Equilibrium Conditions and Algorithm for Numerical Solution of Kaplan, Moll and Violante (2017) HANK Model.

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1 Introduction

This document describes the equilibrium conditions of Kaplan, Moll and Violante (2017) and the computational algorithm used.

2 Equilibrium Conditions: Steady State

We start with the steady state equilibrium conditions. To emphasize that the time subscript is removed.

2.1 Household Maximization

Households discount time by rate ρ, must decide savings on an illiquid and a liquid asset, a and b, are subject to productivity shocks z with logs following a Markov Process, receive bonuses and commissions given by $\pi(z)$ and have exogenous probability of dying ζ. Because of perfect annuity markets households receive interest $r^b(b)+\zeta$ in the liquid asset and $r^a+\zeta$ in the illiquid asset.

Household's value function is given by

$$
\begin{aligned}
(\rho+\zeta)V(a,b,z) = & \max_{c,\ell,d} u(c,\ell) + V_b(a,b,z)\left[(1-\tau)wz\ell + (r^b(b)+\zeta)b + T - d - \chi(d,a) + \pi(z) - c\right] \\
& + V_a(a,b,z)((r^a+\zeta)a+d) + V_y(a,b,z)(-\beta z) \\
& + \lambda\int_{-\infty}^{\infty}\left(V(a,b,x)-V(a,b,z)\right)\phi(x)dx \\
& \text{s.t.} \\
& b \geq \bar{b}; a \geq 0; 0 \leq \ell \leq 1; r^b(b) = r^b + I\{b \leq 0\}\kappa
\end{aligned} \tag{1}
$$

The respective state-constraint boundary conditions are then given by

$$V_b(a,\bar{b},z) \geq u_c(c,\ell) \tag{2}$$
$$V_a(0,b,z) \geq u_c(c,\ell) \tag{3}$$

The FOC can then be written as

$$u_c = V_b \tag{4}$$
$$V_b(1+\chi_d(d,a)) = V_a \tag{5}$$
$$u_\ell = -V_b(1-\tau)wy \tag{6}$$
$$\tag{7}$$

Note that there are two kinks in the budget constraint. The first one is at $b = 0$, which imply that 4 will not hold with equality, the second one is due to the fixed cost of adjustment in the illiquid account, which implies that 5 will not hold with equality. Moreover, 6 may not hold because of the boundary condition on ℓ.[1]

Using that $u(c, \ell) = \frac{c^{1-\gamma}}{1-\gamma} - \varphi \frac{\ell^{1+\nu}}{1+\nu}$ one can write in an interior solution

$$c = (V_b)^{-1/\gamma} \tag{8}$$

$$V_b(1 + \chi_d(d, a)) = V_a \tag{9}$$

$$\ell = \left(\frac{V_b(1-\tau)wy}{\varphi} \right)^{1/\nu} \tag{10}$$

Using the functional form for χ_d we finally have

$$d = \left\{ \left(\left[\frac{V_a}{V_b} - 1 - \chi_0 \right] \frac{1}{\chi_1 \chi_2} \right)^{1/(\chi_2 - 1)} a \right\}^+ + \left\{ \left(- \left[\frac{V_a}{V_b} - 1 + \chi_0 \right] \frac{1}{\chi_1 \chi_2} \right)^{1/(\chi_2 - 1)} a \right\}^- \tag{11}$$

2.2 Kolmogorov Forward Equation

Let $\mu(a, b, z)$ be the stationary distribution of households and $g(a, b, z)$ the corresponding density function. Let $s^a(a, b, z) = r^a a + d$ and $s^b(a, b, z) = (1-\tau)wz\ell + r^b(b)b + T - d - \chi(d, a) + \pi(z) - c$. The Kolmogorov Forward Equation is given by

$$\begin{aligned} 0 = &- \partial_a \left(s^a(a, b, z) g(a, b, z) \right) \\ &- \partial_b \left(s^b(a, b, z) g(a, b, z) \right) - \partial_y (-\beta y g(a, b, z)) \\ &- \lambda g(a, b, z) + \lambda \phi(y) \int_{-\infty}^{\infty} g(a, b, x) dx - \zeta g(a, b, z) + \zeta \delta(a - a_0) \delta(b - b_0) g^*(y) \end{aligned} \tag{12}$$

where δ is the Dirac delta function, (a_0, b_0) are starting assets and income and $g^*(y)$ is the stationary distribution of y.

