Lecture 1: Overview
Hamiltonians and Phase Diagrams

ECO 521: Advanced Macroeconomics I

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Course Overview

Two Parts:

(1) Tools: continuous time methods in macroeconomics

(2) Substance: stochastic models of distribution and growth

- Everything is flexible, feedback very useful!

Substance: Where I'm going

- Stochastic models of distribution and growth

  (or "heterogeneous agent models" or "micro to macro")
- Want to get you started on building these kind of models
- Why should you be interested in this?
  - Fertile area of research, excellent dissertation topics!
  - Many open questions
  - Hard – high entry barriers

Substance: Where I'm going

- Some questions we will try to answer:
  - Where does the firm size distribution come from?
  - Where do income and wealth distribution come from?
  - How important is firm heterogeneity and reallocation for the aggregate economy?
  - What, if any, are the interactions between the aggregate economy and income and wealth distribution?

Substance: Where I'm going

- Macro theories can broadly be classified as follows:
  (1) Models with aggregate shocks (e.g. RBC, New Keynesian): very well developed
  (2) Models with idiosyncratic shocks (Aiyagari-Bewley-Hugget): relatively well developed numerically though not theoretically
  (3) Models with both idiosyncratic and aggregate shocks (Krusell-Smith): underdeveloped KS on accuracy of their algorithm
- One of the big unanswered questions in macro: how to get a tractable micro to macro model ((2) and especially (3))
- This course: briefly do (1), spend most of the time on (2), not much on (3)
- Main reason for teaching you this continuous time stuff: think it's incredibly useful for building better theories of distribution and growth ((2) and (3))

Plan of Lecture

(1) Hamiltonians

(2) Phase diagrams

(3) Finite difference methods and shooting algorithm

Hamiltonians

- Pretty much all deterministic optimal control problems in continuous time can be written as

$$V(x_0) = \max_{u(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h(x(t), u(t))\, dt$$

subject to the law of motion for the state

$$\dot{x}(t) = g(x(t), u(t)) \text{ and } u(t) \in U$$

for $t \geq 0,\ x(0) = x_0$ given.

- $\rho \geq 0$: discount rate
- $x \in X \subseteq \mathbb{R}^m$: state vector
- $u \in U \subseteq \mathbb{R}^n$: control vector
- $h : X \times U \to \mathbb{R}$: instantaneous return function

Example: Neoclassical Growth Model

$$V(k_0) = \max_{c(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} U(c(t)) dt$$

subject to

$$\dot{k}(t) = F(k(t)) - \delta k(t) - c(t)$$

for $t \geq 0$, $k(0) = k_0$ given.

- Comes from

$$\dot{k} = i - \delta k, \quad c + i = F(k)$$

- Here the state is $x = k$ and the control $u = c$
- $h(x, u) = U(u)$
- $g(x, u) = F(x) - \delta x - u$

Hamiltonian: General Formulation

- Consider the general optimal control problem two slides back.
- Can obtain necessary and sufficient conditions for an optimum using the following procedure ("cookbook")
- Current-value Hamiltonian

$$\mathcal{H}(x, u, \lambda) = h(x, u) + \lambda g(x, u).$$

- $\lambda \in \mathbb{R}^m$: "co-state"

Hamiltonian: General Formulation

- Necessary and sufficient conditions:

$$H_u(x(t), u(t), \lambda(t)) = 0$$
$$\dot{\lambda}(t) = \rho\lambda(t) - H_x(x(t), u(t), \lambda(t))$$
$$\dot{x}(t) = g(x(t), u(t))$$

  for all $t \geq 0$.

- Initial value for state variable(s): $x(0) = x_0$.
- Boundary condition for co-state variable(s) $\lambda(t)$, called "Transversality condition"

$$\lim_{T\to\infty} e^{-\rho T}\lambda(T)\,x(T) = 0.$$

- Note: initial value of the co-state variable $\lambda(0)$ not predetermined.

Example: Neoclassical Growth Model

- Recall: $h(x, u) = U(x)$ and $g(x, u) = F(k) - \delta k - c$
- Using the "cookbook"

$$\mathcal{H}(k, c, \lambda) = U(c) + \lambda[F(k) - \delta k - c]$$

- We have

$$\mathcal{H}_c(k, c, \lambda) = U'(c) - \lambda$$
$$\mathcal{H}_k(k, c, \lambda) = \lambda(F'(k) - \delta)$$

- Therefore conditions for optimum are:

$$\dot{\lambda} = \lambda(\rho + \delta - F'(k))$$
$$\dot{k} = F(k) - \delta k - c \qquad \text{(ODE)}$$
$$U'(c) = \lambda$$

with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T}\lambda(T)k(T) = 0$.

