of **A** by solving characteristic polynomial

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = -\lambda(\rho - \lambda) + (\phi - 1)\kappa$$
$$0 = \lambda^2 - \rho\lambda + (\phi - 1)\kappa$$

- This is a simple quadratic with two solutions (“roots”)

$$\lambda_1 = \frac{\rho + \sqrt{\rho^2 - 4(\phi - 1)\kappa}}{2}, \qquad \lambda_2 = \frac{\rho - \sqrt{\rho^2 - 4(\phi - 1)\kappa}}{2}$$

- Have two jump variables $\Rightarrow$ want two roots with positive real parts
- Real part of $\lambda_1 > 0$ always. Real part of $\lambda_2 > 0$ if $\phi > 1$.
- If $\rho^2 - 4(\phi - 1)\kappa < 0$, eigenvalues have imaginary parts $\Rightarrow$ spirals

Intuition for Indeterminacy with $\phi < 1$

- Continue considering case $i^* = r$

$$\dot{x} = (\phi - 1)\pi$$
$$\dot{\pi} = \rho\pi - \kappa x \quad \text{(ODE)}$$

- Key idea: if $\phi < 1$ can construct self-fulfilling equilibria
- Let's construct one: suppose households and firms expect

$$\pi(t) = \pi_0^e e^{\lambda t}$$

  for some π_0^e and some $\lambda < 0$, e.g. $\pi_0^e = 0.1$ and $\lambda = -1$

- Integrating the Euler equation and assuming $\lim_{T\to\infty} x(T) = 0$

$$\begin{aligned} x(t) &= (1-\phi)\int_t^\infty \pi(s)ds \\ &= (1-\phi)\pi_0^e e^{\lambda t}\int_t^\infty e^{\lambda(s-t)}ds \quad \text{(EE)} \\ &= (1-\phi)e^{\lambda t}\frac{\pi_0^e}{-\lambda} \end{aligned}$$

Intuition for Indeterminacy with $\phi < 1$

- From Phillips curve, inflation at $t = 0$ is

$$\pi(0) = \kappa \int_0^\infty e^{-\rho t} x(t) dt = (1-\phi)\kappa \left(\frac{\pi_0^e}{-\lambda}\right) \int_0^\infty e^{(\lambda-\rho)t} dt$$
$$= \frac{(1-\phi)\kappa}{-\lambda(\rho-\lambda)} \pi_0^e$$

- Hence if λ is such that $\frac{(1-\phi)\kappa}{-\lambda(\rho-\lambda)} = 1$, then $\pi(0) = \pi_0^e$
- But this is just our quadratic from last slide $0 = \lambda^2 - \rho\lambda + (\phi - 1)\kappa$
- Hence if we set $\lambda = \lambda_2 < 0$, then any π_0^e is an equilibrium, i.e. we have just constructed a continuum of self-fulfilling equilibria
- Now let's understand why $\phi > 1$ rules out self-fulfilling equilibria
  - construction requires $\lambda < 0$ for integral in (EE) to converge
  - but if $\phi > 1$, $\lambda < 0$, then $\frac{(1-\phi)\kappa}{-\lambda(\rho-\lambda)} < 0$, i.e. $\pi_0^e > 0 \Rightarrow \pi(0) < 0$
- Fed says: "if you ever expect inflation, we'll raise nominal rate so aggressively that we'll have negative output gap & hence deflation"

Monetary Policy: Summary

- Can achieve $\pi = 0$ and $x = 0$ by setting $i^* = r$ (and $\phi > 1$) ("divine coincidence")
- Scenario 1: suppose economy is in $(x, \pi) = (0, 0)$ equilibrium. But at $t = T$, r increases once and for all, e.g. because TFP growth increases (recall $r = \rho + \dot{A}/A$)
- Scenario 2: suppose economy is in $(x, \pi) = (0, 0)$ equilibrium. But at $t = T$, someone at the Fed goes crazy and increases i^* (e.g. because mistakenly think that TFP growth goes up)
- Draw time paths for $(x(t), \pi(t))$ for both scenarios
- Key: model has no state variables $\Rightarrow$ no dynamics

Monetary Policy Shock ("MIT Shock")

- Consider linearized 3 eq model, but with innovation to Taylor rule ϵ

$$\dot{x} = \frac{1}{\sigma}(i - r - \pi)$$

$$\dot{\pi} = \rho\pi - \kappa x$$

$$i = r + \phi\pi + \epsilon, \quad \dot{\epsilon} = -\eta\epsilon, \quad \eta > 0$$

- Consider $\epsilon_0 < 0$, then $\epsilon(t)$ mean-reverts to steady state
- Nothing stochastic, shock is zero-probability event ("MIT shock")...
- ... but can still learn a lot about model's behavior
- For simplicity, assume no dynamics in "natural" interest rate $r(t) = \rho$
- See section 3.4.1 in Gali's book for discrete-time version

Monetary Policy Shock ("MIT Shock")

