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Studies of the Diffusion with the Increasing Quantity of the Substance; Its Application to a Biological Problem*

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Introduction

Our starting point is the diffusion equation

$$\frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \qquad k > 0\,. \tag{1}$$

For the sake of simplicity, we limit ourselves to the case of two spatial dimensions. Here x and y are the coordinates of the generic point on the plane, t is the time variable, and v is the mass density at the point (x, y) at the moment t. Assume that, in addition to diffusion, there is also a growth of the quantity of the substance, and at each point and each moment this growth is occurring at a rate depending on the density then observed. Thus, we come to the equation

$$\frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + F(v)\,. \tag{2}$$

It is quite natural that we are interested only in the values of $F(v)$ for $v \geq 0$. Assume that $F(v)$ is a continuous function of v, $F(v)$ is sufficiently smooth and satisfies the following conditions:

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$$F(0) = F(1) = 0\,; \tag{3}$$

$$F(v) > 0\,, \quad 0 < v < 1\,; \tag{4}$$

$$F'(0) = \alpha > 0\,, \quad F'(v) < \alpha\,, \quad 0 < v \leq 1\,. \tag{5}$$

Thus, it is assumed that for small v the growth rate $F(v)$ of the density v is proportional to v (with ratio α); and as v becomes close to 1, the state of "saturation" occurs and the growth of v ceases. Accordingly, we limit ourselves to solutions of equation (2) which satisfy the inequality

$$0 \leq v \leq 1\,. \tag{6}$$

Any given initial values of v at $t = 0$, which satisfy (6), determine one and only one solution[1] of equation (2) for $t > 0$ subject to the same condition (6).

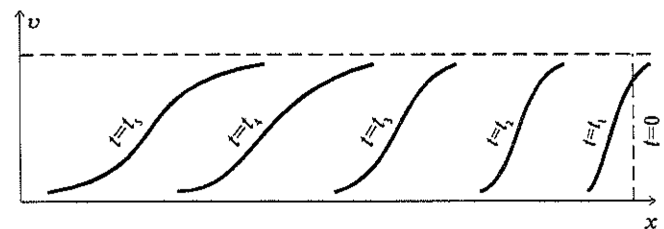

Fig. 1

It is assumed, in what follows, that the density v does not depend on the coordinate y. In this case, the basic equation (2) becomes

$$\frac{\partial v}{\partial t} = k\,\frac{\partial^2 v}{\partial x^2} + F(v)\,. \tag{7}$$

Suppose now that at the initial moment $t = 0$ we have $v = 0$ for $x < a$, and for $x > b \geq a$ the density v attains its largest possible value $v = 1$. Naturally, with the increase of t, the region of densities that are close to 1 will travel from the right to the left, driving the region of small densities to the left. In the special case $a = b$, the behavior of v looks like that in Fig. 1. That part of the density curve (the density is regarded as a function of x) on which the essential density drop from 1 to 0 happens is moving from the right to the left with the increase of t. As $t \to \infty$, the shape of the density curve tends to assume some limit configuration. The problem is to determine this limit shape of the density curve and find the limit velocity

[1]This fact is proved in §3.

of its movement from the right to the left. This limit velocity turns out to be equal to

$$\lambda_0 = 2\sqrt{k\alpha}\,, \tag{8}$$

and the limit shape of the density curve is given by the solution v of the equation

$$\lambda_0 \frac{\mathrm{d}v}{\mathrm{d}x} = k\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + F(v)\,, \tag{9}$$

supplemented by the conditions: $v = 0$ for $x = -\infty$, and $v = 1$ for $x = +\infty$. Such a solution always exists and is unique to within the transformation $x' = x + c$, which leaves intact the shape of the curve.

Note that equation (9) can be obtained in the following way. Let us seek a solution of equation (7) such that with the increase of t the curve, which represents the dependence of v on x, moves from the right to the left with velocity λ, whereas the shape of the curve does not change with the variation of t. This solution has the form

$$v(x,t) = v(x + \lambda t)\,. \tag{10}$$

Now, if we regard v as a function of single variable $z = x + \lambda t$, we obtain the equation

$$\lambda \frac{\mathrm{d}v}{\mathrm{d}z} = k\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + F(v)\,.$$

For any $\lambda \geq \lambda_0$, this equation admits a solution satisfying the conditions specified above for equation (9). But it is only for $\lambda = \lambda_0$ that we get the required limit shape of the curve under the said initial conditions. In order to have a better understanding of a seemingly strange phenomenon that equation (7) has solutions of type (10) for $\lambda > \lambda_0$, i.e., solutions for which the expansion of the high density region (densities close to 1) proceeds with a velocity larger than λ_0, let us consider the limit case $k = 0$. In this case there is no diffusion, and equation (7) can be integrated quite easily. Under the above initial conditions, at the points $x < a$ where the initial density is equal to zero, it remains equal to zero for any $t > 0$. However, simple calculations show that for any $\lambda > 0$ there exist solutions of equation (7) having the form (10) and satisfying all the conditions specified above. The apparent drift of the substance from the right to the left is caused, in this case, by the increase of its density at each point, independently of what happens at all other points.

In §1, the results described in this Introduction are applied to the study of some biological problems; a proof of these results is given in §2 and §3.

§1

Consider an area inhabited by some species. Assume, first, that a dominant gene A is distributed with constant concentration p $(0 \leq p \leq 1)$ over this area. Assume, further, that the members of the species possessing the trait A (i.e., belonging to the genotypes AA or Aa) have an advantage in their efforts to survive over the members lacking that trait (i.e., belonging to the genotype aa); namely, it is assumed that the ratio of the survival probability for a member with trait A to the survival probability for a member without that trait is equal to

$$1 + \alpha\,,$$

where α is a small positive number. Then, for the increment of concentration p in one generation we obtain the following value (see [1]):

$$\Delta p = \alpha p(1 - p)^2 \tag{11}$$

to within the terms of the order α^2.

Now, let us assume that the concentration p varies over the area inhabited by the species, i.e., p depends on the coordinates of a point on the (x, y)-plane. If the members of the species were firmly fixed to their respective places on the territory, the relation (11) would still be valid. We assume, however, that each member, during the period between its birth and its reproduction, travels some distance in a random direction (all directions are equiprobable). Let $f(r)\,\mathrm{d}r$ denote the probability of moving a distance between r and $r + \mathrm{d}r$, and let

$$\rho = \sqrt{\int_0^\infty r^2 f(r)\,\mathrm{d}r}$$

be the mean square displacement. Then, instead of (11), we obtain the following formula:

$$\Delta p(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} p(\xi,\eta)\frac{f(r)}{2\pi r}\,\mathrm{d}\xi\,\mathrm{d}\eta - p(x,y) + \alpha p(x,y)\left\{1 - p(x,y)\right\}^2\,, \tag{12}$$

where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$.

Now we make the following assumptions: $p(x,y)$ is differentiable with respect to x, y, and also t (the latter accounts for the change of generations); α and ρ are very small; the third moment

$$\mathrm{d}^3 = \int_0^\infty |r^3| f(r)\,\mathrm{d}r$$

is small as compared to ρ^2. Then, taking the Taylor expansion in $\xi - x$ and $\eta - y$ for $p(\xi,\eta)$ in (12), and limiting ourselves to the terms of the second order (the first order terms vanish), we obtain[2] the following approximate equation for p:

$$\frac{\partial p}{\partial t} = \frac{\rho^2}{4}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) + \alpha p(1-p)^2 . \tag{13}$$

In order to study this equation, we can use the approach applicable to the general equation (2).

