Lecture 2: Growth Model, Dynamic Optimization in Discrete Time

ECO 503: Macroeconomic Theory I

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Fall 2014

Recall from last lecture

- Economy $\Leftrightarrow$ resource allocation problem $\Leftrightarrow$ primitives
- where primitives =
  - preferences
  - technology
  - endowments
- This lecture: economy = growth model
- Next slide: complete description of economy in terms of primitives

Growth Model: Setup

- **Preferences:** a single household with preferences defined by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)$$

with $u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$

- **Technology:**

$$\begin{aligned} y_t &= F(k_t, h_t), \quad F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \\ c_t + i_t &= y_t \\ k_{t+1} &= i_t + (1 - \delta)k_t \\ c_t &\geq 0, \quad i_t \geq -(1 - \delta)k_t \end{aligned}$$

- **Endowments:**
  - 1 unit of time each period
  - $\hat{k}_0$ units of capital at time 0

Assumptions

- **Preferences:** $0 < \beta < 1$ and u is
    - strictly increasing
    - strictly concave
    - C^2 (twice continuously differentiable)
- **Technology:** $0 < \delta \leq 1$ and F is
    - constant returns to scale
    - strictly increasing
    - weakly concance in (k, h) jointly, strictly concave in each argument individually
    - $F(0, h) = 0$ for all h.
    - C^2
    - ("Inada conditions")

$$\lim_{k \to 0} F_k(k, h) = \infty, \quad \forall h > 0,$$
$$\lim_{k \to \infty} F_k(k, h) = 0, \quad \forall h > 0,$$

Comments

- **Tradeoffs** in the model
  - consumption today c_t vs. consumption tomorrow c_{t+1}
  - consumption c_t vs. leisure $1 - h_t$
- Model assumes **"representative household"** and **"representative firm"** (jointly = "representative agent")
- When is this justified? If at least one of following 3 conditions are satisfied
  1. all individuals in economy are identical
  2. particular assumptions on preferences ("homotheticity", "Gorman aggregation")
  3. perfect markets
     - representative firm ⇔ perfect factor markets (capital, labor), equalize marginal products
     - representative HH ⇔ perfect insurance markets, equalize marginal utilities
- Do we believe these conditions are satisfied? **No**, but...

General Comment: Modeling in (Macro)economics

- Objective is **not** to build one big model that we use to address all issues
  - descriptive realism is not the objective
  - instead make modeling choices that are dependent on the issue
  - whether a model is "good" is context dependent
- Approach to modeling in macro(economics) well summarized by following two statements
  - "All models are false; some are useful"
  - "If you want a model of the real world, look out the window" (kidding, but only half kidding)

General Comment: Modeling in (Macro)economics

- But: growth model is **"the"** benchmark model of macro
- Why is this the benchmark model?
  - minimal model of y where $y = F(k, h)$
- Also, growth model = great laboratory for teaching you tools of macro...
- ... and many other models in macroeconomics build on growth model. Examples:
  - Real business cycle (RBC) model = growth model with aggregate productivity shocks
  - New Keynesian model = RBC model + sticky prices
  - Incomplete markets model (Aiyagari-Bewley-Huggett) = growth model + heterogeneity in form of uninsurable idiosyncratic shocks

What issues is growth model useful for?

- Growth model is designed to be model of capital accumulation process
- Growth model is **not** a "good" model of
  - growth (somewhat ironically given its name)
  - income and wealth distribution (given rep. agent assumption)
  - inflation and monetary policy
  - unemployment
  - financial crises
- But some of growth model's extensions (e.g. those mentioned on previous slide) are "good" models of these issues

Some Concepts

- **Definition:** A **feasible allocation** for the growth model is a list of sequences $\{c_t, h_t, k_t\}$ such that

$$c_t + k_{t+1} = F(k_t, h_t) + (1 - \delta)k_t$$

$$0 \leq h_t \leq 1, \quad c_t \geq 0, \quad k_t \geq 0, \quad k_0 = \hat{k}_0$$

Analysis of Growth Model

- Consistent with there being two key tradeoffs, captured by the model, there are two choices to be made each period
  - c_t vs. k_{t+1}
  - c_t vs. h_t
- Will analyze
  1. Pareto efficient allocations
  2. decentralized equilibrium allocations
- Start with Pareto efficient allocations