2.3 Firms

2.3.1 Final Good Producer

There is a competitive representative final good producer with aggregating technology given by

$$Y = \left(\int_0^1 y_j^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

that solves

[1] To determine consumption when $b = 0$ one must calculate household's optimal solution when $\dot{b} > 0$ and she faces $r^b(b) = r^b$, when $\dot{b} < 0$ and she faces $r^b(b) = r^b + \kappa$ and when $\dot{b} = 0$. The household pick the solution that renders maximum utility. Moreover, at the state constraint 2 and when $\dot{b} = 0$ the consumption-leisure condition determine labor supply

$$\frac{u_\ell((1-\tau)wz\ell + (r^b(b) + \zeta)b + T + \pi(z), \ell)}{u_c((1-\tau)wz\ell + (r^b(b) + \zeta)b + T + \pi(z), \ell)} = (1-\tau)wy$$

$$\min_{y_j} \int_0^1 p_j y_j dj$$
s.t.
$$Y = \left(\int_0^1 y_j^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

That gives the demand for each intermediate good j

$$y_j(p_j) = \left(\frac{p_j}{P} \right)^{-\epsilon} Y$$

where P is the natural price index given by $P = \left(\int_0^1 p_j^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}$.

2.3.2 Intermediate Good Producer

Each intermediate good j is produced by a monopolistic competitive producer with Cobb-Douglas technology and quadratic price adjustment cost. Each intermediate producer choose prices and inputs to maximize discounted present value profits and satisfy the demand $y_j(p_j)$ by the final good producer.

Given price p_j the intermediate good producer chooses k_j and n_j to solve

$$\min_{k_j, n_j} w n_j + r^k k_j$$
s.t.
$$y_j = k_j^{\alpha} n_j^{1-\alpha}$$
$$y_j = \left(\frac{p_j}{P} \right)^{-\epsilon}$$

Which imply the factor demands

$$k_j = y_j \left(\frac{\alpha}{1-\alpha} \frac{w}{r^k} \right)^{1-\alpha}$$
$$n_j = \frac{y_j}{\left(\frac{\alpha}{1-\alpha} \frac{w}{r^k} \right)^{\alpha}}$$

With operational real profits (Π_j) and marginal cost (m) given by

$$m = \left(\frac{w}{1-\alpha} \right)^{1-\alpha} \left(\frac{r^k}{\alpha} \right)^{\alpha} \tag{13}$$
$$\Pi_j = \left(\frac{p_j}{P} - m \right) y_j = \left(\frac{p_j}{P} - m \right) \left(\frac{p_j}{P} \right)^{-\epsilon}$$

Firms are subject to quadratic adjustment cost given by

$$\Theta(\pi_j) = \frac{\theta}{2} (\pi_j)^2 Y$$

Therefore, firm's dynamic price setting problem is then given by

$$r^a J(p_j) = \max_{\pi} \left(\frac{p_j}{P} - m\right)\left(\frac{p_j}{P}\right)^{-\epsilon} Y - \frac{\theta}{2}\pi^2 Y + J_p(p_j)p_j\pi_j$$

Which implies that in steady state

$$m = \frac{\epsilon - 1}{\epsilon} \tag{14}$$

Finally, using this condition and that firms are identical, one can write

$$K = Y\left(\frac{\alpha}{1-\alpha}\frac{w}{r^k}\right)^{1-\alpha} \tag{15}$$

$$N = \frac{Y}{\left(\frac{\alpha}{1-\alpha}\frac{w}{r^k}\right)^{\alpha}} \tag{16}$$

$$Y = K^{\alpha}N^{1-\alpha} \tag{17}$$

$$\Pi = (1-m)Y \tag{18}$$

Where capital letters denote aggregates.