Let us once again clarify our assumptions: The concentration p is varying smoothly with respect to the position in space and time (differentiability in x, y, t); this variation is caused by the selection with ratio $(1+\alpha) : 1$ to the advantage of the dominant characteristic A, and also by random motion of individual members with mean square displacement ρ of one member during the time from its birth to its reproduction. Finally, α and ρ are small; in particular, ρ is small with respect to the distances over which there are substantial changes of concentration p. In this case, taking one generation as a unit of time, we obtain equation (13).

Now, consider the case of a large area already occupied by the gene A with concentration p close to 1. Along the border of this area it is natural to expect a transition zone of intermediate concentrations. Beyond that zone, we assume p to be close to 0. Owing to the positive selection, the region occupied by A will expand; in other words, the border of that region will move in the direction of the territories still clear of the gene A; along the border, there will always be a strip of intermediate concentrations. Our first problem is to determine the *propagation speed of the gene A*, i.e., the speed with which the border of the domain occupied by A moves in the direction of the normal to that border. Formula (8) gives a ready answer to this question: since in our case $k = \rho^2/4$, the propagation speed is given by

$$\lambda = \rho\sqrt{\alpha} . \tag{14}$$

Naturally, our next problem is to find the width of the transition zone. Because of (9), in the direction of the normal to the border the concentration p satisfies the equation

$$\lambda \frac{dp}{dn} = \frac{\rho^2}{4}\frac{d^2p}{dn^2} + \alpha p(1-p)^2 ,$$

which, being divided by α, with λ replaced by its value (14), becomes

$$\frac{\rho}{\sqrt{\alpha}}\frac{dp}{dn} = \frac{1}{4}\frac{\rho^2}{\alpha}\frac{d^2p}{dn^2} + p(1-p)^2 .$$

[2] As regards the transition from (12) to (13), cf., for instance, a similar approach taken by A.Ya. Khinchin in [2].

Introducing a new variable ν by

$$n = \frac{\rho}{\sqrt{\alpha}}\,\nu , \tag{15}$$

we obtain the following equation

$$\frac{dp}{d\nu} = \frac{1}{4}\frac{d^2p}{d\nu^2} + p(1-p)^2 , \tag{16}$$

which contains neither α nor ρ. The boundary conditions for this equation are the same as for (9):

$$p(-\infty) = 0 , \qquad p(+\infty) = 1 .$$

From (15) we conclude that the width of the transition zone is proportional to

$$L = \frac{\rho}{\sqrt{\alpha}} . \tag{17}$$

For instance, let $\rho = 1$, $\alpha = 0.0001$; then $\lambda = 0.01$, $L = 100$.

§2

In this section we consider the equation

$$\lambda \frac{dv}{dx} = k\frac{d^2v}{dx^2} + F(v) , \tag{18}$$

where λ and k are assumed positive, and $F(v)$ satisfies the conditions specified in the Introduction.

Our immediate aim is to find the relations between λ, k, and $\alpha = F'(0)$, which ensure the existence of a solution for (18) satisfying the conditions:

$$0 \le v(x) \le 1 , \qquad \lim_{x\to+\infty} v(x) = 1 , \qquad \lim_{x\to-\infty} = 0 .$$

Set $dv/dx = p$. Then

$$\frac{d^2v}{dx^2} = \frac{dp}{dv}\frac{dv}{dx} = \frac{dp}{dv}p .$$

Substituting this into (18), we get

$$\frac{dp}{dv} = \frac{\lambda p - F(v)}{kp} . \tag{19}$$

The object of our interest are the integral curves of the above equation which pass, on the (p, v)-plane, between the straight lines $v = 0$ and

$v = 1$. In general, among these integral curves the following types can be distinguished:

1. Integral curves separated by a distance larger than $\varepsilon > 0$ from one of the straight lines $v = 0$ or $v = 1$.

2. Integral curves that go to infinity, away from the v axis, while asymptotically approaching one of the straight lines $v = 0$ or $v = 1$.

3. Integral curves crossing one of these straight lines at a finite point outside the v axis.

4. Integral curves that do not belong to any of the above types and approach the points $v = 0$, $p = 0$ and $v = 1$, $p = 0$.

It is easy to see that no integral curve of type 1 may correspond to a solution of equation (18) with the above conditions, since v cannot come infinitely close to both 0 and 1 for such curves.

Integral curves of type 2 do not exist, since such curves must have points at which $|\,\mathrm{d}p/\,\mathrm{d}v|$ is very large for very large $|p|$. But the ratio $(\lambda p - F(v))/kp$ is close to λ/k for large $|p|$, since $F(v)$ is bounded on the interval $(0, 1)$.

Integral curves of type 3 correspond to solutions of (18) whose values do not always remain within the limits 0 and 1. Indeed, suppose that there is a curve of this type which approaches the point $v = 1$, $p = p_1 \neq 0$. In the vicinity of the straight line $v = 1$, we have

$$\frac{\mathrm{d}p}{\mathrm{d}v} \approx \frac{\lambda}{k} \neq 0 ,$$

and therefore, p can be regarded here as a function of v. Let $p = \varphi(v)$. Since $\varphi(1) = p_1 \neq 0$, it follows that on a small interval $(1-\varepsilon,\ 1+\varepsilon)$ the function $|\varphi(v)|$ remains larger than a positive constant C. Denote by x_0 the value of x at which $v = 1 - \varepsilon$. Then, integrating the equation $\mathrm{d}v/\mathrm{d}x = \varphi(v)$, we find that

$$\int_{x_0}^{x} \mathrm{d}x = x - x_0 = \int_{1-\varepsilon}^{v} \frac{\mathrm{d}v}{\varphi(v)} .$$

Hence, we can see that while v varies from $1-\varepsilon$ to $1+\varepsilon$, the variation of x does not exceed $2\varepsilon/C$ in absolute value. Therefore, while x varies from x_0 to $x_0 + 2\varepsilon/C$, the function v must somewhere exceed 1.

It remains to consider the curves of type 4. Both $(v, p) = (0, 0)$ and $(v, p) = (1, 0)$ are singular points for equation (19). The integral curve of type 4 must approach each of those points without crossing the straight lines $v = 0$ and $v = 1$, and therefore, no winding may occur. Thus, for the existence of such curves it is necessary that the characteristic equation for each of these points has real roots. Let us write $F(v)$ in the form

$$F(v) = \alpha v + \varphi_1(v) .$$

Then we obviously have $\varphi_1(v) = o(v)$. Therefore, the characteristic equation for the point $v = 0$, $p = 0$ has the form

$$\begin{vmatrix} \lambda - \rho & -\alpha \\ k & -\rho \end{vmatrix} = 0 ,$$

or

$$\rho^2 - \lambda k + \alpha k = 0 . \tag{20}$$

This equation has real roots if

$$\lambda^2 \geq 4\alpha k .$$

In order to obtain the characteristic equation for the point $v = 1$, $p = 0$, let us change the variables, setting $v = 1 - u$. We get

$$\frac{\mathrm{d}p}{\mathrm{d}u} = \frac{-\lambda p + \Phi(u)}{kp} ,$$

where

$$\Phi(u) = F(1-u) .$$

Clearly, $F'(1) \leq 0$ and $\Phi'(0) = -F'(1) = A \geq 0$. It follows that

$$\Phi(u) = Au + o(u) ,$$

and the characteristic equation for the point $v = 1$, $p = 0$ has the form

$$\begin{vmatrix} -\lambda - \rho & A \\ k & -\rho \end{vmatrix} = 0 ,$$

or

$$\rho^2 + \lambda\rho - Ak = 0 . \tag{21}$$

This equation has real roots if

$$\lambda^2 \geq -4Ak .$$

Since $\alpha > 0$, equation (20) has real roots of the same sign. Consequently, $(0, 0)$ is a nodal point. All integral curves entering a sufficiently small neighborhood of that point must pass through that point. Equation (21) has roots of different signs if $A > 0$. Therefore, if $A > 0$, there are only two integral curves passing through the point $(1, 0)$, and their directions are uniquely determined. Let these directions be given by the equations

$$m_1 u + n_1 p = 0 , \qquad m_2 u + n_2 p = 0 . \tag{22}$$

The coefficients m_1, n_1, m_2, n_2 are known[3] to be defined by the equations

[3]See, for instance, [3].

