Lecture 10

Firm Heterogeneity, Distribution and Dynamics
Stopping Time Problems

Distributional Macroeconomics

Part II of ECON 2149

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Plan

Firm heterogeneity, distribution and dynamics

1. motivating facts
2. workhorse model of firm dynamics: Hopenhayn (1992)
3. stopping time problems
4. Luttmer (2007)

Motivating Facts

- So far: income and wealth distribution in macroeconomics
- Firm size distribution shares many similarities with income, wealth distributions
  - extremely skewed
  - lots of heterogeneity conditional on other observables
  - e.g. Chad Syverson: within typical 4-digit SIC industries 90th percentile firm is twice as productive as 10th percentile firm
  - other key references: work by John Haltiwanger, Steve Davis and co-authors
- Tools for theoretically modeling heterogeneous firms are exactly the same as those for modeling heterogeneous individuals
  - state variable = cross-sectional distribution
  - key ideas: stationary distribution & distributional dynamics

Firm Size Distribution: Very Skewed and Fat Right Tail

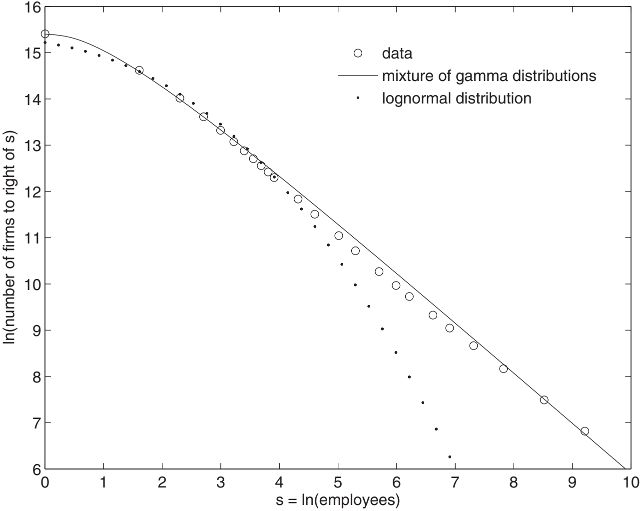


FIGURE I
Size Distribution of U. S. Firms in 2002

Workhorse Model: Hopenhayn (1992)

- Will present my own version
  - notes: `http://www.princeton.edu/~moll/HACTproject/hopenhayn.pdf`
  - code: `http://www.princeton.edu/~moll/HACTproject/hopenhayn.m`
- For some good, concise lecture notes on original see
  `https://web.stanford.edu/~jdlevin/Econ%20257/Industry%20Dynamics.pdf`
  Also good discussion of Jovanovic 82, Olley-Pakes 96
- Before I forget, potentially confusing notation in Hopenhayn 92
  - p.1130: “the total mass $M_t = \mu_t(S)$”
  - p.1132:“Let M_t denote the mass of entrants in period t”
  - latter is one that’s used throughout
- Only dynamic decisions in Hopenhayn model: entry and exit
- Will walk you through two versions
  1. mechanical entry (= assumption in Luttmer: “return process”)
  2. optimal entry (= assumption in Hopenhayn)

Hopenhayn Model with Mechanical Entry

- Continuum of firms, heterogeneous in productivity $z \in [0,1]$, solve

$$v(z) = \max_{\{n_t\}_{t\geq 0},\tau} \mathbb{E}_0 \left[\int_0^\tau e^{-\rho t}(pf(z_t, n_t) - wn_t - c_f)dt + e^{-\rho\tau} v^* \right]$$

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t, \quad z_0 = z.$$

  - n: employment, w: wage rate
  - $f(z,n)$: production, p: price of final goods
  - c_f: per-period operating cost, v^*: scrap value

- Assumption: for each exiting firm, new entrant with $z_0 \sim \psi(z)$
  - $\Rightarrow$ mass of active firms constant, normalize to 1
  - assume lowest z in support of ψ s.t. don't immediately exit

- Equilibrium: exogenous product demand, labor supply to industry

$$p = D(Q), \quad w = W(N), \quad Q := \int_0^1 q(z)g(z)dz, \quad N := \int_0^1 n(z)g(z)dz$$

Write this more compactly

- Continuum of firms, heterogeneous in productivity $z \in [0, 1]$, solve

$$v(z) = \max_{\tau} \mathbb{E}_0 \left[\int_0^{\tau} e^{-\rho t} \pi(z_t) dt + e^{-\rho \tau} v^* \right]$$

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t, \quad z_0 = z,$$

$$\pi(z) = \max_{n} \{pf(z, n) - wn\} - c_f$$

- Assumption: for each exiting firm, new entrant with $z_0 \sim \psi(z)$
  - $\Rightarrow$ mass of active firms constant, normalize to 1
  - assume lowest z in support of ψ s.t. don't immediately exit

- Equilibrium: exogenous product demand, labor supply to industry

$$p = D(Q), \quad w = W(N), \quad Q := \int_0^1 q(z)g(z)dz, \quad N := \int_0^1 n(z)g(z)dz$$

Hopenhayn Model with Optimal Entry

- Continuum of firms, heterogeneous in productivity $z \in [0, 1]$, solve

$$v(z) = \max_{\tau} \mathbb{E}_0 \left[\int_0^{\tau} e^{-\rho t} \pi(z_t) dt + e^{-\rho \tau} v^* \right]$$

$$dz_t = \mu(z_t) dt + \sigma(z_t) dW_t, \quad z_0 = z,$$

$$\pi(z) = \max_{n} \{ pf(z, n) - wn \} - c_f$$

- Previous slide: flow of entrants determined mechanically

- Now: flow of entrants satisfies free entry condition

$$\int_0^1 v(z) \psi(z) dz = c_e$$

- $\Rightarrow$ total mass of firms endogenous, cannot normalize it to one

3-Slide Discussion of Hopenhayn (1992)

