A Reconsideration of the Theory of Value

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Part II.—A Mathematical Theory of Individual Demand Functions

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THE established definition of the way in which a pair of goods can be related in an individual's scale of preferences is due, in its precise form, to Edgeworth and Pareto. The definition assumes the existence of a utility function giving the utility to the individual of any combination of the set of consumers' goods, $X, Y, Z, \ldots\ldots$, which enter into the individual's budget. Denoting the function by $u=\phi(x, y, z, \ldots\ldots)$, the Edgeworth-Pareto definition of the relation between any pair of goods, X and Y, depends on the sign of $\frac{\partial^2 u}{\partial x \partial y}$. The goods are *complementary* or *competitive* according as $\frac{\partial^2 u}{\partial x \partial y}$ is positive or negative. If the individual possesses an increased amount of one good, the marginal utility of the other good increases in the complementary case and decreases in the competitive case.

This definition ignores one fundamental fact. Even if the utility function exists at all, it is by no means unique and it can serve only as an *index*, and not as a *measure*, of individual utility. Pareto himself established this fact, but failed to deduce the logical corollary that the derivative $\frac{\partial^2 u}{\partial x \partial y}$ is also indeterminate both in sign and in magnitude. To prove this corollary we proceed as follows. The utility function index is the integral of the " indifference " differential equation:

$$\phi_x dx+\phi_y dy+\phi_z dz+\ldots\ldots=0,$$

where ϕ_x, ϕ_y, ϕ_z, ... are the marginal utility functions, determinate only as to their ratios. If $u=\phi(x, y, z, \ldots\ldots)$ is *one* form of the integral, and if $\phi_x, \phi_y, \phi_z, \ldots\ldots$ are arranged to be the partial derivatives of this function, then the general utility function index is :

$$u=F\ \{\phi(x,y,z,\ldots\ldots)\}$$

where F is an arbitrary function with a positive derivative.[1]

The partial derivatives of the function index are

$$\frac{\partial u}{\partial x}=F'(\phi).\phi_x;\ \frac{\partial u}{\partial y}=F'(\phi).\phi_y;\ \frac{\partial u}{\partial z}=F'(\phi).\phi_z;\ldots\ldots$$

The signs of these derivatives are the signs of $\phi_x, \phi_y, \phi_z, \ldots\ldots$ respectively and their ratios are determinate (as required). On the other hand, the sign of the second-order derivative

$$\frac{\partial^2 u}{\partial x \partial y}=F'(\phi).\phi_{xy}+F''(\phi).\phi_x.\phi_y$$

cannot be taken as determinate and it depends entirely on the form adopted for the arbitrary function.[2] The above definition of complementary and competitive goods is thus indefinite; for a given combination $(x,y,z,\ldots\ldots)$, the same pair of goods can be sometimes complementary and sometimes competitive according to the form taken for the utility function index. The Edgeworth-Pareto definition is not adequate for the distinction of relations between a pair of goods and it must be rejected in any precise theory of individual choice. The development which follows will decide, amongst other things, what can be put in its place.

I. INDIVIDUAL DEMAND IN THE CASE OF TWO GOODS

1. In the present section we shall consider the case where only two goods, X and Y, enter into the budget of a given individual. The case can be interpreted in two ways. *Either* all goods other than the pair XY are possessed by the individual in known amounts, and he considers his expenditure on X and Y independently, *or* the pair XY represent two broad classes of goods (e.g. food and other items) which together make up the individual's complete budget. In either case, it is seen that the theory of demand for two goods is subject to severe limitations in possible applications.

[1] The positive derivative is only necessary to ensure that u is a genuine index of utility in the sense that all its forms increase or decrease together.

[2] The only case in which $\frac{\partial^2 u}{\partial x \partial y}$ is definite in sign is when either ϕ_x or ϕ_y is zero, i.e. when the marginal utility ratio $\phi_x : \phi_y$ is zero or infinite and the individual is "saturated" with one of the goods. This possibility does not concern us in our consideration of the individual under market conditions.

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2. *The individual's scale of preferences.*—The individual possesses the combination (x,y) of amounts of the two goods X and Y.[1] The fundamental postulate of the theory is that there exists a unique " indifference direction " for variations from the combination (x,y) defined by a differential equation:

$$dx + R^y_x dy = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation expresses a relation between increments dx and dy (one positive and one negative) which just compensate each other as far as the individual is concerned. The expression $R^y_x = -dx/dy$ is the (limiting) ratio of compensating increments in x and y,[2] i.e. R^y_x is *the marginal rate of substitution of X for Y.* As the combination (x,y) varies, so does the individual's indifference direction and the value of R^y_x. In fact, R^y_x is a function of x and y, and its values for various combinations (x,y) describes *the scale of preferences* of the individual. Only the function R^y_x is needed, since the marginal rate of substitution of Y for X is $R^x_y = 1/R^y_x$.

Three assumptions are made about the scale of preferences:

(1) R^y_x is a continuous function of x and y.

(2) R^y_x is positive at all points (x,y).

(3) For the variation of the indifference direction from any point, the expression $dx + R^y_x dy$ always decreases:
$d(dx + R^y_x dy) < 0$ subject to $dx + R^y_x dy = 0$,

i.e. $$\begin{vmatrix} 1 & R^y_x \\ \frac{\partial}{\partial x} R^y_x & \frac{\partial}{\partial y} R^y_x \end{vmatrix} = \frac{\partial}{\partial y} R^y_x - R^y_x \frac{\partial}{\partial x} R^y_x < 0 \quad \ldots\ldots (2).$$

The differential equation (1) is always integrable, and from it is obtained a system of *indifference curves* in the OXY plane. The tangent to the indifference curve at any point has gradient $(-R^y_x)$ referred to OY, or gradient $(-R^x_y)$ referred to OX. The first two assumptions imply that each indifference curve has a continuously variable tangent which is always downward sloping. The third assumption implies that the indifference curves are everywhere convex to O. The numerical value of the tangent gradient referred to OY (or to OX) increases as

[1] The problem is a completely static one and the time element is abstracted by taking x and y as amounts that come into the possession of the individual (to be disposed of by him) per unit of time.