$$km_1 - \rho_1 n_1 = 0\,, \qquad km_2 - \rho_2 n_2 = 0\,, \tag{23}$$

where ρ_1 and ρ_2 are the roots of the characteristic equation (21). Since these roots are of opposite signs, the straight lines (22) will also have their slope of opposite signs.[4] Therefore, within each angle formed by the crossing of straight lines $v = 1$ and $p = 0$ there is only one integral curve of equation (19) passing through the point $v = 1$, $p = 0$. An approximate position of these curves is shown in Fig. 2. The curve II crosses the p-axis below the origin; indeed, equation (19) shows that $dp/dv > 0$ in that part of the strip between the lines $v = 0$ and $v = 1$ which lies beneath the v-axis. Therefore, the curve II should be excluded from consideration. It remains to examine[5] the curve I.

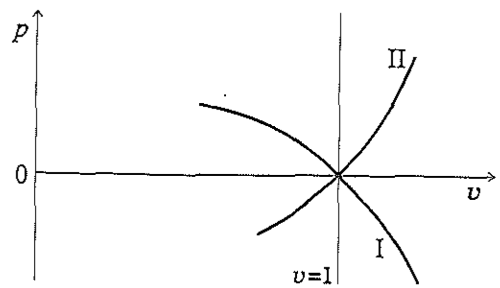

Fig. 2

Our aim is to prove that every curve of type I crosses the p-axis at the origin. First of all, we show that this curve cannot cross the p-axis below the origin. For this purpose, consider the isoclines for equation (19). The family of these curves is given by the equation

$$\frac{\lambda p - F(v)}{kp} = C\,. \tag{24}$$

Here C is the value of dp/dv at the point (v, p). Hence

$$p = \frac{F(v)}{\lambda - Ck}\,. \tag{24'}$$

Equation (24) defines a family of curves passing through the points $(0,0)$ and $(1,0)$. This family is outlined in Fig. 3. The respective value of C is indicated next to each curve. The curve corresponding to $C = 0$ is

[4]In exactly the same manner, the integral curves for equation (19) can be shown to have positive slope.

[5]For $A = 0$, it can only be claimed that there is at least one integral curve of type I approaching the point $(1,0)$ in a certain direction of negative slope (see [4]).

drawn in bold line. As the apex of a curve goes up, the respective value of C increases and tends to λ/k, which corresponds to the straight lines $v = 0$ and $v = 1$. In the region enclosed between the curve $C = 0$ and the v-axis (shaded area in Fig. 3), we have $C < 0$, and at the points close to the v-axis the parameter C has very large absolute values. Beneath the v-axis, we have $C > 0$, and as the apex of the curve goes down from the level of the v-axis to $-\infty$, the values of C decrease from $+\infty$ to λ/k.

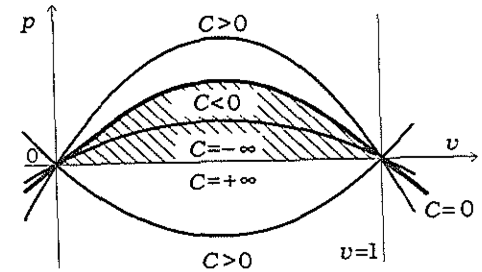

Fig. 3

Now it becomes clear that the integral curve I (see Fig. 2) cannot cross the axis Op below the origin. Indeed, assume the contrary. Then the curve I must cross the v-axis. Since $dp/dv = -\infty$ on the upper side of this axis, and $dp/dv = +\infty$ on its lower side, the convexity of the integral curve I, at its point of intersection with the v-axis, is directed towards the straight line $v = 1$. Therefore, in order that this curve could pass through the point $(1,0)$, it is necessary that dp/dv turn into ∞ above the v-axis, which is impossible. Similar arguments show that the curve I cannot cross the straight line $v = 1$ above the v-axis.

Let us show that the integral curve I cannot cross the p-axis above the origin. To this end it suffices to establish the existence of a half-line passing through the origin in the first coordinate quarter and having no common points with any of the integral curves crossing the p-axis in its positive half. It follows from equation (24′) that

$$\left.\overline{\frac{dp}{dv}}\right|_{v=0} = \frac{\alpha}{\lambda - Ck}\,,$$

where $\overline{dp}/dv$ denotes the derivative of the function $p = p(v)$ defined by equation (24′). Let us define C such that

$$\left.\overline{\frac{dp}{dv}}\right|_{v=0} = C\,.$$

For this purpose we can use the equation

$$\frac{\alpha}{\lambda - Ck} = C\,,$$

or $kC^2 - C\lambda + \alpha = 0$; hence

$$C = \frac{\lambda \pm \sqrt{\lambda^2 - 4\alpha k}}{2k}\,. \tag{25}$$

By assumption, we have $\lambda^2 \geq 4\alpha k$; therefore, both values of C given by (25) are real and positive. Denote one of these values by C_0 and consider the straight line

$$p = C_0 v\,. \tag{26}$$

It is easy to see that for all those points of the strip between the lines $v = 0$ and $v = 1$ whose position is above the line (26), or even on that line (except at the origin), we have

$$\frac{dp}{dv} > C_0\,,$$

where p is the function of v given by equation (18).

Therefore, no integral curve passing through a point on the p-axis above the origin can ever cross that part of the straight line (26) which is above the v-axis. Thereby, we have proved that every integral curve of type I (see Fig. 2) passes through the origin.

Let us show that there exists only one curve of type I. (Of course, this proof is necessary only in the case $A = 0$.) Indeed, all integral curves of type I pass through the origin, as shown above. On the other hand, it follows from (19) that for $p > 0$ and v fixed the derivative dp/dv increases together with p. Consequently, there cannot be two integral curves issuing from the origin and passing through the point $(1, 0)$.

Next we show that the curve I corresponds to the solution of equation (18) with the conditions stated at the beginning. First of all, note that any straight line perpendicular to the v-axis crosses the integral curve I for equation (19) at a single point; otherwise, above the v-axis, dp/dv would turn to ∞. Therefore, along this curve, p is a function of v: $p = \varphi(v)$. Recall also that the curve I crosses the v-axis at the point $(1, 0)$, the tangent of the angle between the curve and the v-axis being negative; it also crosses the same axis at the origin, this time the tangent being positive. Therefore, for small values of v we have

$$p = k_1 v + o(v)\,; \tag{27}$$

and for small values of $1 - v$ we have

$$p = k_2(1 - v) + o(1 - v)\,, \tag{28}$$

where $k_1 > 0$ and $k_2 > 0$.