Stopping Time Problems

Stopping Time Problems

- In lots of problems in economics, agents have to choose an optimal stopping time
- Quite often these problems entail some form of non-convexity
- Examples:
  - how long should a low productivity firm wait before it exits an industry?
  - how long should a firm wait before it resets its prices?
  - when should you exercise an option?
  - etc... Stokey's book is all about these kind of problems
- These problems are very awkward in discrete time because you run into integer problems
- Big payoff from working in continuous time
- Next: flexible algorithm for solving such problems, also works if don't have simple threshold rules and with states > 1

Exercising an Option: Deterministic Warmup

- Problem from chapter 6 of Stokey's "Economics of Inaction"
- Plant has profits
$$\pi(z(t))$$
- $z(t)$: state variable = stand in for demand, plant capacity etc
$$z(t) = z_0 + \mu t \quad \Leftrightarrow \quad \dot{z}(t) = \mu$$
- Can shut down plant at any time, get scrap value S, but cannot reopen
- Problem: choose stopping time τ to solve
$$v(z_0) = \max_{\tau \geq 0} \left[\int_0^\tau e^{-rt} \pi(z(t)) dt + e^{-r\tau} S \right]$$
- Assumptions to make sure $\tau^* < \infty$:
$$\pi'(z) > 0, \quad \mu < 0, \quad \lim_{z \to -\infty} \pi(z) < rS < \lim_{z \to +\infty} \pi(z)$$

Exercising an Option: Deterministic Warmup

- FOC

$$e^{-\rho\tau^*}[\pi(z(\tau^*)) - rS] \leq 0, \quad \text{with equality if } \tau^* > 0$$

- Can write this in terms of cutoff $b^* = z(\tau^*)$

$$\pi(b^*) = rS$$

- Optimal stopping time is

$$\tau^* = \begin{cases} 0, & \text{if } z < b^*, \\ (b^* - z)/\mu, & \text{if } z \geq b^* \end{cases}$$

Exercising an Option: Stochastic Problem

- Problem: choose stopping time τ to solve

$$v(z) = \max_{\tau \geq 0} \mathbb{E}_0 \left[\int_0^\tau e^{-\rho t} \pi(z_t) dt + e^{-\rho \tau} S(z_\tau) \right]$$

$$dz_t = \mu(z_t) dt + \sigma(z_t) dW_t, \quad z_0 = z$$

- Same assumptions as before to ensure $\tau^* < \infty$
- Analytic solution if $\mu(z) = \bar{\mu}, \sigma(z) = \bar{\sigma}, S(z) = \bar{S}$, but not in general
- Two approaches for tackling this problem
  1. standard approach: "smooth pasting"
  2. more powerful approach: HJB "Variational Inequality"
- Discuss these in turn

Exercising an Option: Standard Approach

- Assume scrap value is independent of z: $S(z) = \bar{S}$
- Optimal policy = threshold rule: exit if z_t falls below b
- Standard approach (see e.g. Stokey, Ch.6):

$$\rho v(z) = \pi(z) + \mu(z)v'(z) + \frac{\sigma^2(z)}{2}v''(z), \qquad z > b$$

  with "value matching" and "smooth pasting" at b:

$$v(b) = \bar{S}, \qquad v'(b) = 0$$

- Derivation? See Appendix
- But things more complicated if
  - S depends on z...
  - ... or if dimension > 1
- $\Rightarrow$ can't use threshold property
- want algorithm that works also in those cases

Exercising an Option: HJBVI Approach

- Denote $\mathcal{Z}$ = set of z such that don't exit:

$$z \in \mathcal{Z}: \quad v(z) \geq S(z), \quad \rho v(z) = \pi(z) + \mu(z)v'(z) + \frac{\sigma^2(z)}{2}v''(z)$$

$$z \notin \mathcal{Z}: \quad v(z) = S(z), \quad \rho v(z) \geq \pi(z) + \mu(z)v'(z) + \frac{\sigma^2(z)}{2}v''(z)$$

- Can write compactly as:

$$\min\left\{\rho v(z) - \pi(z) - \mu(z)v'(z) - \frac{\sigma^2(z)}{2}v''(z), v(z) - S(z)\right\} = 0 \quad (*)$$

- Note: have used that following two statements are equivalent
  1. for all z, either $f(z) \geq 0, g(z) = 0$ or $f(z) = 0, g(z) \geq 0$
  2. $\min\{f(z), g(z)\} = 0$ for all z
- $(*)$ is called "HJB variational inequality" (HJBVI)
- Important: did not impose smooth pasting
  - instead, it's a result: can prove that $(*)$ implies $v'(b) = S'(b)$
  - see e.g. Oksendal `http://th.if.uj.edu.pl/~gudowska/dydaktyka/Oksendal.pdf` (who calls "smooth pasting" "high contact (or smooth fit) principle")