[2] The " preference direction " of the individual (i.e. his most preferred direction of acquisition of the goods) is at right angles to the " indifference direction." Hence, $R^y_x = dy/dx$ along the preference direction. There is thus a second interpretation of R^y_x, the marginal rate of increase of y with respect to x for a change in the preference direction.

we move along an indifference curve away from OY (or away from OX). The marginal rate of substitution of X for Y increases as we continue to substitute X for Y. This is *the principle of increasing marginal rate of substitution.*[1]

There are three characteristics or *indices of the individual's scale of preferences* which are sufficient to describe the complete form of the scale or of the indifference curve system. All the indices are expressed in terms of first-order variations of R_x^y; they refer only to the scale of preferences itself and not to market conditions. It will be found, further, that the indices are sufficient for the description of the individual's reaction to market conditions.

The first index refers to a single indifference curve only. *The elasticity of substitution* between X and Y is defined as

$$\sigma = \frac{d\left(\frac{x}{y}\right)}{\frac{x}{y}} \bigg/ \frac{dR_x^y}{R_x^y} = \frac{d\left(\frac{y}{x}\right)}{\frac{y}{x}} \bigg/ \frac{dR_y^x}{R_y^x}$$

taken along the indifference direction at (x,y). Hence:

$$\sigma = -\frac{R_x^y}{xy} \frac{x + R_x^y y}{\frac{\partial}{\partial y} R_x^y - R_x^y \frac{\partial}{\partial x} R_x^y} > 0$$

from the condition (2). The elasticity of substitution is independent of units and is symmetrical with respect to x and y. It is a measure of the curvature of the indifference curve at (x,y), varying in value from zero when the curve is in the form of a right angle at (x,y) to very large values when the curve is flat.

The other two indices refer to the relation of one indifference curve to another and adjacent curve, and they can be called the *coefficients of income-variation*:

$$\rho_x = -\frac{y}{R_x^y} \frac{\partial}{\partial y} R_x^y \text{ and } \rho_y = \frac{x}{R_x^y} \frac{\partial}{\partial x} R_x^y.$$

Both coefficients are expressed in " elasticity " form and are

[1] Of the three assumptions made here, the first is introduced simply for mathematical convenience, but, apart from this, there is no reason to assume away discontinuities in the indifference curve system. The other two assumptions are on a different footing, but they are not *necessarily* satisfied. They must certainly be relaxed when the individual possesses so much of one (or more) of the goods that he would pay to get rid of it. Further, it is not contended that they apply to a position on the preference scale where the individual finds himself below the " subsistence level." It *is* maintained, however, that the assumptions serve to describe all positions in which the individual is likely to find himself under market conditions.

independent of units. It follows from (2) that ρ_x and ρ_y cannot both be negative; they are both positive in the " normal " case and one is positive and the other negative in the " exceptional " case.

3. *The individual demand functions.*—If the individual spends a given money income μ on the two goods X and Y at given uniform market prices p_x and p_y, then his equilibrium purchases are given by the following conditions for equilibrium:

$$xp_x + yp_y = \mu \text{ and } \frac{1}{p_x} = \frac{R_x^y}{p_y} \quad \text{............(3).}$$

In diagrammatic terms, the equations correspond to the fact that the purchases of the individual in equilibrium are represented by the co-ordinates of the point in XY space where the given price line (i.e. $xp_x+yp_y=\mu$) touches an indifference curve. The solution of the equations (3) gives x and y as functions of μ, p_x and p_y—the individual *demand functions.* By the assumption of increasing marginal rate of substitution (indifference curve system convex to the origin), a single equilibrium position exists for each set of μ, p_x and p_y, and the demand functions are single-valued.[1]

Let the proportions of total income spent on X and on Y be denoted by $\kappa_x = \frac{xp_x}{\mu}$ and $\kappa_y = \frac{yp_y}{\mu}$ (where $\kappa_x+\kappa_y=1$). From the equations (3) the values of the three indices of the scale of preferences in the equilibrium position are:

$$\sigma = -\frac{\frac{\mu}{xy} \quad \frac{p_y}{p_x}}{p_x\frac{\partial}{\partial y}R_x^y - p_y\frac{\partial}{\partial x}R_x^y}\,; \quad \rho_x = -\frac{yp_x}{p_y}\frac{\partial}{\partial y}R_x^y\,;$$

$$\rho_y = \frac{xp_x}{p_y}\frac{\partial}{\partial x}R_x^y.$$

The problem is to trace the variation of the equilibrium position and of the demand functions as income or prices vary.

(*a*) *Income-elasticities of demand.*—Let μ vary while p_x and p_y are kept fixed, and denote the income-elasticities of the demand functions by

$$E_\mu(x) = \frac{\mu}{x}\frac{\partial x}{\partial \mu} \text{ and } E_\mu(y) = \frac{\mu}{y}\frac{\partial y}{\partial \mu}.$$

[1] The relaxing of the third assumption made above leads to multiple positions of equilibrium and to multi-valued demand functions. The introduction of this complication is not necessary unless, and until, the simpler theory, based on the third assumption, fails to describe the phenomena of the market.

Differentiate (3) partially with respect to μ:

$$\left.\begin{aligned} xp_x\, E_\mu(x) \quad &+ \quad yp_y\, E_\mu(y) &= \mu \\ x\frac{\partial}{\partial x} R^y_x\, E_\mu(x) &+ y\frac{\partial}{\partial y} R^y_x\, E_\mu(y) &= 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(4).$$

The solution of the equations (4), making use of the equilibrium values of σ, ρ_x and ρ_y, appears quite easily in the form:

$$E_\mu(x) = \sigma \,.\, \rho_x \text{ and } E_\mu(y) = \sigma \,.\, \rho_y \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

From the first equation of (4), it follows at once that

$$\kappa_x E_\mu(x) + \kappa_y E_\mu(y) = 1.$$

and hence that $\kappa_x \rho_x + \kappa_y \rho_y = \dfrac{1}{\sigma}$.