Recall now that $p = dv/dx$. Therefore, $dv/dx = \varphi(v)$, or equivalently, $dx = dv/\varphi(v)$. Integrating the last relation, we get

$$x - x_0 = \int_{v_0}^{v} \frac{dv}{\varphi(v)}\,, \qquad 0 < v_0 < 1\,.$$

By virtue of (27) and (28), it follows that $x \to -\infty$ as $v \to 0$, and $x \to +\infty$ as $v \to 1$, *q.e.d.*

§3

In this section, instead of equation (7) mentioned in the Introduction, we consider the following equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = F(v) \tag{29}$$

with $F(v)$ having the following properties:

$$F(0) = F(1) = 0\,; \tag{30}$$

$$F(v) > 0 \quad \text{for} \quad 0 < v < 1\,; \tag{31}$$

$$F'(0) = 1\,; \tag{32}$$

$$F'(v) < 1 \quad \text{for} \quad 0 < v \leq 1\,; \tag{33}$$

$$F'(v) \quad \text{is continuous and bounded on} \quad (0, 1)\,.$$

Moreover, $F(v)$ is assumed sufficiently smooth. Equation (7) of general type can always be reduced to the form (29) by changing the variables

$$x = \sqrt{\frac{k}{\alpha}}\,\bar{x}\,, \qquad t = \frac{\bar{t}}{\alpha}\,.$$

Our primary aim in this section is the proof of the following facts: the fragment of the density curve $v(x, t)$ (regarded as a function of x) bearing the major part of the density drop from 1 to 0 is moving to the left as t grows to ∞; the speed of this motion tends to 2 (from below), whereas the density curve itself tends to assume the shape of the graph of the function $u(x)$ which is a solution of the equation

$$\frac{d^2u}{dx^2} - 2\frac{du}{dx} + F(u) = 0\,, \tag{34}$$

and satisfies the conditions: $u \to 0$ as $x \to -\infty$, $u \to 1$ as $x \to +\infty$ (the existence of such a solution has been established in §2).

Before we turn to the proof of the main results of this section, consider the equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = F(x,t,v)\,, \tag{35}$$

of which (29) is a special case. For this equation, we prove the existence of a solution taking given initial values at $t = 0$, and study some of its properties.

Theorem 1. *Assume that the function $F(x,t,v)$ in (35) is bounded, continuous, and satisfies the Lipschitz condition with respect to v and x, i.e.,*

$$|F(x_2,t,v_2) - F(x_1,t,v_1)| < k|v_2 - v_1| + k|x_2 - x_1|\,, \tag{36}$$

where k is a constant independent of x, t, v. Let $f(x)$ be a bounded function defined for all x. Assume, for simplicity, that f can have only a finite number of discontinuities. Then there exists one and only one function $v(x,t)$ which is bounded for bounded values of t, satisfies equation (35) for $t > 0$, and turns into f for $t = 0$ at every point of continuity of f.

For the sake of brevity, when saying that $v(x,t)$ turns into $f(x)$ at $t = 0$, we always imply the points of continuity of $f(x)$.

Proof. Let $v_0(x,t)$ be a bounded function satisfying the equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \tag{37}$$

for $t > 0$ and turning into $f(x)$ at $t = 0$. Substituting this function for v in the right-hand side of equation (35) and using the formula

$$\tilde{v}_1(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^t \mathrm{d}\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} F(\xi,\eta,v_0(\xi,\eta))\,\mathrm{d}\xi\,, \tag{38}$$

we find the solution of this equation vanishing on the x-axis (see [5]). The function

$$v_1(x,t) = v_0(x,t) + \tilde{v}_1(x,t)$$

turns into $f(x)$ at $t = 0$ and satisfies the equation

$$\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} = F(x,t,v_0(x,t)) \quad \text{for} \quad t > 0\,.$$

In general, the formula

$$v_{i+1}(x,t) = v_0(x,t) + \frac{1}{2\sqrt{\pi}} \int_0^t \mathrm{d}\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} F(\xi,\eta,v_i)\,\mathrm{d}\xi\,, \tag{39}$$

yields the function $v_{i+1}(x,t)$ satisfying the equation

$$\frac{\partial v_{i+1}}{\partial t} - \frac{\partial^2 v_{i+1}}{\partial x^2} = F(x,t,v_i) \tag{40}$$

for $t > 0$ and turning into $f(x)$ at $t = 0$.

Let us show that the functions $v_i(x,t)$ form a uniformly convergent sequence. Indeed, taking into account (36), we find from (39) that

$$M_{i+1}(t) = \sup_{\eta \le t} |v_{i+1}(x,\eta) - v_i(x,\eta)| \le$$

$$\le \frac{1}{2\sqrt{\pi}} \int_0^t \mathrm{d}\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} |F(\xi,\eta,v_i(\xi,\eta)) - F(\xi,\eta,v_{i-1}(\xi,\eta))|\,\mathrm{d}\xi \le$$

$$\le \int_0^t kM_i(\eta)\,\mathrm{d}\eta\,, \tag{41}$$

since

$$\int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}}\,\mathrm{d}\xi = 2\sqrt{\pi}\,.$$

But, denoting by M_0 the upper bound for $|f(x)|$ and $|F(x,t,0)|$, we get

$$|v_0(x,t)| \le M_0\,,$$

and by (38)

$$M_1 \le \int_0^t (k+1)M_0\,\mathrm{d}t = (k+1)M_0 t = Mt\,.$$

Hence, using the inequality (41), we easily find that

$$M_i \le \frac{Mk^{i-1}t^i}{i!}\,,$$

which makes the uniform convergence of v_i quite clear.

Set

$$v(x,t) = \lim_{i\to\infty} v_i(x,t)\,.$$

The function $v(x,t)$ turns into $f(x)$ at $t = 0$. Moreover, it obviously satisfies the equation

$$v(x,t) = v_0(x,t) + \frac{1}{2\sqrt{\pi}} \int_0^t \mathrm{d}\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} F(\xi,\eta,v(\xi,\eta))\,\mathrm{d}\xi\,. \tag{42}$$

Hence we easily see that $v(x,t)$ is a continuous function of x and t for $t>0$. In Gevrey's memoir [5] (pp. 343–344), referred to above, it has been shown that for any bounded F the second term in the right-hand side of (42) has a bounded derivative in x. Because of (36), it follows that the function $F(x,t,v(x,t))$, $t>0$, has bounded derivative numbers with respect to x, and therefore, equation (35) holds for $v(x,t)$ (see [5], p. 351).

The uniqueness of a bounded solution can be established as follows. Assume that there are two bounded functions $v_1(x,t)$ and $v_2(x,t)$ taking equal values at $t=0$. Then

$$v_2(x,t)-v_1(x,t)=\frac{1}{2\sqrt{\pi}}\int_0^t d\eta\int_{-\infty}^{+\infty}\frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}}[F(\xi,\eta,v_2)-F(\xi,\eta,v_1)]\,d\xi\,. \tag{43}$$

Set

$$M(t)=\sup_{t\geq\eta}|v_2(x,\eta)-v_1(x,\eta)|\,.$$

Then, using (36), we find from (43) that

$$M(t)\leq k\int_0^t M(\eta)\,d\eta\,,$$

which is impossible.

Remark. As shown in [5], for a domain bounded on its sides by two curves of the form $x=\varphi_1(t)$, $x=\varphi_2(t)$, and by the straight lines $t=t_0$, $t=t_1>t_0$ from above and from below, there exists a unique bounded function satisfying equation (35) inside the domain and taking given continuous and bounded data on the lines $x=\varphi_1(t)$, $x=\varphi_2(t)$, and $t=0$. Likewise, it is possible to show that for a domain G bounded in the horizontal direction, on one side only, by the curve $x=\varphi(t)$, and in the vertical direction by the straight lines $t=t_0$ and $t=t_1>t_0$, there exists a unique bounded function satisfying equation (35) inside G and taking given continuous and bounded data on the curve $x=\varphi(t)$ and the line $t=t_0$.

Theorem 2. *If $F(x,t,v)$ be replaced by another function $F_1(x,t,v)$ such that at every point we have*

$$F_1(x,t,v)\geq F(x,t,v)\,,$$

then the corresponding $v(x,t)$ does not become smaller, provided that the initial data are left intact.