Finite Difference Scheme for solving HJBVI

- Codes
  `http://www.princeton.edu/~moll/HACTproject/option_simple_LCP.m`,
  `http://www.mathworks.com/matlabcentral/fileexchange/20952`
- Main insight: discretized HJBVI = Linear Complementarity Problem (LCP) `https://en.wikipedia.org/wiki/Linear_complementarity_problem`
- Prototypical LCP: given matrix **B** and vector **q**, find **x** such that

$$
\begin{aligned}
\mathbf{x}^{\mathsf{T}}(\mathbf{B}\mathbf{x}+\mathbf{q}) &= 0 \\
\mathbf{x} &\geq 0 \\
\mathbf{B}\mathbf{x}+\mathbf{q} &\geq 0
\end{aligned}
$$

- There are many good LCP solvers in Matlab and other languages
- Best one I've found if **B** large but sparse (Newton-based):
  `http://www.mathworks.com/matlabcentral/fileexchange/20952`

Finite Difference Scheme for solving HJBVI

- Recall HJBVI

$$\min \left\{ \rho v(z) - \pi(z) - \mu(z) v'(z) - \frac{\sigma^2(z)}{2} v''(z), v(z) - S(z) \right\} = 0$$

- Without exit, discretize as

$$\rho v_i = \pi_i + \mu_i (v_i)' + \frac{\sigma_i^2}{2} (v_i)'' \qquad \Leftrightarrow \qquad \rho \mathbf{v} = \pi + \mathbf{A}\mathbf{v}$$

- With exit:

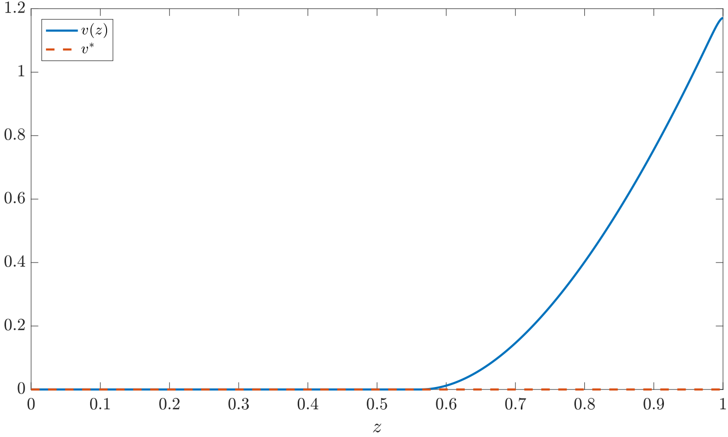
$$\min\{\rho \mathbf{v} - \pi - \mathbf{A}\mathbf{v}, \mathbf{v} - \mathbf{S}\} = 0$$

- Equivalently:

$$(\mathbf{v} - \mathbf{S})^{\mathrm{T}} (\rho \mathbf{v} - \pi - \mathbf{A}\mathbf{v}) = 0$$
$$\mathbf{v} \geq \mathbf{S}$$
$$\rho \mathbf{v} - \pi - \mathbf{A}\mathbf{v} \geq 0$$

- But this is just an LCP with $\mathbf{x} = \mathbf{v} - \mathbf{S}$, $\mathbf{B} = \rho \mathbf{I} - \mathbf{A}$, $\mathbf{q} = -\pi + \mathbf{B}$!!

The solution satisfies smooth pasting even though we didn't impose it!



An Impulse Control Problem: Buying & Selling a Car

- Flow utility $u(c_t) + \kappa d_t$, $d_t \in \{0,1\}$ (car or no car)
- Buy car at p_0, sell at p_1 with $p_1 < p_0$
- When not buying/selling, wealth accumulates in standard fashion

$$\dot{a}_t = y + ra_t - c_t$$

- Notation: $v_d(a)$ = value of wealth a, car ownership state $d \in \{0,1\}$
- Problem of individual without car: choose c_t and stopping time τ

$$v_0(a) = \max_{\{c_t\}_{t\geq 0}, \tau} \int_0^\tau e^{-\rho t} u(c_t) dt + e^{-\rho \tau} v_0^*(a_\tau)$$

$$\dot{a}_t = y + ra_t - c_t, \quad a_t \geq \underline{a}, \quad a_0 = a.$$

where $v_0^*(a)$ = value of buying car

$$v_0^*(a) = \begin{cases} v_1(a - p_0), & \text{if } a - p_0 \geq \underline{a} \\ -\infty, & \text{if } a - p_0 < \underline{a} \end{cases}$$

- Symmetric problem for individual with car, value $v_1(a)$

A Problem with an Indivisible Durable (a.k.a. a Car)

- System of HJBVI's

$$0 = \min\{\rho v_0(a) - \max_c \left\{u(c) + v_0'(a)(y + ra - c)\right\}, v_0(a) - v_0^*(a)\},$$
$$0 = \min\{\rho v_1(a) - \max_c \left\{u(c) + \kappa + v_1'(a)(y + ra - c)\right\}, v_1(a) - v_1^*(a)$$

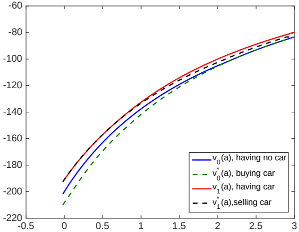
- Discretize as

$$0 = \min\{\rho v_0 - u(v_0) - \mathbf{A}(v_0)v_0, v_0 - v_0^*(v_1)\},$$
$$0 = \min\{\rho v_1 - u(v_1) + \kappa - \mathbf{A}(v_1)v_1, v_1 - v_1^*(v_0)\}$$