It is not possible for both $E_\mu(x)$ and $E_\mu(y)$ to be negative: in the " normal " case both are positive and the demands for X and Y increase with increasing income. In the " exceptional " case, on the other hand, one income-elasticity is negative and the demand for this good decreases with increasing income. If the income-elasticity of demand for a good is negative, this good is said to be *inferior* to the other good.

Notice that $E_\mu(x)$ is a positive multiple (σ) of the second index of the individual's preference scale and that $E_\mu(y)$ is the same multiple of the third index. The income-elasticities of demand can thus be used instead of these two indices.

(*b*) *Price-elasticities of demand.*—Let p_x vary while μ and p_y are kept fixed, and denote the p_x-elasticities of demand by

$$E_{px}(x) = -\frac{p_x}{x}\frac{\partial x}{\partial p_x} \text{ and } E_{px}(y) = -\frac{p_x}{y}\frac{\partial y}{\partial p_x}.$$

Differentiate (3) partially with respect to p_x:

$$\left.\begin{aligned} xp_x\, E_{px}(x) \quad &+ \quad yp_y\, E_{px}(y) &= xp_x \\ x\frac{\partial}{\partial x} R^y_x\, E_{px}(x) &+ y\frac{\partial}{\partial y} R^y_x\, E_{px}(y) &= \frac{p_y}{p_x} \end{aligned}\right\} \qquad \ldots\ldots(6).$$

$$\text{So } \frac{xE_{px}(x)}{xp_x\dfrac{\partial}{\partial y}R^y_x - \dfrac{p_y^2}{p_x}} = \frac{yE_{px}(y)}{xp_x\dfrac{\partial}{\partial x}R^y_x - p_y} = \frac{1}{p_x\dfrac{\partial}{\partial y}R^y_x - p_y\dfrac{\partial}{\partial x}R^y_x}.$$

$$\text{Hence } E_{px}(x) = \frac{yp_x}{\mu p_y}\,\sigma\left(\frac{xp_y}{y}\rho_x + \frac{p_y^2}{p_x}\right) = \kappa_x E_\mu(x) + (1 - \kappa_x)\,\sigma$$

$$\text{and} \quad E_{px}(y) = \frac{xp_x}{\mu p_y}\,\sigma\,(p_y\rho_y - p_y) = \kappa_x E_\mu(y) - \kappa_x\sigma,$$

using the results (5) and the equilibrium values of σ, ρ_x and ρ_y.

Similar results hold for the p_y-elasticities of demand. Hence:

$$\left.\begin{aligned} E_{px}(x) &= \kappa_x E_\mu(x) + (1-\kappa_x)\,\sigma \\ E_{px}(y) &= \kappa_x E_\mu(y) - \kappa_x\,\sigma \\ E_{py}(x) &= \kappa_y E_\mu(x) - \kappa_y\,\sigma \\ E_{py}(y) &= \kappa_y E_\mu(y) + (1-\kappa_y)\,\sigma \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Two relations exist between the four price-elasticities of demand given by (7). From the first equation of (6)

$$\kappa_x E_{px}(x) + \kappa_y E_{px}(y) = \kappa_x.$$

Similarly $$\kappa_x E_{py}(x) + \kappa_y E_{py}(y) = \kappa_y.$$

The conclusions that can be derived from the equations (7) are set out fully by Dr. Hicks.[1] There is, however, one point that cannot be over-emphasised. The substitution term in $E_{py}(x)$ and in $E_{px}(y)$ is always negative, and a fall in the price of one good causes a substitution of this good for the other. Hence, two goods must always be regarded as substitutes, or as " competitive," when they stand by themselves; complementarity is a characteristic which does not appear until at least three goods are considered.[2] Further, if the elasticity of substitution is a more important (i.e. numerically larger) quantity than the income-elasticities, then both the " cross " price-elasticities $E_{px}(y)$ and $E_{py}(x)$ are negative, i.e. both $\frac{\partial y}{\partial p_x}$ and $\frac{\partial x}{\partial p_y}$ are positive. This is the traditional characteristic of substitute or " competitive " goods in a general sense. But, if the income-elasticities are at least as important in magnitude as the elasticity of substitution, then the derivatives $\frac{\partial y}{\partial p_x}$ and $\frac{\partial x}{\partial p_y}$ can be of either sign, and their signs need not agree.

II. INDIVIDUAL DEMAND IN THE CASE OF THREE OR MORE GOODS

1. We turn now to the general case where any number of inter-related goods enter into the individual's budget. In order to simplify the mathematical analysis, only the case of three goods is considered in detail. It is a difficult step from the case of two goods to that of three goods, but no additional difficul-

[1] Hicks, ECONOMICA, February 1934, pp. 65 *et seq.* This is the first part of the present joint article.

[2] The coefficient of κ_x or κ_y in the second term of $E_{px}(y)$ or $E_{py}(x)$ is what corresponds to the " elasticity of complementarity " in the general case considered in the following section. Here the coefficient is equal in magnitude but opposite in sign to the elasticity of substitution.

ties or complications are encountered in the generalisation of the latter to the case of n goods.

2. *The individual's complex of preferences.*—The fundamental postulate is that, for variations from any combination (x,y,z) of amounts of three goods, X, Y and Z, possessed by the individual, there exist certain definite "indifference directions" defined by the differential equation:

$$dx + R_x^y dy + R_x^z dz = 0 \quad \text{.............................(8)},$$

where the increments dx, dy and dz can take any values that compensate for each other as far as the individual is concerned. The expression $R_x^y = -dx/dy$ (z constant) is *the marginal rate of substitution of X for Y*, and $R_x^z = -dx/dz$ (y constant) is *the marginal rate of substitution of X for Z*. Both R_x^y and R_x^z are functions of x, y and z, and together their values make up the individual's *complex of preferences*.[1] Other marginal rates of substitution exist but can be obtained from the two written above:

$$R_y^x = 1/R_x^y;\ R_z^x = 1/R_x^z;\ R_y^z = R_x^z . R_y^x;\ R_z^y = R_x^y . R_z^x = 1/R_y^z.$$

Three assumptions[2] are made about the individual's complex of preferences:

(1) R_x^y and R_x^z are continuous functions of x, y and z.