Remark. For (35) interpreted as the heat equation, the function $F(x,t,v)$ characterizes the power of the heat source, and from the physical standpoint Theorem 2 becomes evident.

Proof. Let $v(x,t)$ be the solution of (35), and let $v_1(x,t)$ be the solution of

$$\frac{\partial v_1}{\partial t}-\frac{\partial^2 v_1}{\partial x^2}=F_1(x,t,v_1)\,.$$

Subtracting one equation from another, we find that

$$w(x,t)=v_1(x,t)-v(x,t)$$

satisfies the equation

$$\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}=F_1(x,t,v_1)-F(x,t,v)\,.$$

Set

$$w(x,t)=\bar{w}(x,t)\,e^{-kt}\,,$$

where k is the same as in (36). Then

$$\frac{\partial\bar{w}}{\partial t}-\frac{\partial^2\bar{w}}{\partial x^2}=k\bar{w}+e^{kt}[F_1(x,t,v_1)-F(x,t,v)]\,.$$

Hence

$$\bar{w}(x,t)=$$

$$=\frac{1}{2\sqrt{\pi}}\int_0^t d\eta\int_{-\infty}^{+\infty}\frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}}\left\{k\bar{w}+e^{k\eta}[F_1(\xi,\eta,v_1)-F(\xi,\eta,v)]\right\}d\xi=$$

$$=\frac{1}{2\sqrt{\pi}}\int_0^t d\eta\int_{-\infty}^{+\infty}\frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}}\{k\bar{w}+e^{k\eta}[F_1(\xi,\eta,v_1)-F(\xi,\eta,v_1)+$$

$$+F(\xi,\eta,v_1)-F(\xi,\eta,v)]\}d\xi\geq$$

$$\geq\frac{1}{2\sqrt{\pi}}\int_0^t d\eta\int_{-\infty}^{+\infty}\frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}}\left\{k\bar{w}+e^{k\eta}[F(\xi,\eta,v_1)-F(\xi,\eta,v)]\right\}d\xi\geq$$

$$\geq\frac{1}{2\sqrt{\pi}}\int_0^t d\eta\int_{-\infty}^{+\infty}\frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}}(k\bar{w}-k|\bar{w}|)\,d\xi\,. \tag{44}$$

The last expression in parentheses vanishes if $\bar{w}\geq 0$ and is equal to $2k\bar{w}$ if $\bar{w}\leq 0$. Set

$$-m(t)=\inf_{\eta\leq t}\{\bar{w}(\xi,\eta)-|\bar{w}(\xi,\eta)|\}\,.$$

Clearly, in order to prove our theorem, it suffices to show that $m(t)\equiv 0$. To this end we note that it follows from (44) that

$$\bar{w}(x,t) \geq -k \int_0^t m(\eta)\,\mathrm{d}\eta\,,$$

and therefore,

$$m(t) \leq k \int_0^t m(\eta)\,\mathrm{d}\eta\,,$$

which is possible only if $m(t) \equiv 0$, *q.e.d.*

Theorem 3. *The function $v(x,t)$ does not become smaller if $f(x)$ is replaced with a larger function.*

Physically, the statement of this theorem is as clear as that of the previous one, provided that (35) is regarded as the heat conduction equation for a rod. The function $f(x)$ specifies the initial temperature of the rod. With the increase of the initial temperature, subsequent temperature becomes higher.

Proof of Theorem 3. Assume that $v_1(x,t)$ and $v_2(x,t)$ satisfy equation (35) and, at $t=0$, turn into $f_1(x)$ and $f_2(x)$, respectively, where $f_2(x) \geq f_1(x)$. Let us show that $v_2 \geq v_1$.

The function $w = v_2 - v_1$ satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = F(x,t,v_2) - F(x,t,v_1)\,.$$

By virtue of (36) we have

$$F(x,t,v_2) - F(x,t,v_1) \geq -k|w|\,.$$

Thus, according to Theorem 2, the function $w(x,t)$ is not less than $v^*(x,t)$, where $v^*(x,t)$ is equal to $f_2(x) - f_1(x) \geq 0$ for $t=0$ and, for $t>0$, satisfies the equation

$$\frac{\partial v^*}{\partial t} - \frac{\partial^2 v^*}{\partial x^2} = -k|v^*|\,.$$

The solution of this equation which is bounded for bounded t and takes the initial values $f_2 - f_1$ at $t=0$ (by Theorem 1 this solution is unique) has the form $\mathrm{e}^{-kt}v^{**}$, where $v^{**}(x,t)$ satisfies (37) and the initial condition $v^{**}(x,0) = f_2(x) - f_1(x)$ (obviously this function is non-negative). Consequently,

$$w = v_2 - v_1 \geq 0\,, \quad \text{q.e.d.}$$

Theorem 4. *Assume that $f(x) \geq 0$ and $F(x,t,0) = 0$ for all x,t. Then*

$$v(x,t) \geq 0\,.$$

Proof. According to Theorem 3, with the decrease of $f(x)$ the function $v(x,t)$ does not become larger. For $f(x) \equiv 0$, we have $v(x,t) \equiv 0$. Therefore, $v(x,t) \geq 0$ if $f(x) \geq 0$, *q.e.d.*

Theorem 5. *Assume that besides the conditions of Theorem 4 we also have $f(x) > 0$ on some interval of positive length. Then*

$$v(x,t) > 0 \quad \text{for} \quad t > 0\,.$$

The proof of this theorem is obtained from that of Theorem 3 if we set $v_2 = v$, $v_1 = 0$ and take into account that the function $v^{**}(x,t)$ represented by Poisson's integral is positive for $t > 0$.

Theorem 6. *If $F(x,t,1) \equiv 0$ and $f(x) \leq 1$, then $v(x,t) \leq 1$.*

Proof. By Theorem 3, the function $v(x,t)$ does not become smaller with the increase of $f(x)$. For $f(x) \equiv 1$, we have $v(x,t) \equiv 1$. Hence we obtain the needed result.

Theorem 7. *Assume that for $t=0$ the function $v(x,t)$ turns into a monotonically increasing differentiable function $f(x)$ and for $t>0$ satisfies the equation*

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = F(t,v)\,. \tag{45}$$

Then $v(x,t)$ is a non-decreasing function of x for any $t>0$.

Proof. By Theorem 1, we have

$$v(x,t) = v_0(x,t) + \frac{1}{2\sqrt{\pi}} \int_0^t \mathrm{d}\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} F(\eta, v(\xi,\eta))\,\mathrm{d}\xi\,, \tag{46}$$

where $v_0(x,t)$ satisfies the equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \tag{47}$$

for $t>0$ and turns into $f(x)$ at $t=0$. If $f(x)$ is differentiable, we have $v'_{0x}(x,t) \to f'(x)$ as $(x,t) \to (x,0)$ (see [5], pp. 330–331). On the other hand, the partial derivative in x of the second term in the right-hand side of (46) has its absolute value bounded by $(4/\sqrt{\pi})Mt^{1/2}$, if $|F| \leq M$ (see [5], p. 344). Therefore, $v'_x(x,t) \to f'(x)$ as $t \to 0$. If we assume, in addition, that $v(x,t)$ has the derivatives $\partial^2 v/\partial t \partial x$ and $\partial^3 v/\partial x^3$ (which is the case if $F(t,v)$ is three times differentiable in v), then the function $w(x,t) = v'_x(x,t)$ satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = \frac{\partial F}{\partial v} w . \tag{48}$$

Now, applying Theorem 4, we find that $w(x,t) \geq 0$, *q.e.d.*

Theorem 8. *Assume that*

$$f^{(\varepsilon)}(x) \to f^{(0)}(x) \quad \text{as} \quad \varepsilon \to 0 ,$$

and also

$$\int_{-\infty}^{+\infty} \left| f^{(\varepsilon)} - f^{(0)} \right| \mathrm{d}x \to 0 .$$