Proposition

The equilibrium output gap, inflation, nominal and real interest rates are

$$x = -\frac{\rho + \eta}{(\phi - 1)\kappa + \sigma\eta(\rho + \eta)}\epsilon$$

$$\pi = -\frac{\kappa}{(\phi - 1)\kappa + \sigma\eta(\rho + \eta)}\epsilon$$

$$i = \rho + \frac{\sigma\eta(\rho + \eta) - \kappa}{(\phi - 1)\kappa + \sigma\eta(\rho + \eta)}\epsilon$$

$$i - \pi = \rho + \frac{\sigma\eta(\rho + \eta)}{(\phi - 1)\kappa + \sigma\eta(\rho + \eta)}\epsilon$$

- Observations: in response to $\epsilon(0) < 0$
  - output gap $x(0) \uparrow$
  - inflation $\pi(0) \uparrow$
  - nominal interest rate $i(0)$ ambiguous
  - real interest rate $i(0) - \pi(0) \downarrow$

Proof via Method of Undetermined Coefficients

- Substitute Taylor rule into Euler equation

$$\sigma \dot{x} = (\phi - 1)\pi + \epsilon, \quad \dot{\epsilon} = -\eta\epsilon$$
$$\dot{\pi} = \rho\pi - \kappa x$$

- Guess

$$x = \psi_x \epsilon, \quad \pi = \psi_\pi \epsilon \quad \Rightarrow \quad \dot{x} = -\psi_x \eta\epsilon, \quad \dot{\pi} = -\psi_\pi \eta \epsilon$$

- Plugging in

$$-\sigma\psi_x\eta = (\phi - 1)\psi_\pi + 1$$
$$-\psi_\pi\eta = \rho\psi_\pi - \kappa\psi_x$$

- From second equation $\psi_x = \frac{\rho+\eta}{\kappa}\psi_\pi$
- Plugging into first equation gives

$$\psi_\pi = -\frac{\kappa}{(\phi - 1)\kappa + \sigma\eta(\rho + \eta)}$$

- Some more algebra/substitutions ⇒ remaining coefficients.□

Optimal Monetary Policy with “Cost Push Shocks”

- Woodford (2003): approximate welfare with quadratic loss function

$$\frac{1}{2}\int_0^\infty e^{-\rho t}(\pi(t)^2 + \alpha x(t)^2)dt \qquad (*)$$

- Optimal monetary policy: minimize $(*)$ subject to

$$\rho\pi = \kappa x + \dot{\pi}$$

- Solution obvious: $(x(t), \pi(t)) = (0, 0)$ for all t
- Reason: Phillips curve always consistent with $x = \pi = 0$, i.e. there is no tradeoff
- Clarida, Gali and Gertler (1999): introduce “cost push shocks” $u(t)$

$$\rho\pi = \kappa x + u + \dot{\pi}$$

  where $u(t) \to 0$ as $t \to \infty$, e.g. $u(t) = e^{-\eta t}u_0, \eta > 0$
- Can no longer achieve $x = \pi = 0 \Rightarrow$ problem more interesting

Optimal Monetary Policy with "Cost Push Shocks"

- Planner's problem with cost-push shocks:

$$\min_{\{x(t)\}_{t\geq 0}} \frac{1}{2}\int_0^\infty e^{-\rho t}(\pi(t)^2 + \alpha x(t)^2)dt \quad \text{s.t.} \quad \rho\pi = \kappa x + u + \dot{\pi}$$

- Hamiltonian:

$$\mathcal{H} = \frac{1}{2}(\pi^2 + \alpha x^2) + \mu(\rho\pi - \kappa x - u)$$

- Optimality conditions:

$$\dot{\mu} = \rho\mu - \mathcal{H}_\pi = -\pi \tag{1}$$

$$\alpha x = \kappa\mu \tag{2}$$

- Differentiate (2) and substitute in (1)

$$\dot{x} = -\frac{\kappa}{\alpha}\pi \tag{3}$$

- (3) captures what Gali dubs "leaning against the wind": decrease output gap in face of inflationary pressures

Intuition for “Leaning Against the Wind”

- Cost-push shock $u > 0 \Rightarrow$ firms want to increase prices but this is bad for welfare (loss function features π^2)
- Planner’s response: $x \downarrow \Rightarrow$ marginal costs $W/P \downarrow \Rightarrow$ offset inflationary pressures
- Optimality condition (3) balances welfare loss due to $\pi > 0$ and welfare loss due to $x < 0$ (π^2 vs αx^2)

Full Solution of Optimal Policy with Cost Push Shocks

- Given any time path $u(t)$, solve for optimal $x(t), \pi(t)$ as follows
- Strategy is continuous-time analogue of p.104 in Gali's book
- Differentiate Phillips curve

$$\rho\dot{\pi} = \kappa\dot{x} + \dot{u} + \ddot{\pi}$$

- Substitute in from (3)

$$\rho\dot{\pi}(t) = -\pi(t)/\alpha + \dot{u}(t) + \ddot{\pi}(t) \tag{4}$$

- Given time path $u(t)$, (4) is a second-order ODE for $\pi(t)$ that can be solved (e.g. plug into Mathematica)
  - for instance: homogeneous part is exponential

    $$\pi(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

    where $\lambda_{1/2}$ are roots of quadratic (from $\pi(t) = ce^{\lambda t}$ into (4))

    $$\rho\lambda = -1/\alpha + \lambda^2$$