(2) R_x^y and R_x^z are positive at all points (x,y,z).

(3) For a variation in any indifference direction from any point, the expression $dx + R_x^y dy + R_x^z dz$ decreases:

$$d(dx + R_x^y dy + R_x^z dz) < 0 \quad \text{subject to} \quad dx + R_x^y dy + R_x^z dz = 0,$$

i.e. $\begin{vmatrix} 1 & R_x^y \\ \frac{\partial}{\partial x} R_x^y & \frac{\partial}{\partial y} R_x^y \end{vmatrix}$ and similar determinants are *negative*

and $\begin{vmatrix} 1 & R_x^y & R_x^z \\ \frac{\partial}{\partial x} R_x^y & \frac{\partial}{\partial y} R_x^y & \frac{\partial}{\partial z} R_x^y \\ \frac{\partial}{\partial x} R_x^z & \frac{\partial}{\partial y} R_x^z & \frac{\partial}{\partial z} R_x^z \end{vmatrix}$ is *positive* ....(9).

[1] The term "complex" of preferences is a better description of this idea in the general case than the term "scale" of preferences which suffices in the simpler two goods case. In the general case (without the integrability condition) it is not possible to "integrate" the preferences of an individual into anything like a complete and ordered scale. The "preference direction" (the most preferred direction) of the individual is still unique, and it is at right angles to each of the indifference directions. It is given by

$$\frac{dx}{1} = \frac{dy}{R_x^y} = \frac{dz}{R_x^z}.$$

Hence, $R_x^y = dy/dx$ and $R_x^z = dz/dx$ for a change in the preference direction.

[2] As before, the assumptions are not *necessarily* satisfied but serve to describe all situations in which the individual is likely to find himself under market conditions.

If the differential equation (8) is integrable (which is not always, or even usually, the case), then a complete indifference surface system exists in XYZ space. The first two assumptions imply that each indifference surface has a continuously variable tangent plane which is always downward sloping both in the OX direction and in the OY direction. The third assumption implies that the indifference surfaces are everywhere convex to O. In particular, the second-order determinants with negative values in (9) give the principle of increasing marginal rate of substitution—the marginal rate of substitution of one good for another increases as we continue to substitute these goods, the third good remaining fixed in amount.

The following notations are required:

$$\sigma = \frac{R_x^y R_x^z}{xyz} \frac{x + R_x^y y + R_x^z z}{\begin{vmatrix} 1 & R_x^y & R_x^z \\ \frac{\partial}{\partial x} R_x^y & \frac{\partial}{\partial y} R_x^y & \frac{\partial}{\partial z} R_x^y \\ \frac{\partial}{\partial x} R_x^z & \frac{\partial}{\partial y} R_x^z & \frac{\partial}{\partial z} R_x^z \end{vmatrix}}.$$

The coefficient σ is positive from the condition (9); it is independent of units and symmetrical with respect to x, y and z. Further, it denotes the mutual substitutability of the three goods and, if the indifference surface system exists, it measures the curvature of the surface passing through any point.

Consider now one good, say X, apart from the other two. The elasticity of substitution between Y and Z is

$$\frac{d\left(\frac{y}{z}\right)}{\frac{y}{z}} \quad \text{divided by} \quad \frac{dR_y^z}{R_y^z},$$

where the differential can be taken along any one of the three perpendicular indifference directions at the point (x,y,z). There are thus three elasticities of substitution between Y and Z which we can denote by ${}_{yz}\sigma_{yz}$, ${}_{xz}\sigma_{yz}$ and ${}_{xy}\sigma_{yz}$, according as it is taken along the YZ indifference direction (X constant), along the XZ indifference direction (Y constant), or along the XY indifference direction (Z constant). Evaluating:

$$ {}_{yz}\sigma_{yz} = -\frac{R_y^z}{yz} \frac{y + R_y^z z}{\begin{vmatrix} 1 & R_y^z \\ \frac{\partial}{\partial y} R_y^z & \frac{\partial}{\partial z} R_y^z \end{vmatrix}} \text{ which is } \textit{positive}, $$

$$ {}_{xz}\sigma_{yz} = \frac{R_y^z}{z \begin{vmatrix} 1 & R_x^z \\ \frac{\partial}{\partial x} R_y^z & \frac{\partial}{\partial z} R_y^z \end{vmatrix}} \text{ and } {}_{xy}\sigma_{yz} = \frac{-R_y^z}{y \begin{vmatrix} 1 & R_x^y \\ \frac{\partial}{\partial x} R_y^z & \frac{\partial}{\partial y} R_y^z \end{vmatrix}}. $$

All these elasticities are independent of units. The first ${}_{yz}\sigma_{yz}$ is symmetrical with respect to y and z and measures the ordinary elasticity of substitution between Y and Z (X being now fixed in amount). The other two elasticities are new and can be positive or negative; ${}_{xz}\sigma_{yz}$ measures the elasticity of substitution between Y and Z when the relation between these goods varies on account of a substitution of X for Z, Y remaining fixed in amount, and similarly for ${}_{xy}\sigma_{yz}$. There are three similar elasticities when Y is considered apart from XZ and three more when Z is considered apart from XY.