2.4 Government

The government taxes labor supply, issue debt B^g, redistribute amount T to households and expend on exogenous goods G. So the government budget constrain in steady state is

$$G + T = \tau w N + r^b B^g \tag{19}$$

2.5 Distribution of Profits

Profits are going to be distributed among share holders and workers. A share $1-\omega$ of total profits Π are given to workers according to their productivity

$$\pi(z) = \frac{z}{\bar{z}}(1-\omega)\Pi \tag{20}$$

where $\bar{z}$ is the average productivity.

2.6 Illiquid Asset

The illiquid asset is allocated efficiently and without costs in capital or equity share: for each household $a = k + qs$. Normalizing the aggregate amount of equity to one ($\int sd\mu = 1$) one can write the following relation between aggregate illiquid assets, A, aggregate capital, K, and equity, q:

$$A = K + q \tag{21}$$

Because resources can be allocated efficiently between the two assets, returns must equalize

$$\frac{\Pi\omega}{q} = r^k - \delta \equiv r^a$$

where we have used that $\dot{q}_t = 0$ on steady state. Which imply that

$$q = \frac{\Pi\omega}{r^a} = \frac{\omega(1-m)Y}{r^a} \tag{22}$$

Using 21, 22 and 14 we have an expression for steady state capital

$$K = A - \frac{\omega(1-m)Y}{r^a} = A - \omega\frac{Y}{r^a\epsilon} \tag{23}$$

Note that an individual household's illiquid portfolio composition, i.e., the split of a into k and qs, is indeterminate. In contrast, the aggregate illiquid portfolio, the split of A into K and q, is pinned down.

2.7 Liquid Asset

The government is the only source of liquid asset. Therefore in equilibrium

$$B^h + B^g = 0 \tag{24}$$

where B^h is the aggregate liquid asset in the hands of households.

2.8 Stationary Equilibrium

The set of aggregate equilibrium objects are $\{K, N, Y, \Pi, m, w, r^k, T, \pi(z), A, q, B^h, r^a\}$ that with 13 to 15, 19, 20, 21, 22, 24 and

$$A = \int a d\mu$$
$$N = \int \ell d\mu$$

provides a system of 13 equations and 13 unknowns.

Definition 2.1 (Stationary Equilibrium)**.** An **Stationary Equilibrium** is defined by a solution to households' problem $\{a, b, c, d, \ell, V\}$, a stationary distributions $\{\mu\}$, a solution to firm's problem $\{n_j, k_j, \Pi_j\}$, prices $\{w, r^k, r^b, r^a, q\}$, government fiscal policy $\{\tau, T, G, B^g\}$ and aggregate quantities $\{K, N, Y, \Pi, A, B^h\}$ such that

1. Given prices $\{w, r^b, r^a\}$, profits Π and government fiscal policy $\{\tau, T\}$, $\{a, b, c, d, \ell, V\}$ solves household's problem characterized by 1-11;
2. Given the solution for agent's problem $\{a, b, c, d, \ell, V\}$, $\{\mu\}$ satisfy 12;
3. The aggregate variables $\{A, B^h, N\}$ are compatible with individual households' policy functions and stationary distribution;
4. Given prices $\{w, r^b, r^a\}$, $\{n_j, k_j\}$ solve firm's problem 13 with profit given by Π_j. Moreover
$$k_j = K, n_j = N, \Pi_j = \Pi$$
5. The government budget constrain 19 is satisfied;
6. The illiquid asset markets clears 21 with conditions 23 and 22 satisfied;
7. The liquid asset markets clears 24.

2.9 Algorithm

1. Guess K, N and r^b;
2. Calculate implied prices:

$$
\begin{aligned}
Y &= K^\alpha N^{1-\alpha} \\
m &= \frac{\epsilon - 1}{\epsilon} \\
r^k &= \frac{\alpha m}{(K/N)^{1-\alpha}} \\
w &= (1-\alpha) m (K/N)^\alpha \\
r^a &= r^k - \delta
\end{aligned}
$$