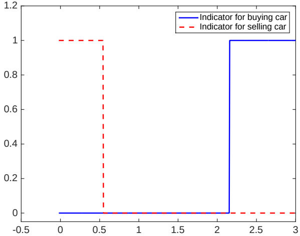
- Solve using LCP solver

- Code: `http://www.princeton.edu/~moll/HACTproject/car.m`

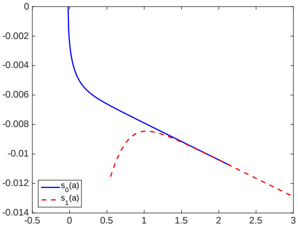
A Problem with an Indivisible Durable (a.k.a. a Car)



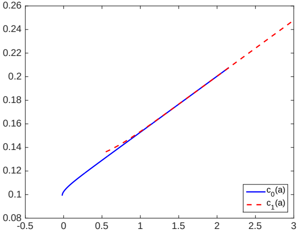
(a) Value Function



(b) Value Function



(c) Saving Policy Function



(d) Cons Policy Function

Numerical Solution of Hopenhayn Model

`http://www.princeton.edu/~moll/HACTproject/hopenhayn.pdf`

`http://www.princeton.edu/~moll/HACTproject/hopenhayn.m`

Hopenhayn Model with Mechanical Entry

- Write more compactly

$$v(z) = \max_{\tau} \mathbb{E}_0 \left\{ \int_0^{\tau} e^{-\rho t} \pi(z_t) dt + e^{-\rho \tau} v^* \right\}$$

$$dz_t = \mu(z_t) dt + \sigma(z_t) dW_t, \quad z_0 = z,$$

$$\pi(z) = \max_{n} \{ pf(z, n) - wn \} - c_f$$

- Assumption: for each exiting firm, new entrant with $z_0 \sim \psi(z)$
  - $\Rightarrow$ mass of active firms constant, normalize to 1
  - assume lowest z in support of ψ s.t. don't immediately exit

Equations for Stationary Equilibrium, Mechanical Entry

- Denote $\mathcal{Z}$ = inaction region, i.e. set of z's such that don't exit...
- ... and m = entry rate (by assumption also = exit rate)

$$0 = \min\left\{\rho v(z) - v'(z)\mu(z) - \frac{1}{2}v''(z)\sigma^2(z) - \pi(z), v(z) - v^*\right\}, \quad \text{all } z \in (0,1)$$

$$0 = -(\mu(z)g(z))' + \frac{1}{2}\left(\sigma^2(z)g(z)\right)'' + m\psi(z), \quad \text{all } z \in \mathcal{Z},$$

$$p = D(Q), \quad w = W(N), \quad Q = \int_{\mathcal{Z}} q(z)g(z)dz, \quad N = \int_{\mathcal{Z}} n(z)g(z)dz$$

- Remains to determine m, find it from $\int_{\mathcal{Z}} g(z,t)dz = 1$ for all t

$$\partial_t g = \mathcal{A}^* g + m(t)\psi(z) \quad \text{and} \quad \int_{\mathcal{Z}} \partial_t g(z,t)dz = 0$$

$$\Rightarrow \quad m = -\int_{\mathcal{Z}} (\mathcal{A}^* g)(z)dz$$

- If threshold rule (stay when $z \geq b$), then $m = -\frac{1}{2}\partial_z\left(\sigma^2(b)g(b)\right)$

Equations for Stationary Equilibrium with Optimal Entry

Now: Mass of entrants m pinned down by free entry condition

$$0 = \min \left\{ \rho v(z) - v'(z)\mu(z) - \frac{1}{2}v''(z)\sigma^2(z) - \pi(z), v(z) - v^* \right\}, \quad \text{all } z \in (0,1)$$

$$0 = -(\mu(z)g(z))' + \frac{1}{2}\left(\sigma^2(z)g(z)\right)'' + m\psi(z), \quad \text{all } z \in \mathcal{Z},$$

$$c_e = \int_0^1 v(z)\psi(z)dz$$

$$p = D(Q), \quad w = W(N), \quad Q = \int_{\mathcal{Z}} q(z)g(z)dz, \quad N = \int_{\mathcal{Z}} n(z)g(z)dz$$

Equations for Stationary Equilibrium with Optimal Entry

Free-entry condition not particularly well behaved numerically $\Rightarrow$ replace

$$0 = \min\left\{\rho v(z) - v'(z)\mu(z) - \frac{1}{2}v''(z)\sigma^2(z) - \pi(z), v(z) - v^*\right\}, \quad \text{all } z \in (0,1)$$

$$0 = -(\mu(z)g(z))' + \frac{1}{2}\left(\sigma^2(z)g(z)\right)'' + m\psi(z), \quad \text{all } z \in \mathcal{Z},$$

$$m = \bar{m}\exp\left(\eta\left(\int_0^1 v(z)\psi(z)dz - c_e\right)\right), \quad \eta, \bar{m} > 0$$

$$p = D(Q), \quad w = W(N), \quad Q = \int_{\mathcal{Z}} q(z)g(z)dz, \quad N = \int_{\mathcal{Z}} n(z)g(z)dz$$

- $\int_0^1 v(z)\psi(z)dz = c_e$ is special case $\eta \to \infty$
- to see this, write as $\frac{\log(m/\bar{m})}{\eta} = \int_0^1 v(z)\psi(z)dz - c_e$
- that is, Hopenhayn model has infinitely elastic supply of entrants