Solow Model

- Historically, much interest in allocations that resulted from specific "ad hoc" decision rules

$$c_t = sy_t$$
$$h_t = \bar{h}$$

- = "Solow model" you may know from your undergraduate courses
- See homehork 1

Pareto Efficient Alloc. in Growth Model

- To simplify analysis and focus on dynamics considerations, begin with extreme case: leisure not valued, or (with slight abuse of notation)

$$u(c_t, 1 - h_t) = u(c_t)$$

- Assume (Inada condition akin to those on F)

$$\lim_{c \to 0} u'(c) = \infty$$

- Also define

$$f(k_t) = F(k_t, 1)$$

Social Planner's Problem

- Only one person in economy $\Rightarrow$ our life is simple.
- Pareto efficient allocation = max. utility of household subject to feasibility
- Think of this as problem of fictitious "social planner":

$$V(\hat{k}_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$
$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$
$$c_t \geq 0, \quad k_{t+1} \geq 0, \quad k_0 = \hat{k}_0.$$

- Alternatively, substitute resource constraint into objective

$$V(\hat{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \quad \text{s.t.}$$
$$0 \leq k_{t+1} \leq f(k_t) + (1-\delta)k_t, \quad k_0 = \hat{k}_0.$$

Dynamic Optimization: General Theory

- There's a general theory for solving these types of problems
  - let's first work out more general theory
  - then apply to growth model
  - purpose: teach you some tools that are also applicable for solving other models
- In general will encounter two different formulations of dynamic optimization problems
  1. control-state formulation
  2. state-only formulation
- see previous slide, return to this momentarily

Dynamic Optimization: General Theory

Control-State Formulation

- Recall discussion of two formulations
  - do state-control formulation first
  - then do state-only formulation

- Pretty much all deterministic optimal control problems in discrete time can be written as

$$V(\hat{x}_0) = \max_{\{z_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t h(x_t, z_t)$$

subject to the law of motion for the state

$$x_{t+1} = g(x_t, z_t) \text{ and } z_t \in Z, \quad x_0 = \hat{x}_0.$$

- $\beta \in (0,1)$: discount factor
- $x \in X \subseteq \mathbb{R}^m$: state vector
- $z \in Z \subseteq \mathbb{R}^k$: control vector
- $h : X \times Z \to \mathbb{R}$: instantaneous return function

Example: Growth Model

$$V(\hat{k}_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$c_t \geq 0, \quad k_{t+1} \geq 0, \quad k_0 = \hat{k}_0.$$

- Here the state is $x_t = k_t$ and the control $z_t = c_t$
- $h(x, z) = u(z)$
- $g(x, z) = f(x) + (1-\delta)x - z$

Dynamic Optimization: General Theory

State-only Formulation

- Alternatively, can write the same problem in terms of states only

$$V(\hat{x}_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \quad \text{s.t.}$$
$$x_{t+1} \in \Gamma(x_t), \quad x_0 = \hat{x}_0.$$

- $\beta \in (0,1)$: discount factor
- $x \in X \subseteq \mathbb{R}^m$: state vector
- $U : X \times X \to \mathbb{R}$: instantaneous return function
- $\Gamma : X \to X$: correspondence describing feasible values for state

Example: Growth Model

$$V(\hat{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \quad \text{s.t.}$$

$$k_{t+1} \in [0, f(k_t) + (1-\delta)k_t], \quad k_0 = \hat{k}_0.$$

- Here the state is $x_t = k_t$
- $U(x, y) = f(x) + (1-\delta)x - y$
- $\Gamma(x) = [0, f(x) + (1-\delta)x]$

Dynamic Optimization: General Properties

- **Existence** of a solution
  - Extreme Value Theorem (or "Weierstrass Theorem"): continuous function on compact set has a maximum
- Satisfied in growth model?
  - objective continuous? Yes
  - constraint set compact? Yes. Result: there exists a "maximum maintainable capital stock" $\hat{k}$ s.t. $k_t > \hat{k} \Rightarrow k_{t+1} - k_t < 0$, and we can restrict attention to $k_t \in [0, \hat{k}]$.
    - Inada conditions $\Rightarrow f'(k_t) - \delta < 0$ for k_t large enough $\Rightarrow$ there exists $\hat{k}$ satisfying $0 = f(\hat{k}) - \delta\hat{k}$ and $f(k_t) - \delta k_t < 0, k_t > \hat{k}$
    - $k_t > \hat{k} \Rightarrow k_{t+1} - k_t = f(k_t) - \delta k_t - c_t \leq f(k_t) - \delta k_t < 0$
  - $\Rightarrow$ in growth model, there exists an optimal $\{k_{t+1}\}_{t=0}^{\infty}$
- **Uniqueness** of a solution
  - strictly concave objective & convex constraint set $\Rightarrow$ unique solution
- Satisfied in growth model? Yes