3. The form of the individual's complex of preferences is described by twelve indices, all of which are expressed in terms of first-order variations of R_x^y and R_x^z. The indices refer only to the complex of preferences and not to market conditions. The twelve *indices of the complex of preferences* can be divided into three sets:

(1) *The elasticity of substitution between X and the pair YZ*

$$ = \frac{\sigma}{{}_{yz}\sigma_{yz}} $$

and two similar elasticities $\frac{\sigma}{{}_{xz}\sigma_{xz}}$ and $\frac{\sigma}{{}_{xy}\sigma_{xy}}$. All three elasticities are positive. In general terms, $\frac{\sigma}{{}_{yz}\sigma_{yz}}$ is the mutual elasticity of the triad X, Y and Z reduced by the elasticity of substitution of Y and Z between themselves. The elasticity can be large, therefore, if the triad is a highly substitutable one (σ large) or if Y and Z themselves are hardly substitutable at all (${}_{yz}\sigma_{yz}$ small).

(2) *The elasticity of complementarity of Y with X against Z*

$$ = \frac{\sigma}{{}_{xz}\sigma_{yz}} $$

and *the elasticity of complementarity of Z with X against Y*

$$= \frac{\sigma}{{}_{xy}\sigma_{yz}}.$$

In these elasticities σ is again reduced by one of the elasticities of substitution between Y and Z—the new ones in this case. The signs of the elasticities of complementarity can be positive or negative, and these will be interpreted later in terms of the competitive and complementary nature of the relations between X, Y and Z. There are four other and similar elasticities of complementarity:

$$\frac{\sigma}{{}_{yz}\sigma_{xz}}\ ;\ \frac{\sigma}{{}_{xy}\sigma_{xz}}\ ;\ \frac{\sigma}{{}_{yz}\sigma_{xy}}\ ;\ \text{and}\ \frac{\sigma}{{}_{xz}\sigma_{xy}}.$$

(3) *The coefficients of income-variation*:

$$\rho_x = \frac{yz}{R_x^y R_x^z}\begin{vmatrix} \frac{\partial}{\partial y}R_x^y & \frac{\partial}{\partial z}R_x^y \\ \frac{\partial}{\partial y}R_x^z & \frac{\partial}{\partial z}R_x^z \end{vmatrix}\ ;\ \rho_y = -\frac{xz}{R_x^y R_x^z}\begin{vmatrix} \frac{\partial}{\partial x}R_x^y & \frac{\partial}{\partial z}R_x^y \\ \frac{\partial}{\partial x}R_x^z & \frac{\partial}{\partial z}R_x^z \end{vmatrix}$$

and

$$\rho_z = \frac{xy}{R_x^y R_x^z}\begin{vmatrix} \frac{\partial}{\partial x}R_x^y & \frac{\partial}{\partial y}R_x^y \\ \frac{\partial}{\partial x}R_x^z & \frac{\partial}{\partial y}R_x^z \end{vmatrix}.$$

These coefficients are the co-factors (adjusted to be independent of units) of the first row of the positive third-order determinant given in (9). It follows that they cannot be all negative; they are all positive in the "normal" case, and either one or two of them are negative in the "exceptional" cases.

The three indices of (1) and the six indices of (2) refer only to the indifference directions at the point (x,y,z), i.e. to a single indifference surface (if the system exists). The three indices of (3) refer to variations from one set of indifference directions to another and adjacent set, or from one indifference surface to another and adjacent surface.

4. *The individual demand functions.*—If the individual spends a given money income μ on the three goods X, Y and Z at given uniform market prices p_x, p_y and p_z, then his purchases in equilibrium are given by the conditions:

$$xp_x + yp_y + zp_z = \mu \text{ and } \frac{1}{p_x} = \frac{R_x^y}{p_y} = \frac{R_x^z}{p_z} \quad \ldots\ldots\ldots(10).$$

If the indifference surface system exists, these equations correspond to the diagrammatic condition that the individual's equilibrium purchases are the co-ordinates of the point in XYZ space where the given price plane $(xp_x + yp_y + zp_z = \mu)$ touches an indifference surface. The equations (10) suffice to determine x, y and z as functions of μ, p_x, p_y and p_z—the individual *demand functions*. These functions are single-valued since the assumption (9) implies that a unique equilibrium position exists for any set of μ, p_x, p_y and p_z.

Let $\kappa_x = \frac{xp_x}{\mu}$, $\kappa_y = \frac{yp_y}{\mu}$ and $\kappa_z = \frac{zp_z}{\mu}$ denote the proportions of total income spent on X, Y and Z respectively $(\kappa_x + \kappa_y + \kappa_z = 1)$. For convenience, denote

$$D = \begin{vmatrix} p_x & p_y & p_z \\ \frac{\partial}{\partial x} R^y_x & \frac{\partial}{\partial y} R^y_x & \frac{\partial}{\partial z} R^y_x \\ \frac{\partial}{\partial x} R^z_x & \frac{\partial}{\partial y} R^z_x & \frac{\partial}{\partial z} R^z_x \end{vmatrix}.$$

The values, in the equilibrium position, of the various elasticities and coefficients defined above are:

$$\left.\begin{aligned}
&\sigma = \frac{\mu}{xyz}\frac{p_y\,p_z}{p_x^2}\frac{1}{D}\,; \quad {}_{yz}\sigma_{yz} = \frac{-\dfrac{\mu(1-\kappa_x)}{yz}\dfrac{p_z}{p_y}}{\begin{vmatrix} p_y & p_z \\ \frac{\partial}{\partial y} R^z_y & \frac{\partial}{\partial z} R^z_y \end{vmatrix}} \quad \text{etc.} \\
&{}_{xy}\sigma_{yz} = \frac{\dfrac{\mu\kappa_x}{xz}\dfrac{p_z}{p_y}}{\begin{vmatrix} p_x & p_z \\ \frac{\partial}{\partial x} R^z_y & \frac{\partial}{\partial z} R^z_y \end{vmatrix}}\,; \quad {}_{xz}\sigma_{yz} = \frac{-\dfrac{\mu\kappa_x}{xy}\dfrac{p_z}{p_y}}{\begin{vmatrix} p_x & p_y \\ \frac{\partial}{\partial x} R^z_y & \frac{\partial}{\partial y} R^z_y \end{vmatrix}} \quad \text{etc.} \\
&\rho_x = \frac{yzp_x^2}{p_yp_z} D_x\,; \quad \rho_y = \frac{xzp_x^2}{p_yp_z} D_y\,; \quad \rho_z = \frac{xyp_x^2}{p_yp_z} D_z
\end{aligned}\right\} \quad \ldots(11),$$

where D_x, D_y and D_z are the co-factors of the first row of D. The problem of the variation of the demand functions, as income or as prices vary, is treated by the method adopted in the two-goods case.