Discretization of KF equation

- Discretized KF equation is

$$0 = \sum_{j=1}^{I} A_{j,i} g_j + m\psi_i, \quad \text{all } i \in \mathcal{I}$$

$$g_i = 0, \quad \text{all } i \notin \mathcal{I}$$

- Write this in matrix notation as

$$0 = \widetilde{\mathbf{A}}^{\top}\mathbf{g} + m\psi$$

  - where $\widetilde{A}_{i,j} = A_{i,j}$ for all columns in inaction region $j \in \mathcal{I}$ ...
  - ... columns in exit region $j \notin \mathcal{I}$ are replaced by a column of zeros everywhere except for 1 on the diagonal
  - hence $0 = \widetilde{\mathbf{A}}^{\top}\mathbf{g} + m\psi$ implies that $g_i = 0$ for all $i \notin \mathcal{I}$

- $\Rightarrow$ non-singular $\widetilde{\mathbf{A}}^{\top}$ $\Rightarrow$ can simply solve (no eigenvalue problem)

$$\mathbf{g} = -(\widetilde{\mathbf{A}}^{\top})^{-1} m\psi$$

Solution Algorithm

`http://www.princeton.edu/~moll/HACTproject/hopenhayn.m`

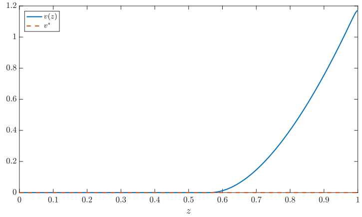
(i) Guess w^0

(ii)
1. Guess p^0
2. Given (p^j, w^k) solve the HJBVI equation. This yields v and exit region $\mathcal{Z}$
3. Given v, compute m from supply of entrants. To approximate perfectly elastic supply of entrants, set $\eta = 1,000$
4. Given exit region $\mathcal{Z}$, and entry rate m, solve KF equation to get g. Note that g will, in general, not integrate to one
5. Given g, compute Q & update p: $p^{j+1} = (1 - \lambda_p)p^j + \lambda_p Q^{-\varepsilon}$
6. If p^{j+1} and $Q^{-\varepsilon}$ are close enough, go to *iii*, otherwise back to 2

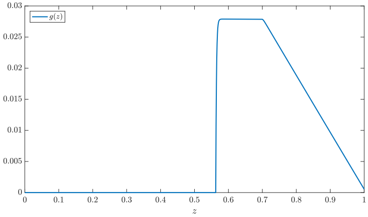
(iii) Given g, compute N & update w: $w^{k+1} = (1 - \lambda_w)w^{k+1} + \lambda_w N^{\phi}$

(iv) If w^{k+1} and N^{ϕ} are close enough, exit, otherwise back to *ii*

Results: Value Function and Size Distribution



(e) Value function $v(z)$



(f) Size distribution of active firms $g(z)$

Luttmer (2007) – Short Version

Luttmer (2007): Overview

- Firms are monopolistic competitors
- Permanent shocks to preferences and technologies associated with firms
- Low productivity firms exit, new firms imitate and attempt to enter
    - selection produces Pareto right tail rather than log-normal
    - population productivity grows faster than mean of incumbents
    - thickness of right tail depends on the difference
    - Zipf tail when entry costs are high or imitation is difficult

Luttmer (2007): Key Mechanism for Pareto Distribution

- Exactly same logic as in Gabaix, Gabaix-Lasry-Lions-Moll
- Logarithm of size s_t follows "return process"/"exit with reinjection"

$$ds_t = \mu dt + \sigma dW_t$$

  - assume $\mu < 0$
  - if s_t ever reaches b, exit and get reinjected at $x > b$
- $\Rightarrow$ exponential tail for log size s, Pareto tail for size e^s
- More precisely, a double-Pareto distribution
- Remaining model ingredients only make economics nicer, model less mechanical

Stationary Size Distribution, $s = \log$ size

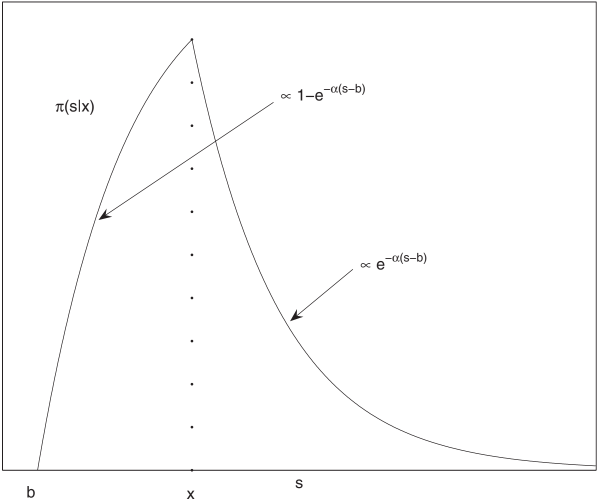


FIGURE II
Size Density Conditional on Initial Size

3-Slide Discussion of Luttmer

Appendix:
Smooth Pasting and All That

Deterministic Problem: HJB Approach

Claim (Stokey, Proposition 6.2): The value function, V, and optimal threshold, b^*, have the following properties:

(i) v satisfies the **HJB equation**

$$
\begin{aligned}
rV(z) &= \pi(z) + V'(z)\mu, \qquad && z \geq b^* \\
V(z) &= S, && z \leq b^*
\end{aligned}
$$