Overview: Solution Methods

- There are different methods for solving dynamic optimization problems
  - not only deterministic ones ...
  - ... but also stochastic ones (= with uncertainty)
  - Table provides an overview of different solution methods

	Discrete	Time	Continuous	Time
	sequence	recursive	sequence	recursive
deterministic	"classical"	Bellman eqn	Hamiltonian	HJB eqn
stochastic		Bellman eqn		HJB eqn

- blue = this class (first six weeks of 503)
- recursive approach also called "dynamic programming"
- blank box
  - **can** solve stochastic problems using sequence formulation...
  - ... but recursive/dynamic programming approach strictly

Classical Solution Method of Sequence Pb.

- Recall general dynamic optimization problem

$$V(\hat{x}_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \quad \text{s.t.} \tag{P}$$
$$x_{t+1} \in \Gamma(x_t), \quad x_0 = \hat{x}_0.$$

- The following are necessary and sufficient conditions for $\{x_{t+1}\}_{t=0}^{\infty}$ to be optimal: if x_{t+1} is in the interior of $\Gamma(x_t)$

$$U_y(x_t, x_{t+1}) + \beta U_x(x_{t+1}, x_{t+2}) = 0, \quad \forall t \tag{EE}$$

$$\lim_{T\to\infty} \beta^T U_y(x_T, x_{T+1}) \cdot x_{T+1} = 0 \tag{TC}$$

and $x_0 = \hat{x}_0$.

- (EE) together with (TC) and initial condition $x_0 = \hat{x}_0$ fully characterizes optimal $\{x_{t+1}\}_{t=0}^{\infty}$

Derivation/Interpretation

- (EE) is called "Euler equation"
  - simply first-order condition (FOC) w/ respect to x_{t+1}
  - derivation: differentiate problem (P) with respect to x_{t+1}
  - "Euler equation" simply means "intertemporal FOC"
- (TC) is called "transversality condition"
  - understanding it is harder than (EE), let's postpone this for now and revisit in a few slides
  - Note: some books (e.g. Stokey-Lucas-Prescott) write (TC) as

$$\lim_{T\to\infty} \beta^T U_x(x_T, x_{T+1}) \cdot x_T = 0 \qquad \text{(TC2)}$$

  - To see that (TC2) is equivalent to (TC), substitute (EE) into (TC), and evaluate at T rather than $T+1$

Example: Growth Model

- Recall social planner's problem in growth model

$$V(\hat{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \quad \text{s.t.} \tag{P'}$$

$$k_{t+1} \in [0, f(k_t) + (1-\delta)k_t], \quad k_0 = \hat{k}_0.$$

- (EE) and (TC) are

$$\begin{aligned} & -u'(f(k_t) + (1-\delta)k_t - k_{t+1}) && \text{(EE')} \\ & + \beta u'(f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2})(f'(k_{t+1}) + 1 - \delta) = 0 \end{aligned}$$

$$\lim_{T \to \infty} \beta^T u'(f(k_T) + (1-\delta)k_T - k_{T+1})k_{T+1} = 0 \tag{TC'}$$

- Get (EE') simply by differentiating (P') w.r.t. k_{t+1} (or by applying formula on previous slide)

Example: Growth Model

- (EE') can be written more intuitively as

$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{MRS} = \underbrace{f'(k_{t+1}) + 1 - \delta}_{MRT}$$

MRS between c_t and c_{t+1} = MRT between c_t and c_{t+1}

- Same logic as in static utility maximization problems, e.g.

$$\max_{c_A, c_B} u(c_A, c_B) \quad \text{s.t.} \quad c_A = f(\ell_A), \quad c_B = f(\ell_B), \quad \ell_A + \ell_B \leq 1$$

where A=apples, B=bananas

$$\Rightarrow \quad \frac{u_{c_A}(c_A, c_B)}{u_{c_B}(c_A, c_B)} = \frac{f'(\ell_A)}{f'(\ell_B)}$$

- growth model: different dates = different goods

Example: Growth Model

- Summarizing all necessary conditions

$$u'(c_t) = \beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta)$$
$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \qquad \text{(DE)}$$

for all t, with

$$k_0 = \hat{k}_0$$
$$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0 \qquad \text{(TC')}$$

- (DE) is system of two difference equations in (c_t, k_t) ...
- ... needs two boundary conditions
  1. initial condition for capital stock: $k_0 = \hat{k}_0$
  2. transversality condition, **plays role of boundary condition**

Where does (TC) come from?