(*a*) *Income-elasticities of demand.*—Denoting the three income-elasticities of demand by

$$E_\mu(x) = \frac{\mu}{x}\frac{\partial x}{\partial \mu}\,; \quad E_\mu(y) = \frac{\mu}{y}\frac{\partial y}{\partial \mu}\,; \quad E_\mu(z) = \frac{\mu}{z}\frac{\partial z}{\partial \mu},$$

differentiate (10) partially with respect to μ:

$$\left.\begin{aligned} xp_x\, E_\mu(x) \quad + yp_y\, E_\mu(y) \quad + zp_z\, E_\mu(z) &= \mu \\ x\frac{\partial}{\partial x}R^y_x E_\mu(x) + y\frac{\partial}{\partial y}R^y_x\, E_\mu(y) + z\frac{\partial}{\partial z}R^y_x\, E_\mu(z) &= 0 \\ x\frac{\partial}{\partial x}R^z_x E_\mu(x) + y\frac{\partial}{\partial y}R^z_x\, E_\mu(y) + z\frac{\partial}{\partial z}R^z_x\, E_\mu(z) &= 0 \end{aligned}\right\}\ldots(12).$$

Solving the equations (12) in determinant form,

$$\frac{xE_\mu(x)}{D_x} = \frac{yE_\mu(y)}{D_y} = \frac{zE_\mu(z)}{D_z} = \frac{\mu}{D}.$$

Substituting the values of the various expressions given by (11),

$$E_\mu(x) = \sigma\,.\,\rho_x\,;\; E_\mu(y) = \sigma\,.\,\rho_y\,;\; E_\mu(z) = \sigma\,.\,\rho_z \ldots\ldots\ldots\ldots(13).$$

From the first equation of (12) we have a relation between the income-elasticities and hence between ρ_x, ρ_y and ρ_z:

$$\left.\begin{aligned} \kappa_x E_\mu(x) + \kappa_y\, E_\mu(y) + \kappa_z\, E_\mu(z) &= 1 \\ \kappa_x\rho_x + \kappa_y\rho_y + \kappa_z\rho_z &= \frac{1}{\sigma} \end{aligned}\right\}\ldots\ldots\ldots\ldots(14).$$

It follows that all three income-elasticities cannot be negative. In the " normal " case they are all positive and each demand increases with increasing income. In the " exceptional " cases one (or two) of the income-elasticities is negative and the demand for one (or two) of the goods decreases with increasing income. A good is said to be *inferior* if its demand decreases with increasing income, and it is possible, therefore, for one or two of a set of three goods to be inferior in this sense.

Since the income-elasticities of demand are positive multiples of the third set of indices, they can be used instead of the latter to characterise the individual's complex of preferences.

(*b*) *Price-elasticities of demand.* Denoting the p_x-elasticities of demand by

$$E_{px}(x) = -\frac{p_x}{x}\frac{\partial x}{\partial p_x};\; E_{px}(y) = -\frac{p_x}{y}\frac{\partial y}{\partial p_x};\; E_{px}(z) = -\frac{p_x}{z}\frac{\partial z}{\partial p_x},$$

differentiate (10) partially with respect to p_x:

$$\left.\begin{aligned} xp_x\, E_{px}(x) \; + yp_y\, E_{px}(y) \quad + zp_z\, E_{px}(z) &= xp_x \\ x\frac{\partial}{\partial x}R^y_x\, E_{px}(x) + y\frac{\partial}{\partial y}R^y_x\, E_{px}(y) + z\frac{\partial}{\partial z}R^y_x\, E_{px}(z) &= \frac{p_y}{p_x} \\ x\frac{\partial}{\partial x}R^z_x\, E_{px}(x) + y\frac{\partial}{\partial y}R^z_x\, E_{px}(y) + z\frac{\partial}{\partial z}R^z_x\, E_{px}(z) &= \frac{p_z}{p_x} \end{aligned}\right\}\ldots(15).$$

Solving the equations (15) in determinant form:

$$\frac{x\,E_{px}(x)}{\begin{vmatrix} xp_x & p_y & p_z \\ \dfrac{p_y}{p_x} & \dfrac{\partial}{\partial y}R_x^y & \dfrac{\partial}{\partial z}R_x^y \\ \dfrac{p_z}{p_x} & \dfrac{\partial}{\partial y}R_x^z & \dfrac{\partial}{\partial z}R_x^z \end{vmatrix}} = \frac{y\,E_{px}(y)}{\begin{vmatrix} p_x & xp_x & p_z \\ \dfrac{\partial}{\partial x}R_x^y & \dfrac{p_y}{p_x} & \dfrac{\partial}{\partial z}R_x^y \\ \dfrac{\partial}{\partial x}R_x^z & \dfrac{p_z}{p_x} & \dfrac{\partial}{\partial z}R_x^z \end{vmatrix}} = \frac{z\,E_{px}(z)}{\begin{vmatrix} p_x & p_y & xp_x \\ \dfrac{\partial}{\partial x}R_x^y & \dfrac{\partial}{\partial y}R_x^y & \dfrac{p_y}{p_x} \\ \dfrac{\partial}{\partial x}R_x^z & \dfrac{\partial}{\partial y}R_x^z & \dfrac{p_z}{p_x} \end{vmatrix}} = \frac{1}{D},$$

i.e.
$$E_{px}(x) = \frac{yz}{\mu}\frac{p_x^2}{p_yp_z}\,\sigma\left[xp_x\,D_x + \frac{p_y}{p_x}\left\{p_z\frac{\partial}{\partial y}R_x^z - p_y\frac{\partial}{\partial z}R_x^z\right\} + \frac{p_z}{p_x}\left\{p_y\frac{\partial}{\partial z}R_x^y - p_z\frac{\partial}{\partial y}R_x^y\right\}\right]$$
$$= \kappa_x\,\sigma\,.\,\rho_x - \frac{yz}{\mu}\frac{p_y}{p_z}\,\sigma\left(p_y\frac{\partial}{\partial z}R_y^z - p_z\frac{\partial}{\partial y}R_y^z\right),$$

using the expressions (11) and making the transformation $R_x^z = R_y^z . R_x^y$ together with the equilibrium equations (10). Hence, from (11) and (13),

$$E_{px}(x) = \kappa_x E_\mu(x) + (1-\kappa_x)\frac{\sigma}{{}_{yz}\sigma_{yz}}.$$