Then

$$v^{(\varepsilon)}(x,t) \to v^{(0)}(x,t) \quad \text{as} \quad \varepsilon \to 0 ,$$

for every $t > 0$, where $v^{(\varepsilon)}(x,t)$ and $v^{(0)}(x,t)$ are solutions of equation (35) *for $t > 0$, with the initial values $f^{(\varepsilon)}(x)$ and $f^{(0)}(x)$ at $t = 0$, respectively.*

Proof. In order to find the functions $v^{(\varepsilon)}(x,t)$ and $v^{(0)}(x,t)$, we use the method of successive approximations, just as in the proof of Theorem 1. The functions $v_0^{(\varepsilon)}$ and $v_0^{(0)}$ are respectively given by

$$v_0^{(\varepsilon)}(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} f^{(\varepsilon)}(\xi) \frac{\exp\left(-\frac{(x-\xi)^2}{4t}\right)}{\sqrt{t}} \mathrm{d}\xi ,$$

$$v_0^{(0)}(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} f^{(0)}(\xi) \frac{\exp\left(-\frac{(x-\xi)^2}{4t}\right)}{\sqrt{t}} \mathrm{d}\xi .$$

Hence, we immediately see that

$$v_0^{(\varepsilon)}(x,t) \to v_0^{(0)}(x,t) \quad \text{as} \quad \varepsilon \to 0$$

for $t > 0$. The difference $\tilde{v}_1^{(\varepsilon)}(x,t) - \tilde{v}_1^{(0)}(x,t)$ (the notation is the same as in the proof of Theorem 1) is given by the formula

$$\frac{1}{2\sqrt{\pi}} \int_0^t \mathrm{d}\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} [F(\xi,\eta,v_0^{(\varepsilon)}) - F(\xi,\eta,v_0^{(0)})] \, \mathrm{d}\xi .$$

It follows that

$$v_1^*(x,t) = \left| \tilde{v}_1^{(\varepsilon)}(x,t) - \tilde{v}_1^{(0)}(x,t) \right| \leq$$

$$\leq \frac{k}{2\sqrt{\pi}} \int_0^t \mathrm{d}\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} \left| v_0^{(\varepsilon)}(\xi,\eta) - v_0^{(0)}(\xi,\eta) \right| \mathrm{d}\xi .$$

Set

$$v_0^*(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \left| f^{(\varepsilon)}(\xi) - f^{(0)}(\xi) \right| \frac{\exp\left(-\frac{(x-\xi)^2}{4t}\right)}{\sqrt{t}} \mathrm{d}\xi .$$

We obviously have

$$v_0^*(x,t) \geq \left| v_0^{(\varepsilon)}(x,t) - v_0^{(0)}(x,t) \right| ,$$

and therefore,

$$v_1^*(x,t) \leq \frac{k}{2\sqrt{\pi}} \int_0^t \mathrm{d}\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} v_0^*(\xi,\eta) \, \mathrm{d}\xi =$$

$$= k \int_0^t v_0^*(x,t) \, \mathrm{d}\eta = ktv_0^*(x,t) .$$

The penultimate equality follows from the fact that $v_0^*(x,t)$ satisfies equation (47). Thus,

$$v_1^*(x,t) \leq ktv_0^*(x,t) .$$

In exactly the same manner, we find that

$$v_i^*(x,t) = \left| v_i^{(\varepsilon)}(x,t) - v_i^{(0)}(x,t) \right| \leq \frac{(kt)^i}{i!} v_0^*(x,t) .$$

It follows that, choosing ε suitably small, we can make the sum $\sum_{i=0}^{\infty} v_i^*(x,t)$, and therefore, $|v^{(\varepsilon)}(x,t) - v^{(0)}(x,t)|$ arbitrarily small, *q.e.d.*

Theorem 9. *Let $v(x,t)$ be a function that satisfies equation* (45) *for $t > 0$, together with the initial conditions: $v = 0$ for $t = 0$ and $x < 0$, and $v = 1$ for $t = 0$ and $x > 0$. Then $v(x,t)$ is a non-decreasing function of x for any $t > 0$; moreover, $v'_x(x,t) > 0$ for $t > 0$.*

Proof. According to the preceding theorem, $v(x,t)$ can be regarded as a limit (for $\varepsilon \to 0$) of functions $v^{(\varepsilon)}(x,t)$ which coincide with $v(x,t)$ on the x-axis for $|x| \geq \varepsilon$, are monotone and continuous, together with their derivative in x, on the entire x-axis. But, as we have just shown in Theorem 7, $v^{(\varepsilon)}(x,t)$ is a monotonically increasing function of x for $t > 0$; therefore, the same is true for $v(x,t)$.

Let us show that $v'_x(x,t) > 0$ for $t > 0$. To this end, it suffices to show that for $t > 0$ we cannot have $v'_x(x,t) = 0$. This fact can be established by the following arguments. For $t > 0$, the function $v'_x(x,t)$ satisfies equation (48). Therefore, the function $\bar{w}(x,t) = \mathrm{e}^{Mt} v'_x(x,t)$, where $M = \sup |\partial F/\partial v|$, satisfies the equation

$$\frac{\partial \bar{w}}{\partial t} - \frac{\partial^2 \bar{w}}{\partial x^2} = \left[\frac{\partial F}{\partial v} + M\right] \bar{w} .$$

We also have

$$\frac{\partial F}{\partial v} + M \geq 0 ;$$

therefore, according to Theorem 2, $\bar{w}(x,t) \geq \bar{\bar{w}}(x,t)$ for $t > t_0 > 0$, where $\bar{\bar{w}}(x,t)$ coincides with $\bar{w}(x,t)$ for $t = t_0$ and satisfies the equation

$$\frac{\partial \bar{\bar{w}}}{\partial t} - \frac{\partial^2 \bar{\bar{w}}}{\partial x^2} = 0$$

for $t > 0$. The function $\bar{\bar{w}}$ is positive for all $t > t_0$, since $\bar{w}(x,t)$ does not vanish identically for $t = t_0$, provided that t_0 is sufficiently small.

In what follows, we always denote by $v(x,t)$ the function that satisfies equation (29) for $t > 0$, together with the conditions: $v = 0$ for $t = 0$ and $x < 0$, $v = 1$ for $t = 0$ and $x > 0$.

Theorem 10. *For any fixed $x < 0$, we have*

$$v(x - 2t, t) \to 0 \quad \text{as} \quad t \to +\infty .$$

Proof. The function $\bar{v}(x,t) = v(x - 2t, t)$ satisfies the equation

$$\frac{\partial \bar{v}}{\partial t} - \frac{\partial^2 \bar{v}}{\partial x^2} = -2\frac{\partial \bar{v}}{\partial x} + F(\bar{v}) ,$$

whereas $v^*(x,t) = v(x - 2t, t)\, e^{-x}$ satisfies the equation[6]

$$\frac{\partial v^*}{\partial t} - \frac{\partial^2 v^*}{\partial x^2} = [F(\bar{v}) - \bar{v}]\, e^{-x} .$$

According to conditions (32) and (33) on $F(v)$, we have $F(v) - v \leq 0$. Therefore, $v^*(x,t)$ is smaller than the function satisfying equation (37) for $t > 0$ and, for $t = 0$, equal to 1 on the half-line $x < 0$, and equal to e^{-x} if $x > 0$. The latter function tends to 0 uniformly in x as $t \to +\infty$.

Theorem 11. *For t fixed, let us consider $v'_x(x,t)$ as a function of v. This is possible on account of Theorem 9. Let*

$$v'_x(x,t) = \psi(v,t) . \tag{49}$$

Then, for v fixed, the function ψ does not increase with the growth of t.