(ii) V is continuous at b^* (**value matching**)

$$\lim_{z \downarrow b^*} V(z) = S$$

(iii) V' is continuous at b^* (**smooth pasting**)

$$\lim_{z \downarrow b^*} V'(z) = 0$$

Intuitive Derivation

- Periods of length Δt,
- Value of a firm with $z_0 = z$:

$$V(z) = \max\{\tilde{V}(z), S\}$$

- S: value of exiting
- $\tilde{V}(z)$: value of staying in industry satisfying

$$\tilde{V}(z) = \pi(z)\Delta t + (1 - r\Delta t)V(z + \mu\Delta t)$$

Derivation: Value Matching $\lim_{z \downarrow b} V(z) = S$

- Consider some (not necessarily optimal) threshold b
- By definition of b:

$$V(z) = \begin{cases} \tilde{V}(z), & z > b \\ S, & z \leq b \end{cases}$$

(Note: could write $z \geq b$ and $z < b$, would need to slightly change argument below; just definition of b in any case.)

- Subtract $(1 - r\Delta t)\tilde{V}(z)$ from both sides and divide by Δt

$$r\tilde{V}(z) = \pi(z) + (1 - r\Delta t)\frac{V(z + \mu\Delta t) - \tilde{V}(z)}{\Delta t}$$

Derivation: Value Matching $\lim_{z \downarrow b} V(z) = S$

- Evaluate $\tilde{V}$ at $z = b - \mu \Delta t$, i.e. at an x just above the threshold (recall $\mu < 0$).

$$r\tilde{V}(b - \mu\Delta t) = \pi(b - \mu\Delta t) + (1 - r\Delta t)\frac{S - \tilde{V}(b - \mu\Delta t)}{\Delta t}$$

- Want to take $\Delta t \to 0$. Note:

$$\lim_{\Delta t \to 0} \tilde{V}(b - \mu\Delta t) = \lim_{z \downarrow b} \tilde{V}(z)$$

- **Proof** by contradiction. Suppose $\lim_{z \downarrow b} \tilde{V}(z) < S$.
  - then $\frac{S - \tilde{V}(b - \mu\Delta t)}{\Delta t} \to \infty$ and hence $r\tilde{V}(b - \mu\Delta t) \to \infty$.
  - but $\lim_{z \downarrow b} \tilde{V}(z) = \infty$ contradicts $\lim_{z \downarrow b} \tilde{V}(z) < S$.
- Symmetric argument for $\lim_{z \downarrow b} \tilde{V}(z) > S$
- Since $V(z) = \tilde{V}(z)$ for $z > b$, also $\lim_{z \downarrow b} V(z) = S$
- Note: this has to hold for any threshold b, also suboptimal ones. Continuous problems have continuous value functions.

Derivation: Smooth Pasting $\lim_{z \downarrow b^*} V'(z) = 0$

- Now consider the optimal threshold choice.
- The value of staying, $\tilde{V}$, satisfies the Bellman equation

$$\tilde{V}(z) = \pi(z)\Delta t + (1 - r\Delta t) \max\left\{\tilde{V}(z + \mu\Delta t), S\right\}$$

- Consider the **optimal** threshold b^*. If it is indeed optimal, then
  1. $\tilde{V}(b^*) = S$
  2. $\tilde{V}(b^* + \mu\Delta t) = S$ (recall that $\mu < 0$ and so $b^* + \mu\Delta t < b^*$)

  and therefore

$$\tilde{V}(b^*) = \pi(b^*)\Delta t + (1 - r\Delta t)S = S$$

  which implies

$$\pi(b^*) = rS \qquad (*)$$

- Observation 1: if we are indifferent between stopping or not, flow payoff from stopping must be same as flow payoff from continuing

Derivation: Smooth Pasting $\lim_{z \downarrow b^*} V'(z) = 0$

- Next, evaluating at $b^* - \mu\Delta t$

$$\tilde{V}(b^* - \mu\Delta t) = \pi(b^* - \mu\Delta t)\Delta t + (1 - r\Delta t)S$$

From value matching $\tilde{V}(b^*) = S$,

$$\tilde{V}(b^* - \mu\Delta t) - \tilde{V}(b^*) = \pi(b^* - \mu\Delta t)\Delta t - r\Delta t S$$

and hence

$$\frac{\tilde{V}(b^* - \mu\Delta t) - \tilde{V}(b^*)}{\Delta t} = \pi(b^* - \mu\Delta t) - rS$$

- Taking $\Delta t \to 0$ and using ($*$) $\Rightarrow$ smooth pasting $V'(b^*) = 0$
- Observation 2: If we are close to stopping we cannot be much better off than stopping now, given Observation 1

Deterministic Problem: Extensions

- Suppose the scrap value is $S(z)$ rather than S.
- And further that drift is $\mu(z)$ rather than μ
- Can use the same approach as above to show that
  - **Value Matching**:

$$\lim_{z \downarrow b^*} V(z) = S(b^*)$$

  - **Smooth Pasting**:

$$\lim_{z \downarrow b^*} V'(z) = S'(b^*)$$

Luttmer (2007) – Long Version

Luttmer (2007)

- Preferences:
  - differentiated commodities with permanent taste shocks
- Technologies:
  - at a cost, entrants draw technologies from some distribution
  - fixed overhead labor, asymptotic constant returns to scale
  - random productivity, quality growth.