By an exactly similar procedure, we obtain

$$E_{px}(y) = \kappa_x E_\mu(y) + \kappa_x\frac{\sigma}{{}_{xz}\sigma_{yz}} \text{ and } E_{px}(z) = \kappa_x E_\mu(z) + \kappa_x\frac{\sigma}{{}_{xy}\sigma_{yz}}.$$

Similar sets of results can be obtained for the p_y-elasticities and for the p_z-elasticities of demand. Hence:

$$\left.\begin{aligned} E_{px}(x) &= \kappa_x E_\mu(x) + (1-\kappa_x)\frac{\sigma}{{}_{yz}\sigma_{yz}} \\ E_{px}(y) &= \kappa_x E_\mu(y) + \kappa_x\frac{\sigma}{{}_{xz}\sigma_{yz}} \\ E_{px}(z) &= \kappa_x E_\mu(z) + \kappa_x\frac{\sigma}{{}_{xy}\sigma_{yz}} \\ &\text{and two similar sets of three equations} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(16).$$

Each price-elasticity of demand thus consists of two terms, the first term being a multiple of the third set of indices of the complex of preferences (the coefficients of income-variation) and the second term being a multiple of the first and second set of indices (the elasticities of substitution and complementarity).

One relation can be found between each of the three sets of price-elasticities of demand. From the first equation of (15),

$$\left.\begin{array}{l} \kappa_x E_{px}(x) + \kappa_y E_{px}(y) + \kappa_z E_{px}(z) = \kappa_x \\ \qquad \text{and two similar relations} \\ \text{Hence, } (1 - \kappa_x)\dfrac{\sigma}{{}_{yz}\sigma_{yz}} + \kappa_y \dfrac{\sigma}{{}_{xz}\sigma_{yz}} + \kappa_z \dfrac{\sigma}{{}_{xy}\sigma_{yz}} = 0 \\ \qquad \text{and two similar relations} \end{array}\right\} \quad \ldots\ldots(17).$$

The three relations of the type (17) between the indices of the first and second sets imply, amongst other things, that the elasticities of substitution can be obtained in terms of the elasticities of complementarity. Further, since $\dfrac{\sigma}{{}_{yz}\sigma_{yz}} > 0$, both $\dfrac{\sigma}{{}_{xz}\sigma_{yz}}$ and $\dfrac{\sigma}{{}_{xy}\sigma_{yz}}$ cannot be positive. There are, therefore, only two possibilities. *Either* both elasticities of complementarity of Y and Z with X are negative *or* one elasticity is negative and the other positive.

5. A number of important conclusions can be derived from the results (16).[1] The increases and decreases in the various demands that follow a change in any one price are made up of two separate changes, the first due to the change in real income and the second to the substitutions made possible by the change in the relative prices.

The effect of a change in the price of a good on the demand for the *same* good is clear. The change is measured by an elasticity of the form $E_{px}(x)$ and this is positive in almost all cases. The demand for a good is thus increased by a fall in its price. It is possible, however, that this result is reversed in very exceptional cases. A price-elasticity of the form $E_{px}(x)$ can be negative and the demand for a good can increase with a *rise* in its price, provided that the income-elasticity of demand for the good is negative and large relative to the substitution effect. Hence, the demand curve for a good X can be rising, in the Giffen-Marshall sense, provided that X is an inferior good and that a large proportion of total income is spent on X for which no ready substitutes are available.

The effect of a change in the price of a good on the demands for other goods is more involved, and it is here that we must look for observable evidence of the " competitive " or " complementary " nature of the relations between the three goods.

[1] For a complete account of these conclusions, see the first part of this article by Dr. Hicks, ECONOMICA, February 1934, p. 67 and pp. 69 *et seq.*

Consider the three-way substitution made possible by the *relative* price changes apart from the effect of the change in the level of real income. The second terms of the results (16) indicate this substitution effect. The effect of substitution following a fall in the price of X is to increase or decrease the demand for Y according as the elasticity of complementarity of Y with X against Z is positive or negative. *A negative elasticity of complementarity implies that Y competes with X against Z and a positive elasticity that Y complements X against Z.* The signs of the elasticities of complementarity determine the competitive and complementary nature of the relations between the three goods and their magnitudes indicate the extent of the relations.[1] Since both elasticities of complementarity of Y and Z with X cannot be positive, it is impossible that both Y and Z complement X. There must be an element of competition between one good and the other pair. In conclusion, it is important to notice that these competitive and complementary relations depend only on the indices of the individual's complex of preferences, and not on market prices or conditions.

III. THE INTEGRABILITY CASE

1. The development of the previous section was perfectly general and, in particular, it was independent of the existence of an integral of the fundamental differential equation (8). The results to be set out in the present section, on the other hand, hold only in cases where the equation (8) is integrable.

The mathematical condition for integrability imposes a restriction on the form of R_x^y and R_x^z.[2] Assuming that the condition is satisfied, there exists a function index of utility

$$u = F\{\phi(x,y,z)\}$$

where $\phi(x,y,z)$ is any one integral of (8) and F denotes the arbitrary function involved in the general integral. Writing the partial derivatives of $\phi(x,y,z)$ by ϕ_x, ϕ_y and ϕ_z, we have

$$R_x^y = \frac{\phi_y}{\phi_x} \text{ and } R_x^z = \frac{\phi_z}{\phi_x}.$$

[1] In the complementary case the demands for both X and Y increase at the expense of the demand for Z (apart from the effect of the change in real income). This is why only competitive goods are possible in the two-goods case; there is no third good to absorb the loss that must occur in substitution.