Proof. Consider the functions $v(x,t)$ and $v(x + c,\, t + t_0) = v_{t_0}(x,t)$, where c is a constant and $t_0 > 0$. Set

[6]It is easy to see that $v^*(x,t)$ remains bounded for bounded $t > 0$.

$$w(x,t) = v(x,t) - v_{t_0}(x,t) .$$

Denote by $\mathcal{M}$ the set formed by the points (x,t) on the plane such that $w(x,t) > 0$. First of all, let us show that this set is bounded only on its left-hand side so that its boundary curve issues from the origin and, moreover, the coordinate t never decreases as we move along the curve. To this end, we note that $w(x,t)$ satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = k(x,t) w , \tag{50}$$

where $k(x,t)$ is a bounded function, namely,

$$k(x,t) = F'(\bar{u}(x,t)) ;$$

$\bar{u}(x,t)$ is a certain number between $v(x,t)$ and $v_{t_0}(x,t)$. Therefore, the set $\mathcal{M}$ cannot contain isolated pieces[7] disjoint from the x-axis. Therefore, $\mathcal{M}$ consists of a single piece joining, of course, the right half of the x-axis. In order to prove that from its left-hand side the set $\mathcal{M}$ is bounded by a curve along which the variable t never decreases, assume the contrary, namely, that this curve has a piece of the form indicated in Fig. 4. For definiteness, let us assume that, as it issues from the point A, the curve goes down. Then $w(x,t)$ must take negative values to the right of the line OA, whereas on OA proper $w(x,t) = 0$, and on the x-axis for $x > 0$ it takes positive values. But this is impossible, which can be shown by exactly the same methods as those used in the proof of Theorem 4.

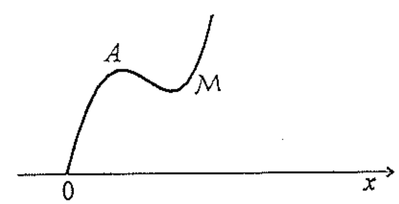

Fig. 4

Similar arguments show that the set $\mathcal{M}$ is unbounded on its right-hand side.

After these remarks, our theorem can be proved quite easily. Indeed, since the constant c can be chosen arbitrary, we can fix it in such a way that for any given t the values $v(x_0,t)$ and $v_{t_0}(x_0,t)$ coincide for some $x = x_0$. Then, on the basis of the above arguments, we conclude that

[7]See [6] for the proof of a similar statement in the case of finite pieces. The same result can also be obtained for infinite pieces. Cf. Remark to Theorem 1.

$$v(x,t) \geq v_{t_0}(x,t) \quad \text{for} \quad x > x_0 ,$$

and therefore,

$$v'_x(x_0,t) \geq v'_{t_0 x}(x_0,t) , \quad \text{q.e.d.}$$

Theorem 12. *For any t, we have*

$$v'_x(x,t) \geq u'(x)$$

if $v(x,t) = u(x)$. Here $u(x)$ is the solution of equation (34) discussed in the beginning of this section.

Proof. This theorem is proved in exactly the same way as Theorem 11. However, in this case we should take $u(x+c)$ instead of $v_{t_0}(x,t)$, and to consider the difference $v(x,t) - u(x+c)$ instead of $w(x,t)$.

Theorem 13. *Let*

$$v^*(x,t) = v(x+\varphi(t),t) ,$$

where the function $\varphi(t)$ is chosen such that

$$v^*(0,t) = c = \text{const} .$$

Then

$$v^*(x,t) \to v^*(x) \quad \text{as} \quad t \to \infty ,$$

and the convergence is uniform with respect to x.

Proof. From (49) we find that

$$x = \int_c^{v^*} \frac{dv}{\psi(v,t)} . \tag{51}$$

According to Theorem 11, the integrand is a monotonically increasing function of $t \to \infty$. Moreover, by Theorem 12, the integral $\int_c^{v^*} (\psi(v,t))^{-1}\, dv$ cannot increase to infinity. Therefore, we can pass to the limit under the sign of the integral. Let

$$\psi(v,t) \to \psi(v) \quad \text{as} \quad t \to +\infty .$$

Then, passing to the limit in (51), we get

$$x = \int_c^{v^*} \frac{dv}{\psi(v)} .$$

By Theorem 12, we have $\psi(v) > 0$; therefore, the above relation defines a function of x, say, $v^*(x)$. It remains to show uniform convergence of $v^*(x,t)$ to $v^*(x)$. To this end, observe that (51) implies uniform convergence of $x(v^*,t)$ to $x(v^*)$ on any interval of the form $\varepsilon < v^* < 1-\varepsilon$. Now, if we take into account that $\psi(v^*,t)$ remains bounded on any such interval (owing to Theorem 11), it follows that $v^*(x,t)$ uniformly converges to $v^*(x)$ for x such that $v(x)$ has its values between ε and $1-\varepsilon$ (ε is arbitrarily small). For x outside that interval, we have uniform convergence $v^*(x,t) \to v^*(x)$, since for sufficiently large t the function $v^*(x,t)$ has its values close to 0 and 1.

Theorem 14. *As $t_0 \to +\infty$, the sequence of functions*

$$v_{t_0}(x,t) = v(x+\varphi(t_0),\, t+t_0)$$

converges to a solution $\bar{v}(x,t)$ of equation (29), uniformly in the domain $t \leq T = \text{const}$. The function $\varphi(t_0)$ is defined in such a way that

$$v_{t_0}(0,0) = c = \text{const} , \quad \text{for all} \quad t_0 .$$

Proof. The function $w(x,t) = v_{t_0}(x,t) - v_{t_0+T}(x,t)$ satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = F'(\bar{v})w , \tag{52}$$

where $\bar{v}(x,t)$ is between $v_{t_0}(x,t)$ and $v_{t_0+T}(x,t)$. According to Theorem 13,

$$|w(x,0)| < \varepsilon \quad \text{for sufficiently large} \quad t_0 ,$$

where ε is arbitrarily small. By Theorems 2 and 3, we have

$$w(x,t) < \bar{w}(x,t) \equiv \varepsilon\, e^{kt} ,$$

where k is the upper bound for the values of $|F'(u)|$, since $\bar{w}(x,t) \geq w(x,0)$ for $t=0$, and for $t>0$, $\bar{w}$ satisfies the equation

$$\frac{\partial \bar{w}}{\partial t} - \frac{\partial^2 \bar{w}}{\partial x^2} = k|\bar{w}| ,$$

whose right-hand side, for $w = \bar{w}$, is not smaller than the right-hand side of equation (52). In exactly the same way we can prove the inequality

$$w(x,t) > -\varepsilon\, e^{kt} .$$

Thereby we have shown that the sequence $v_{t_0}(x,t)$, $t_0 \to +\infty$, uniformly converges, in some domain $t < T$, to a function, which we denote by $\bar{v}(x,t)$. Let us show that equation (29) holds for $\bar{v}(x,t)$.

For this purpose, we can use (42) and write

$$v_{t_0}(x,t) = v_{t_0,0}(x,t) + \frac{1}{2\sqrt{\pi}} \int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} F(v_{t_0}(\xi,\eta))\, d\xi\ . \quad (53)$$

Here we can pass to the limit, after substituting $\bar{v}$ for v_{t_0}. The function satisfying equation (53) is also a solution of equation (29), as shown in the proof of Theorem 1.