Consumers

- A population $He^{\eta t}$ with preferences over per-capita consumption $C_t e^{-\eta t}$:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \frac{(C_t e^{-\eta t})^{1-\gamma}}{1-\gamma} dt$$

- where

$$C_t = \left[\int u^{1-\beta} c_t^\beta(u) dM_t(u) \right]^{1/\beta}$$

- Elasticity of substitution is $\sigma = 1/(1-\beta)$
- Demands

$$c_t(u,p) = \left(\frac{p}{P_t} \right)^{-1/(1-\beta)} uC_t$$

where

$$P_t = \left(\int up^{-\beta/(1-\beta)} dM_t(u) \right)^{-(1-\beta)/\beta}$$

Firms

- Firms indexed by age a and date of birth t.
- Calendar time = $t + a$
- Production function

$$y_{t,a} = z_{t,a} L_{t,a}$$

- Revenues

$$R_{t,a} = C_{t+a}^{1-\beta} (Z_{t,a} L_{t,a})^{\beta}, \quad Z_{t,a} \equiv (u_{t,a}^{1-\beta} z_{t,a}^{\beta})^{1/\beta}$$

- $Z_{t,a}$: combined quality and technology shock

Firms

- $Z_{t,a}$: combined quality and technology shock ("productivity") evolves according to

$$Z_{t,a} = Z \exp(\theta_E t + \theta_I a + \sigma_Z dW_{t,a})$$

- That is, $Z_{t,a}$ is a geometric Brownian motion

$$\frac{dZ_{t,a}}{Z_{t,a}} = \theta_E dt + \theta_I da + \sigma_Z dW_{t,a}, \quad Z_{0,0} = Z$$

- θ_E: growth of productivity of new firms
- θ_I: growth of productivity of incumbent firms
- $\theta_I - \theta_E$ is key parameter.

Firms

- Continuation requires λ_F units of labor per unit of time.
- Value of a firm:

$$V_t(Z) = \max_{L,\tau} \mathbb{E}_t \int_0^{\tau} e^{-ra}(R_{t,a} - w_{t+a}[L_{t,a} + \lambda_F])da$$

- τ: stopping time

Balanced Growth Path

- Will look for equilibria where a bunch of things are growing at a constant growth rate κ
- Aggregate labor supply: $H_t = He^{\eta t}$
- Number of firms: $M_t = Me^{\eta t}$
- Initial productivity $Z_{t,0} = Ze^{\theta_E t}$
- Total consumption $C_t = Ce^{\kappa t}$. Per capita $C_t e^{-\eta t} = Ce^{(\kappa-\eta)t}$.
- Revenues $R_{t,a} = C_{t+a}^{1-\beta}(Z_{t,a}L_{t,a})^{\beta}$ also grow at κ.
- Growth rate

$$\kappa = \theta_E + \left(\frac{1-\beta}{\beta}\right)\eta$$

Production Decisions along BGP

- Firms maximize variable profits $R_{t,a} - w_{t+a}L_{t,a}$. Solution:

$$R_{t,a} - w_{t+a}L_{t,a} = (1-\beta)\left(\frac{\beta Z_{t,a}}{w_{t+a}}\right)^{\beta/(1-\beta)} C_{t+a}$$

- Therefore total profits can be written as

$$R_{t,a} - w_{t+a}L_{t,a} - w_{t+a}\lambda_F = w_{t+a}\lambda_F(e^{s_a} - 1)$$

$$\text{where} \quad s_a \equiv S(Z) + \frac{\beta}{1-\beta}\left[\ln\left(\frac{Z_{t,a}}{Z_{t,0}} - \theta_E a\right)\right]$$

$$\text{and} \quad e^{S(Z)} \equiv \frac{1-\beta}{\lambda_F}\frac{C}{w}\left(\frac{\beta Z}{w}\right)^{\beta/(1-\beta)}$$

- s_a: firm size relative to fixed costs. This is a Brownian motion

$$ds_a = \mu da + \sigma dW_{t,a}$$

$$\text{where} \quad \mu \equiv \frac{\beta}{1-\beta}(\theta_I - \theta_E), \quad \sigma = \frac{\beta}{1-\beta}\sigma_Z$$

Exit Decision: Stopping Time Problem

- Value of a firm is

$$V_t(Z) = w_t \lambda_F V(S(Z))$$

where

$$V(s) = \max_{\tau} \mathbb{E}\left[\int_0^{\tau} e^{-(r-\kappa)a}(e^{s_a} - 1)\right]$$

- Stopping time problem $\Rightarrow$ threshold policy: shut down when s falls below b.

- For $s > b$, the **HJB equation** holds

$$(r-\kappa)V(s) = e^s - 1 + V'(s)\mu + \frac{1}{2}V''(s)\sigma^2$$

- b determined by **value matching** and **smooth pasting**

$$V(b) = 0, \quad V'(b) = 0$$

Exit Decision: Stopping Time Problem

- Can show: exit barrier determined by

$$e^b = \left(\frac{\xi}{1+\xi}\right)\left(1 - \frac{\mu + \sigma^2/2}{r - \kappa}\right)$$

$$\text{where} \quad \xi \equiv \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{r-\kappa}{\sigma^2/2}}$$

and the HJB equation has solution

$$V(s) = \frac{1}{r-\kappa}\left(\frac{\xi}{1+\xi}\right)\left(e^{s-b} - 1 - \frac{1 - e^{-\xi(s-b)}}{\xi}\right), \quad s \geq b$$

- Faster aggregate productivity growth $\theta_E \uparrow \Rightarrow \mu \propto \theta_I - \theta_E \downarrow \Rightarrow b \uparrow$, i.e. incumbents more likely to exit.