3. Calculate implied profits and worker's bonus:

$$\Pi = (1-m)Y$$

4. Find government fiscal policy:
   - If taxes are adjusting set τ to:

$$\tau = \frac{G + T - r^b B^g}{wN}$$

   - If transfers are adjusting set T to:

$$T = \tau w N + r^b B^g - G$$

   - If the exogenous expenditure is adjusting set G to

$$G = \tau w N + r^b B^g - T$$

   - If debt is adjusting set B^g to

$$B^g = \frac{G + T - \tau w N}{r^b}$$

5. Solve the household problem and find the stationary distribution using the method described by Achdou et al. (2017);
6. Aggregating individual's policy function compute implied illiquid asset holdings, A', labor supply, N', and household liquid asset demand $B^{h'}$;
7. Using aggregate illiquid asset A', calculate implied equity, q', and capital, K':

$$
\begin{aligned}
q' &= \frac{(1-m)Y}{r^a} \\
K' &= A' - \frac{Y}{\epsilon}
\end{aligned}
$$

   where Y is obtained from the guess of K and N;
8. Calculate the excess demand

$$\Lambda = \frac{|K - K'|}{K} + \frac{|N - N'|}{N} + \frac{|B^{h'} + B^g|}{B^g}$$

9. If Λ is close enough to zero the equilibrium have been found. Otherwise update the guess and go back to step 2.

3 Equilibrium Conditions: Transition

3.1 Household Maximization

The household value function is now given by

$$
\begin{aligned}
(\rho+\zeta)V_t(a,b,z) = & \max_{c,\ell,d} u(c,\ell) + V_{b,t}(a,b,z)\left[(1-\tau_t)w_t z\ell + (r_t^b(b)+\zeta)b + T_t - d - \chi(d,a) + \pi_t(z) - c\right] \\
& + V_{a,t}(a,b,z)((r_t^a+\zeta)a+d) + V_{y,t}(a,b,z)(-\beta z) \\
& + \lambda\int_{-\infty}^{\infty}\left(V_t(a,b,x)-V_t(a,b,z)\right)\phi(x)dx + \dot{V}_t(a,b,y) \\
& \text{s.t.} \\
& b \geq \bar{b}; a \geq 0; 0 \leq \ell \leq 1; r_t^b(b) = r_t^b + I\{b \leq 0\}\kappa
\end{aligned} \tag{25}
$$

Which implies an equivalent set of equations as in the steady state case: 2-11.

3.2 Kolmogorov Forward Equation

Let $\mu_t(a,b,y)$ be the distribution of households at time t and $g_t(a,b,y)$ the corresponding density function. The Kolmogorov Forward Equation is now given by

$$\partial_t g_t(a,b,y) = -\partial_a\left(s_t^a(a,b,y)g_t(a,b,y)\right) \tag{26}$$
$$-\partial_b\left(s_t^b(a,b,y)g_t(a,b,y)\right) - \partial_y(-\beta y g_t(a,b,y)) \tag{27}$$
$$-\lambda g_t(a,b,y) + \lambda\phi(y)\int_{-\infty}^{\infty} g_t(a,b,x)dx - \zeta g_t(a,b,y) + \zeta\delta(a-a_0)\delta(b-b_0)g_t^*(y) \tag{28}$$

where δ is the Dirac delta function, (a_0,b_0) are starting assets and income and $g^*(y)$ is the stationary distribution of y.

3.3 Firms

3.3.1 Final Good Producer

The final good producer has the same problem as before with implied demand for the intermediate good given by:

$$y_j(p_{j,t}) = \left(\frac{p_{j,t}}{P_t}\right)^{-\epsilon} Y_t$$

3.4 Intermediate goods producers

Given the price $p_{j,t}$ the intermediate good producer chooses $k_{j,t}$ and $n_{j,t}$ to solve

$$
\begin{aligned}
& \min_{k_{j,t},n_{j,t}} w_t n_{j,t} + r_t^k k_{j,t} \\
& \text{s.t.} \\
& y_{j,t} = \Xi_t k_{j,t}^{\alpha} n_{j,t}^{1-\alpha} \\
& y_{j,t} = \left(\frac{p_{j,t}}{P_t}\right)^{-\epsilon} Y_t
\end{aligned}
$$

where Ξ_t is TFP shock at t. The solution for firm's problem imply that

$$k_{j,t} = \frac{y_{j,t}}{\Xi_t}\left(\frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k}\right)^{1-\alpha} \tag{29}$$