Theorem 15. *The first order partial derivatives of $v_{t_0}(x,t)$ in x and t converge to the respective derivatives of $\bar{v}(x,t)$ as $t_0 \to +\infty$; the convergence is uniform in any region $\varepsilon < t < T$, where ε and T are arbitrary positive constants.*

Proof. Uniform convergence of $\partial v_{t_0}/\partial x$ is established on the basis of (53). Indeed, for $t > \varepsilon$, uniform convergence of the derivative in x of the first term in the right-hand side follows from the representation of that term by the Poisson integral. In order to prove this result for the second term with $t < T$, consider the difference of its values for $t_0 = t_0'$ and $t_0 = t_0''$:

$$\frac{1}{2\sqrt{\pi}} \int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\xi)^2}{4(t-\eta)}\right)}{\sqrt{(t-\eta)}} [F(v_{t_0'}) - F(v_{t_0''})]\, d\xi\ . \quad (54)$$

According to Theorem 14,

$$F(v_{t_0'}(\xi,t)) - F(v_{t_0''}(\xi,t))$$

becomes arbitrarily small for large t_0' and t_0''. In this situation, applying the above mentioned result of [5], we find that for sufficiently large t_0' and t_0'' the x-derivative of (54) becomes arbitrarily small (uniformly in x) if $t < T$.

The function $w_{t_0}(x,t) = \partial v_{t_0}(x,t)/\partial x$ satisfies the equation

$$\frac{\partial w_{t_0}}{\partial t} - \frac{\partial^2 w_{t_0}}{\partial x^2} = F'(v_{t_0}) w_{t_0}\ .$$

For $\varepsilon < t < T$, we have already established uniform convergence of the right-hand side of this equation as $t_0 \to \infty$. Therefore, arguments similar to those used for the proof of uniform convergence of $\partial v_{t_0}/\partial x$, can be applied to prove uniform convergence of $\partial w_{t_0}/\partial x = \partial^2 v_{t_0}/\partial x^2$. Since v_{t_0} satisfies (29), it follows that $\partial v_{t_0}/\partial t$ is uniformly convergent.

Theorem 16. *Assume that the function $v_{t_0}(x,t)$ (resp,. $\bar{v}(x,t)$) remains equal to a constant c along the curve $x = \varphi_{t_0}(t)$ (resp., $x = \varphi(t)$). Then*

$$\varphi_{t_0}'(t) \to \varphi'(t) \quad \text{as} \quad t_0 \to \infty\ ,$$

uniformly in t for $\varepsilon < t < T$.

Proof. We have

$$\varphi_{t_0}'(t) = -\frac{\partial v_{t_0}}{\partial t} \Big/ \frac{\partial v_{t_0}}{\partial x} \quad \text{at the point} \quad (\varphi_{t_0}(t),t)\ ;$$

$$\varphi'(t) = -\frac{\partial \bar{v}}{\partial t} \Big/ \frac{\partial \bar{v}}{\partial x} \quad \text{at the point} \quad (\varphi(t),t)\ .$$

By Theorems 12 and 14, for sufficiently large t_0, we have

$$|\varphi_{t_0}(t) - \varphi(t)| < \varepsilon_1$$

everywhere in the domain $\overline{G}$ ($\varepsilon < t < T$), where ε_1 is arbitrarily small. According to Theorem 15, the respective numerators and denominators of the fractions

$$\frac{\partial v_{t_0}}{\partial t} \Big/ \frac{\partial v_{t_0}}{\partial x} \quad \text{and} \quad \frac{\partial \bar{v}}{\partial t} \Big/ \frac{\partial \bar{v}}{\partial x} \quad (55)$$

are arbitrarily close to one another in $\overline{G}$ for the same values of their arguments. Moreover, in the strip

$$\varphi(t) - \varepsilon_2 < x < \varphi(t) + \varepsilon_2\ ,$$

the function $\partial\bar{v}/\partial x$ is larger than a positive constant. Therefore, the fractions (55), for the same values of their arguments and sufficiently large t_0, differ less than by ε_3 on the strip

$$\varepsilon < t < T\ , \qquad \varphi(t) - \varepsilon_2 < x < \varphi(t) + \varepsilon_2\ .$$

If we also take into account that $\partial\bar{v}/\partial t\ (\partial\bar{v}/\partial x)^{-1}$ is uniformly continuous on that strip, and therefore, its values at the points of the strip with the same t are arbitrarily close to one another for small enough ε_3, we get the statement of our theorem.

Theorem 17. *For any t, we have*

$$\bar{v}(x,t) = u(x+2t)\ , \quad \text{and} \quad \frac{d\varphi}{dt} \to -2 \quad \text{as} \quad t \to \infty\ .$$

(The notation here is that of Theorem 14.)

Proof. Consider the function

$$v^*(x,t) = \bar{v}(x + c_1(t), t)\ ,$$

where $c_1(t)$ is chosen such that

$$v^*(0,t) \equiv c = \text{const}\ .$$

Then

$$\frac{\partial v^*}{\partial t} = \frac{\partial^2 v^*}{\partial x^2} + c_1'(t)\frac{\partial v^*}{\partial x} + F(v^*)\,.$$

On the other hand, for any x, $v^*(x,t)$ does not depend on t (this follows from the definition of $\bar{v}(x,t)$). Therefore,

$$\frac{\partial v^*}{\partial t} = 0 \quad \text{and} \quad c_1'(t) = \text{const}\,.$$

According to §2, this constant cannot be larger than −2; and it cannot be smaller than −2, by Theorem 10. Therefore, it is equal to −2, and by Theorem 16 we have

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} \to -2 \quad \text{as} \quad t \to \infty\,, \qquad \text{q.e.d.}$$

Remark. Assume that the initial values of $v(x,t)$ are other than those considered so far, namely, let

1) $v(x,0) = 1$ for $x \geq c_1$;
2) $v(x,0) = 0$ for $x \leq c_2 < c_1$;
3) $0 \leq v(x,0) \leq 1$ for $c_2 < x < c_1$.

Then it is easy to show that the segment of the curve bearing the major part of the drop from 1 to 0 travels with a speed which, nevertheless, tends to 2 as $t \to \infty$, since in this case we have

$$v(x - c_1, t) \leq \bar{v}(x,t) \leq v(x - c_2, t)\,;$$

here $\bar{v}(x,t)$ is the solution of (29) with the new initial conditions.

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7

On the Speed of Propagation of Discontinuities of Displacement Derivatives on the Surface of a Non-Homogeneous Elastic Body of Arbitrary Shape*

1. Let us regard the time variable t as a spatial coordinate and assume that the displacements u, v, w, usually considered in the theory of elasticity, are defined inside and on the boundary of a cylinder C with its generatrix going along the t-axis and its base coinciding with the elastic body in question. To simplify the notation, set $u = u_1$, $v = u_2$, $w = u_3$. We also assume that in the vicinity of a point $M(x^0, y^0, z^0)$ the body is bounded by a surface of the form $z = f(x,y)$, where f has continuous derivatives up to the order $n+2$, or briefly, f has smoothness $n+2$. First we consider the case $n \geq 3$.

In general, we examine a non-homogeneous anisotropic elastic body free of externally applied forces, under the assumption that the coefficients of the elasticity system and those of the boundary conditions do not depend on t and have smoothness n and $n+1$, respectively.

Assume that the functions u_i, in a neighborhood of the point M^0 with coordinates (t^0, x^0, y^0, z^0), have smoothness n, whereas on the surface of C near this point, u_i have continuous derivatives up to the order $n+1$ everywhere, except at the points of a two-dimensional surface S_2 containing M^0 and having smoothness $n+2$; at the points of S_2 the displacements u_i have singularities of the type specified below. Our problem consists in finding the slope of S_2 at the point M^0.

2. In a neighborhood of M^0, let us consider a transformation of the variables t, x, y, z with smoothness $n+2$ and the following properties:

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