Entry

- Labor cost of an arrival rate of ℓ_t entry opportunities per unit of time:

$$L_{E,t} = \lambda_E \ell_t$$

- An entry opportunity yields a draw Z from a distribution J

- Zero profit condition

$$\lambda_E = \lambda_F \int V(S(Z))dJ(Z)$$

- For now: J exogenous

Kolmogorov Forward Equation

- Density of measure of firms of age a and size s at time t

$$f(a, s, t) = m(a, s) I e^{\eta t}$$

- The KFE is

$$\frac{\partial f(a, s, t)}{\partial t} = -\frac{\partial}{\partial a} f(a, s, t) - \frac{\partial}{\partial s}[\mu f(a, s, t)] + \frac{1}{2}\frac{\partial^2}{\partial s^2}[\sigma^2 f(a, s, t)]$$

- Note: unit drift of age $da = dt$

- Substituting in $f(a, s, t) = m(a, s) I e^{\eta t}$ yields

$$\frac{\partial m(a, s)}{\partial a} = -\eta m(a, s) - \frac{\partial}{\partial s}[\mu m(a, s)] + \frac{1}{2}\frac{\partial^2}{\partial s^2}[\sigma^2 m(a, s)]$$

Boundary Conditions

- Denote size distribution of entering firms by $G(s)$, derived from $J(Z) = G(S(Z))$

- First boundary condition: at age zero

$$\int_b^s m(0,x)dx = G(s) - G(b) \quad \text{all } s > b$$

or more intuitively in terms of the density $g(s) = G'(s)$

$$m(0,s) = g(s), \quad \text{all } s > b$$

- Second boundary condition: at the exit threshold

$$m(a,b) = 0, \quad \text{all } a > 0$$

Boundary Conditions

- **Lemma 1** the solution to the KFE subject to the boundary conditions is

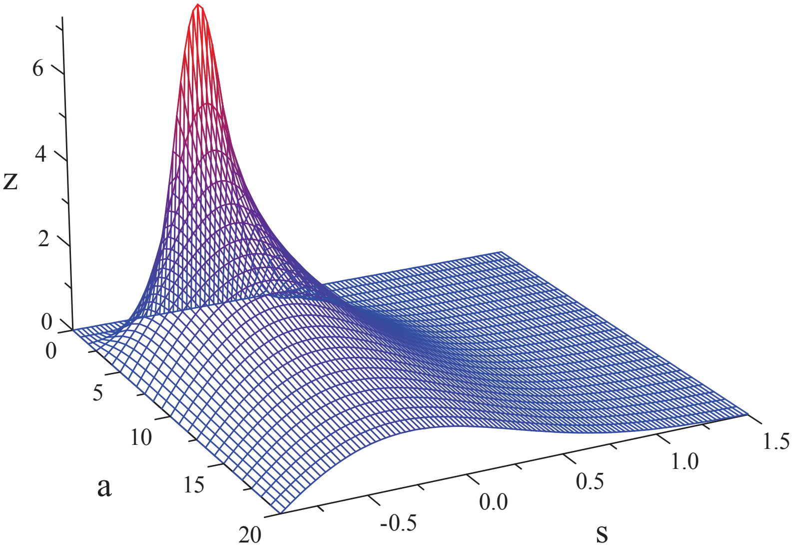
$$m(a,s) = \int_b^\infty e^{-\eta a}\psi(a,s|x)dG(x)$$

$$\psi(a,s|x) = \frac{1}{\sigma\sqrt{a}}\left[\phi\left(\frac{s-x-\mu a}{\sigma\sqrt{a}}\right) - e^{-\mu(x-b)/(\sigma^2/2)}\phi\left(\frac{s+x-2b-\mu a}{\sigma\sqrt{a}}\right)\right]$$

- where ϕ is the standard normal probability density.

- $\psi(a,s|x)$ is the density of survivors at age a with size s of the cohort that entered with the same initial size x (not a p.d.f.)

Life of a Cohort: evolution of $m(a, s)$



Aside: Practical Advice

- Question: how to find solutions for these kinds of ODEs/PDEs?
- Answer: there is a collection of **known solutions** to a big number of ODEs/PDEs. This one apparently from Harrison (1985, p.46)
- if you ever encounter an ODE or PDE that you need to solve, plug into **Mathematica** (function DSolve). Knows **all known solutions**.

Size Distribution

- Want to obtain size distribution. Almost there.

- Denote by $\pi(a, s|x)$ the probability density of survivors at age a with size s of the cohort that entered with the same initial size x (proportional to $\psi(a, s|x)$)

$$\pi(a, s|x) = \left(\frac{1 - e^{-\alpha_*(x-b)}}{\eta}\right)^{-1} e^{-\eta a}\psi(a, s|x)$$

- Integrate this over all ages, a, to get density conditional on initial size

$$\pi(s|x) \propto e^{-\alpha(s-b)} \min\left\{e^{(\alpha+\alpha_*)(s-b)} - 1, e^{(\alpha+\alpha_*)(x-b)} - 1\right\}$$

- Density of e^s is our friend the double Pareto distribution. Can write in a better way.

- From fact: if s has an exponential distribution, then e^s has a Pareto distribution.

Special Case: $\eta = 0$

- when $\eta = 0$, then the tail exponents are $\alpha_* = 0$ and

$$\alpha = -\frac{\mu}{\sigma^2/2} = \frac{\theta_E - \theta_I}{\left(\frac{\beta}{1-\beta}\sigma_Z^2/2\right)}$$