$$n_{j,t} = \frac{y_{j,t}}{\Xi_t\left(\frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k}\right)^{\alpha}} \tag{30}$$

$$m_{j,t} = \frac{1}{\Xi_t}\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{r_t^k}{\alpha}\right)^{\alpha} \tag{31}$$

$$\tilde{\Pi}_{j,t} = \left(\frac{p_{j,t}}{P_t} - m_{j,t}\right) y_{j,t} = \left(\frac{p_{j,t}}{P_t} - m_{j,t}\right)\left(\frac{p_{j,t}}{P_t}\right)^{-\epsilon} \tag{32}$$

$$\tag{33}$$

Where $\tilde{\Pi}_{j,t}$ is firm's operational profit.

Firm's dynamic price setting problem imply

$$\left(r_t^a - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\epsilon}{\theta}\left(m_t - m^*\right) + \dot{\pi}_t \tag{34}$$

$$m^* = \frac{\epsilon - 1}{\epsilon}$$

Firm's profit is given by

$$\Pi_{j,t} = \tilde{\Pi}_{j,t} - \Theta(\pi_{j,t})$$

Using again that firms are homogeneous we have

$$m_t = \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{r_t^k}{\alpha}\right)^{\alpha} \tag{35}$$

$$K_t = Y_t\left(\frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k}\right)^{1-\alpha} \tag{36}$$

$$N_t = \frac{Y_t}{\left(\frac{\alpha}{1-\alpha}\frac{w}{r_t^k}\right)^{\alpha}} \tag{37}$$

$$Y_t = \Xi_t K_t^{\alpha} N_t^{1-\alpha} \tag{38}$$

$$\Pi_t = \left(1 - m_t\right) Y_t - \Theta(\pi_t) \tag{39}$$

34 to 39 characterize the solution to firm's problem.

3.5 Government

The government budget constraint is given by

$$\dot{B}_t^g + G_t + T_t = \tau_t w_t N_t + r_t^b B_t^g \tag{40}$$

3.6 Distribution of Profits

The distribution of profits follows the same scheme as in the steady state scenario. A share $1-\omega$ of total profits Π_t are given to workers as bonuses according to their productivity

$$\pi_t(z) = \frac{z}{\bar{z}}(1-\omega)\Pi_t \tag{41}$$

where $\bar{z}$ is the average productivity. The remaining goes to equity holders.

3.7 Illiquid Asset

The illiquid asset is invested in capital and equity from the intermediary firms as before. This again imply that

$$A_t = K_t + q_t \tag{42}$$

$$\frac{\omega\Pi_t + \dot{q}_t}{q_t} = r_t^k - \delta = r_t^a \tag{43}$$

which are the two equations characterizing equilibrium in the illiquid asset market.

3.8 Liquid Asset

The liquid asset market clears when government debt is equal to household holdings of liquid bonds

$$B_t^h + B_t^g = 0 \tag{44}$$

3.9 Monetary Authority

The Monetary authority follows a Taylor rule affecting the return of the liquid asset

$$i_t = \bar{r}^b + \phi\pi_t + \epsilon_t \tag{45}$$

therefore $r_t^b = i_t - \pi_t$.

3.10 Equilibrium

The set of aggregate equilibrium objects are $\{K_t, N_t, Y_t, \Pi_t, m_t, w_t, r_t^k, T_t, \pi_t(z), A_t, q_t, B_t^h, r_t^a, \pi_t\}$ that with 34 to 39, 40, 41, 42, 43, 44, 45 and the definition of A_t and N_t form a system with 14 equations and 14 unknowns at each point t in time.