- Transversality condition is a bit mysterious
- Best treatments are in various papers by Kamihigashi
  - most intuitive "Transversality Conditions and Dynamic Economic Behavior," New Palgrave Dict. of Economics, 2008 `http://www.dictionaryofeconomics.com/download/pde2008_T000217.pdf`
  - "A simple proof of the necessity of the transversality condition," Economic Theory, 2002
  - "Necessity of transversality conditions for infinite horizon problems," Econometrica, 2001
- Next slide: intuitive but "fake" derivation from finite horizon problem
- Afterwards: necessity proof from Kamihigashi (2002)

Where does (TC) come from?

- Consider finite horizon problem:

$$V(\hat{k}_0, T) = \max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t) \quad \text{s.t.}$$

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \quad k_{t+1} \geq 0.$$

- Lagrangean

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t (f(k_t) + (1-\delta)k_t - c_t - k_{t+1}) + \sum_{t=0}^{T} \mu_t k_{t+1}$$

- Necessary conditions at $t = T$

$$\begin{aligned} \beta^T u'(c_T) &= \lambda_T \\ \lambda_T &= \mu_T \qquad \Rightarrow \quad \beta^T u'(c_T) k_{T+1} = 0 \\ \mu_T k_{T+1} &= 0 \end{aligned}$$

Where does (TC) come from?

- From previous slide: in finite horizon problem

$$\beta^T u'(c_T) k_{T+1} = 0 \tag{$*$}$$

- $(*)$ is really two conditions in one
  1. $\beta^T u'(c_T) > 0$: need $k_{T+1} = 0$
  2. $\beta^T u'(c_T) = 0$: k_{T+1} can be > 0
- Intuition for case 1: if my marginal utility of consumption at T is positive, I want to eat up all my wealth before I die
- (TC) is same condition as $(*)$ in economy with $T \to \infty$

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0 \tag{TC}$$

- Intuition:
  - capital should not grow too fast compared to marginal utility
  - e.g. with $u(c) = \log c$: $\beta^T k_{T+1}/c_{T+1} \to 0$
  - if I save too much/spend too little, I'm not behaving optimally
- (TC) rules out **overaccumulation** of wealth

Proof of Necessity of (TC)

Kamihigashi (2002)

- Consider general optimal control problem

$$V(\hat{x}_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \quad \text{s.t.} \tag{P}$$
$$x_{t+1} \in \Gamma(x_t), \quad x_0 = \hat{x}_0.$$

- **Assumptions:**
  1. $x_t \in X \subset \mathbb{R}_+^m$ (i.e. $x_t \geq 0$)
  2. $Gr(\Gamma) = \{(y, x) : x \in X, y \in \Gamma(x)\}$ is convex, $(0, 0) \in Gr(\Gamma)$
  3. $U : Gr(\Gamma) \to \mathbb{R}$ is C^1 and concave
  4. $\forall (x, y) \in Gr(\Gamma), U_y(x, y) \leq 0$
  5. For any feasible path $\{x_t\}$ $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t U(x_t, x_{t+1})$ exists (i.e. it is bounded)
- (TC) can also be derived under weaker assumptions. But above assumptions yield straightforward proof.

Proof of Necessity of (TC)

Kamihigashi (2002)

- **Definition:** A feasible path $\{x_t^*\}$ is **optimal** if

$$\sum_{t=0}^{\infty} \beta^t U(x_t^*, x_{t+1}^*) \geq \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1})$$

  for any feasible path $\{x_t\}$

- i.e. $\{x_t^*\}$ attains the maximum of (P)

- **Theorem:** Under Assumptions 1 to 5, for any **interior** optimal path $\{x_t^*\}$

$$\lim_{T \to \infty} \beta^T U_y(x_T^*, x_{T+1}^*) \cdot x_{T+1}^* = 0$$

Proof of Necessity of (TC)

Kamihigashi (2002)

- **Useful preliminary fact:** Let $f : [0,1] \to \mathbb{R}$ be a **concave function** with $f(1) > -\infty$. Then

$$\frac{f(1) - f(\lambda)}{1 - \lambda} \leq f(1) - f(0) \qquad (*)$$

- Kamihigashi calls this a Lemma, not sure it deserves the name
- Follows immediately from definition of a concave function:

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y) \quad \forall\ 0 \leq \lambda \leq 1,\ x, y$$

- Letting $x = 1$ and $y = 0$

$$f(1) - f(\lambda) \leq f(1) - (\lambda f(1) + (1-\lambda) f(0))$$

- Rearranging yields $(*)$

Proof of Necessity of (TC)

Kamihigashi (2002)

- Let x_t^* be an interior optimal path. Consider alternative path

$$\{x_0^*, x_1^*, ..., x_T^*, \lambda x_{T+1}^*, \lambda x_{T+2}^*, ...\}, \quad \lambda \in [0,1)$$

- path is feasible by interiority and convexity of constraint set
- By optimality

$$\beta^T[U(x_T^*, \lambda x_{T+1}^*) - U(x_T^*, x_{T+1}^*)] + \sum_{t=T+1}^{\infty} \beta^t \left[U(\lambda x_t^*, \lambda x_{t+1}^*) - U(x_t^*, x_{t+1}^*)\right] \leq 0$$

- Dividing through by $1 - \lambda$

$$\beta^T \frac{U(x_T^*, \lambda x_{T+1}^*) - U(x_T^*, x_{T+1}^*)}{1-\lambda} \leq \sum_{t=T+1}^{\infty} \beta^t \frac{U(x_t^*, x_{t+1}^*) - U(\lambda x_t^*, \lambda x_{t+1}^*)}{1-\lambda}$$

$$\leq \sum_{t=T+1}^{\infty} \beta^t [U(x_t^*, x_{t+1}^*) - U(0,0)]$$

where the last inequality follows from A3 (concavity of U) and $(*)$

Proof of Necessity of (TC)

Kamihigashi (2002)

- Applying $\lim_{\lambda \to 1}$ to the LHS

$$0 \leq -\beta^T U_y(x_T^*, x_{T+1}^*) \cdot x_{T+1}^* \leq \sum_{t=T+1}^{\infty} \beta^t [U(x_t^*, x_{t+1}^*) - U(0,0)]$$

where the first inequality follows from A4 ($U_y(x, y) \leq 0$) and A1 ($x_t \geq 0$)

- Applying $\lim_{T \to \infty}$ to both sides

$$\begin{aligned} 0 &\leq -\lim_{T \to \infty} \beta^T U_y(x_T^*, x_{T+1}^*) \cdot x_{T+1}^* \\ &\leq \lim_{T \to \infty} \sum_{t=T+1}^{\infty} \beta^t [U(x_t^*, x_{t+1}^*) - U(0,0)] = 0 \end{aligned}$$

where the equality follows from A5 (boundedness)

- (TC) now follows.□

(TC) in Practice

- In practice, often don't have to impose (TC) exactly
- Instead, just have to make sure trajectories "don't blow up."
- E.g. consider growth model: since $\beta < 1$, easy to see that

$$\lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} = 0$$

whenever

$$\lim_{T \to \infty} c_T = c^*, \quad \lim_{T \to \infty} k_T = k^*$$

with $0 < c^*, k^* < \infty$ which is satisfied if $\{c_t, k_t\}$ converge to steady state.

Steady State

- **Definition:** a steady state is a point in the state space x^* such that $x_0 = x^*$ implies $x_t = x^*$ for all $t \geq 1$. ("if you start there, you stay there")
- Steady state in general model: any $x^* \in X$ such that

$$U_y(x^*, x^*) + \beta U_x(x^*, x^*) = 0$$

- Steady state in growth model: (c^*, k^*) satisfying

$$\begin{aligned} 1 &= \beta(f'(k^*) + 1 - \delta) \\ c^* &= f(k^*) - \delta k^* \end{aligned} \qquad (*)$$

  comes from (DE) with $c_{t+1} = c_t = c^*$ and $k_{t+1} = k_t = k^*$
- For example, if $f(k) = Ak^\alpha, \alpha < 1$. Then

$$k^* = \left(\frac{\alpha A}{\beta^{-1} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}$$

Dynamics

- What else can we say about dynamics of $\{c_t\}_{t=0}^{\infty}$ and $\{k_{t+1}\}_{t=0}^{\infty}$?
- Turns out answering this is easier in continuous time
    - phase diagram
    - can also do discrete-time phase diagram, but a bit awkward
    - so rather do it properly
- See next lecture