Example: Neoclassical Growth Model

- Interpretation: continuous time Euler equation
- In discrete time

$$\lambda_t = \beta \lambda_{t+1}(F'(k_{t+1}) + 1 - \delta)$$
$$k_{t+1} = F(k_t) + (1 - \delta)k_t - c_t$$
$$U'(c_t) = \lambda_t$$

- (ODE) is continous-time analogue

Phase Diagrams

- How analyze (ODE)? In one-dimensional case (scalar x): use phase-diagram
- Two possible phase-diagrams:
  (i) in (λ, k)-space: more general strategy.
  (ii) in (c, k)-space: nicer in terms of the economics.
- For (i), use $U'(c) = \lambda$ or $c = (U')^{-1}(\lambda)$ to write (ODE) as

$$\begin{aligned} \dot{\lambda} &= \lambda(\rho + \delta - F'(k)) \\ \dot{k} &= F(k) - \delta k - (U')^{-1}(\lambda) \end{aligned} \qquad \text{(ODE')}$$

  with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T}\lambda(T)k(T) = 0$.
- Homework 1: draw phase-diagram in (λ, k)-space.

Phase Diagrams

- For (ii), assume CRRA utility

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

- Not necessary but makes algebra easier.

$$c^{-\sigma} = \lambda \quad \Rightarrow \quad -\sigma \log c(t) = \log \lambda(t) \quad \Rightarrow \quad -\sigma \frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda}$$

- Therefore write (ODE) as

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\sigma}(F'(k) - \rho - \delta) \\ \dot{k} &= F(k) - \delta k - c \end{aligned} \qquad \text{(ODE'')}$$

with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T} c(T)^{-\sigma} k(T) = 0$.

Steady State

- In steady state $\dot{k} = \dot{c} = 0$. Therefore

$$F'(k^*) = \rho + \delta$$

$$c^* = F(k^*) - \delta k^*$$

- Same as in discrete time with $\beta = 1/(1+\rho)$.
- For example, if $F(k) = Ak^\alpha, \alpha < 1$. Then

$$k^* = \left(\frac{\alpha A}{\rho + \delta}\right)^{\frac{1}{1-\alpha}}$$

Phase Diagram

- See graph that I drew in lecture by hand or Figure 8.1 in Acemoglu's textbook.
- Obtain saddle path.
- Prove stability of steady state.
- Important: saddle path is **not** a "knife edge" case in the sense that the system only converges to steady state if $(c(0), k(0))$ happens to lie on the saddle path and diverges for all other initial conditions.
- In contrast to the state variable $k(t)$, $c(t)$ is a "jump variable." That is, $c(0)$ is free and **always** adjusts so as to lie on the saddle path.

Numerical Solution: Finite-Difference Methods

- By far the simplest and most transparent method for numerically solving differential equations.
- Approximate $k(t)$ and $c(t)$ at N discrete points in the time dimension, $t^n, n = 1, ..., N$. Denote distance between grid points by Δt.
- Use short-hand notation $k^n = k(t^n)$.
- Approximate derivatives

$$\dot{k}(t^n) \approx \frac{k^{n+1} - k^n}{\Delta t}$$

- Approximate (ODE") as

$$\frac{c^{n+1} - c^n}{\Delta t} \frac{1}{c^n} = \frac{1}{\sigma}(F'(k^n) - \rho - \delta)$$
$$\frac{k^{n+1} - k^n}{\Delta t} = F(k^n) - \delta k^n - c^n$$

Finite-Difference Methods/Shooting Algorithm

- Or
$$
\begin{aligned}
c^{n+1} &= \Delta t c^n \frac{1}{\sigma}(F'(k^n) - \rho - \delta) + c^n \\
k^{n+1} &= \Delta t (F(k^n) - \delta k^n - c^n) + k^n
\end{aligned}
\tag{FD}
$$
with $k^0 = k_0$ given.
- Homework 2: draw phase diagram/saddle path in MATLAB.
- Assume $F(k) = Ak^{\alpha}$, $A = 1$, $\alpha = 0.3$, $\sigma = 2$, $\rho = \delta = 0.05$, $k_0 = \frac{1}{2}k^*$, $\Delta t = 0.1$, $N = 700$.
- Algorithm:
  - (i) guess c^0
  - (ii) obtain $(c^n, k^n), n = 1, ..., N$ by running (FD) forward in time.
  - (iii) If the sequence converges to (c^*, k^*), then you have obtained the correct saddle path. If not, back to (i) and try different c^0.
- This is called a "shooting algorithm"

Krusell-Smith on Accuracy of their Algorithm ▸ Back

- p.897: *"Like most numerical procedures, the present one does not provide bounds on how far the approximate equilibrium deviates from an exact equilibrium. In particular, one might imagine that there are self-fulfilling approximate equilibria: because agents perceive a simple law of motion, they behave accordingly."*
- Another problem: this R^2 business. See
  - Den Haan, Wouter (2010), "Assessing the Accuracy of the Aggregate Law of Motion in Models with Heterogeneous Agents"