Definition 3.1 (Equilibrium)**.** Equilibrium is defined by a solution to households' problem $\{a_t, b_t, c_t, d_t, \ell_t, V_t\}$, a path of distributions $\{\mu_t\}$, a solution to firm's problem $\{n_{j,t}, k_{j,t}, \pi_{j,t}\}$, prices $\{w_t, r_t^k, r_t^b, r_t^a, q_t\}$, the inflation rate $\{\pi_t\}$, shocks $\{\Xi_t, \epsilon_t\}$, government fiscal policy $\{\tau_t, T_t, G_t, B_t^g\}$ and aggregate quantities $\{K_t, N_t, Y_t, \Pi_t, A_t, B_t^h\}$ such that

1. Given prices $\{w_t, r_t^b, r_t^a\}$ profits Π_t and government taxes $\{\tau_t, T_t\}$, $\{a_t, b_t, c_t, d_t, \ell_t, V_t\}$ solves household's problem characterized by 25;
2. Given the solution for agent's problem $\{a_t, b_t, c_t, d_t, l_t, V_t\}$, $\{\mu_t\}$ satisfy 26;
3. The aggregate quantities $\{N_t, A_t, B_t^h\}$ are consistent with households' maximization and the path for distribution $\{\mu_t\}$;

4. Given prices $\{w_t, r_t^b, r_t^a\}$, $\{n_{j,t}, k_{j,t}\}$ solve firm's problem 13 with profit given by Π_t. Moreover

$$k_t = K_t, n_t = N_t$$

   and π_t satisfy ;

5. The government budget constrain 40 is satisfied;
6. The illiquid asset markets clears 42 and 43;
7. The liquid asset markets clears 44.
8. The Taylor rule 45 is satisfied;

3.11 Algorithm

Let's say that the economy is initially at the steady state equilibrium and then is hit by a transitory monetary shock ϵ_0 or a transitory TFP shock A_0. The algorithm to calculate the transition dynamics is given by the following:

1. Compute the steady state equilibrium;
2. Guess a path for capital, K_t, liquid asset holdings, B_t^h, labor supply, N_t, and illiquid asset returns,r_t^a;
3. Calculate the excess demand of liquid assets, $B^w = B_t^h + B_t^g$, and use it to derive the implied guess for interest rate in the liquid asset

$$r_t^b = r^* + \frac{\frac{1}{\phi}\dot{B}_t^w + B_t^w - \bar{B}}{\eta}$$

   where r^* is the stationary liquid asset return and η, ϕ and $\bar{B}$ are parameters;

4. Using this guess compute the implied prices and equilibrium objects
   - Inflation using the Taylor Rule:

$$\pi_t = \frac{r_t^b - \bar{r}^b - \epsilon_t}{\phi - 1}$$

   - Marginal cost by solving the Phillips Curve backward in time with the terminal condition $m_T = m^*$:

$$m_t = m^* + \frac{\theta}{\epsilon}\left[\left(r_t^a - \frac{\dot{Y}_t}{Y_t}\right)\pi_t - \dot{\pi}_t\right]$$

   - Wage:

$$w_t = (1-\alpha)\Xi_t m_t \left(\frac{K_t}{N_t}\right)^{\alpha}$$

   - Profits:

$$\Pi_t = (1 - m_t)\, Y_t - \Theta(\pi_t)$$

- Equity by solving the following equation backward with terminal condition that $q_T = q^*$:

$$\dot{q}_t = q_t r^a_t - \omega \Pi_t$$

  Because q_0 drops and K_0 is fixed, aggregate illiquid holdings A_0 drop on impact. Similarly, individual households' a_0 drops. For simplicity, we assume that in steady state before impact all households have the same share of illiquid assets invested in equity $q^*s^*/a^* = q^*/A^*$ and $k^*/a^* = K^*/A^*$, where an asterisk denotes variables before impact. Hence each households' a_0 drops proportionately to A_0:

$$a_0 = k^* + q_0 s^* = \frac{K^*}{A^*}a^* + q^0 \frac{1}{A^*}a^* = \frac{A_0}{A^*}a^*$$

- Use government budget constrain 40 to calculate the implied tax, transfer or debt depending on what is adjusting;

5. Solve household's transition problem and find the path of distributions using the method described by Achdou et al. (2017);
6. Using the path of distributions compute the implied path for liquid assets $B^{h'}_t$, illiquid assets, A'_t and labor supply, N'_t;
7. Use equilibrium on the illiquid asset to calculate the implied capital:

$$K'_t = A'_t - q_t$$

8. Calculate the error

$$\Lambda = |K'_t - K_t| + |B^{h'}_t - B^h_t|$$

9. If Λ is close to zero, the equilibrium have been found. Otherwise adjust the guess and go back to step 2.