[2] The condition is $\frac{\partial}{\partial z} R_x^y - \frac{\partial}{\partial y} R_x^z + R_x^y \frac{\partial}{\partial x} R_x^z - R_x^z \frac{\partial}{\partial x} R_x^y = 0.$

The analysis can now be expressed in terms of the function $\phi(x,y,z)$ and its partial derivatives of the first and second orders, remembering that only the ratios of ϕ_x, ϕ_y and ϕ_z are definite. All the results given below, however, can be shown to involve only these ratios or the derivatives of the ratios, and not the functions themselves.

2. *Simplification of the competitive and complementary relations.*—In the integrability case the values of the six elasticities of complementarity are

$$\frac{\sigma}{{}_{xz}\sigma_{yz}} = \frac{x\phi_x + y\phi_y + z\phi_z}{xy}\,\frac{\Phi_{xy}}{\Phi} = \frac{\sigma}{{}_{yz}\sigma_{xz}}$$

$$\frac{\sigma}{{}_{xy}\sigma_{xz}} = \frac{x\phi_x + y\phi_y + z\phi_z}{yz}\,\frac{\Phi_{yz}}{\Phi} = \frac{\sigma}{{}_{xz}\sigma_{xy}}$$

$$\frac{\sigma}{{}_{yz}\sigma_{xy}} = \frac{x\phi_x + y\phi_y + z\phi_z}{xz}\,\frac{\Phi_{xz}}{\Phi} = \frac{\sigma}{{}_{xy}\sigma_{yz}},$$

where $-\Phi$ stands for the negative and symmetrical determinant

$$\begin{vmatrix} \phi_{xx} & \phi_{xy} & \phi_{xz} & \phi_x \\ \phi_{xy} & \phi_{yy} & \phi_{yz} & \phi_y \\ \phi_{xz} & \phi_{yz} & \phi_{zz} & \phi_z \\ \phi_x & \phi_y & \phi_z & 0 \end{vmatrix}$$

and Φ_{xy}, Φ_{yz} and Φ_{xz} are the co-factors of ϕ_{xy}, ϕ_{yz} and ϕ_{xz} in the determinant.

Symmetry is, therefore, introduced into the relations between the three goods when the integrability condition is satisfied. For the relation between any pair of goods X and Y (with respect to the third good Z), the elasticity of complementarity of Y with X against Z is equal to the elasticity of complementarity of X with Y against Z. It is now possible to speak of the elasticity of complementarity of the pair XY (against Z):

$$\sigma_{xy} = \frac{\sigma}{{}_{xz}\sigma_{yz}} = \frac{\sigma}{{}_{yz}\sigma_{xz}},$$

and there are only three of these elasticities, σ_{xy}, σ_{yz} and σ_{xz}, instead of the full set of six.[1] The signs of σ_{xy}, σ_{yz} and σ_{xz} determine the competitive and complementary relations of the whole set of three goods. If σ_{xy} is negative X and Y are competitive, if positive X and Y are complementary (with respect to Z in each case). Similar criteria apply to the relation between the other pairs.

[1] The integrability condition is clearly *sufficient* for this symmetry; it also appears to be a *necessary* condition.

The equations (17) now take the form:

$$(1-\kappa_x)\frac{\sigma}{{}_{yz}\sigma_{yz}}+\kappa_y\sigma_{xy}+\kappa_z\sigma_{xz}=0.$$

and two similar relations

Hence, at least two of σ_{xy}, σ_{yz} and σ_{xz} are negative, and it is not possible for more than one pair of the three goods to be complementary. Further, the equations can be solved either to give the elasticities of substitution of the goods in pairs (${}_{yz}\sigma_{yz}$, ${}_{xz}\sigma_{xz}$ and ${}_{xy}\sigma_{xy}$) in terms of the elasticities of complementarity of the goods in pairs (σ_{xy}, σ_{yz} and σ_{xz}), or conversely. The latter is the more interesting solution. We obtain:

$$\sigma_{xy}=\frac{1}{2}\sigma\left\{\frac{1}{{}_{xy}\sigma_{xy}}\left(\frac{\kappa_z}{\kappa_x}+\frac{\kappa_z}{\kappa_y}\right)-\frac{1}{{}_{yz}\sigma_{yz}}\frac{1-\kappa_x}{\kappa_y}-\frac{1}{{}_{xz}\sigma_{xz}}\frac{1-\kappa_y}{\kappa_x}\right\} \quad \ldots(18).$$

and two similar expressions

Hence, the goods X and Y can only be markedly complementary if the term $\frac{1}{{}_{xy}\sigma_{xy}}\left(\frac{\kappa_z}{\kappa_x}+\frac{\kappa_z}{\kappa_y}\right)$ is large compared with the other two terms of the expression (18), i.e. if the elasticity of substitution of the pair XY is small compared with the two similar elasticities, or if κ_z is large compared with κ_x and κ_y, or both. The results stated by Dr. Hicks on this point follow at once.[1]

Consider, finally, the two actual variations in demand:

$$\frac{\partial x}{\partial p_y}=-\frac{xy}{\mu}\{E_\mu(x)+\sigma_{xy}\} \text{ and } \frac{\partial y}{\partial p_x}=-\frac{xy}{\mu}\{E_\mu(y)+\sigma_{xy}\}.$$

Both $\frac{\partial x}{\partial p_y}$ and $\frac{\partial y}{\partial p_x}$ are independent of units and can be compared directly. The two variations are equal only if $E_\mu(x)=E_\mu(y)$, i.e. if an increase in income has the same proportional effect on the demands for the two goods. This may approximate to the actual state of affairs in many cases, but it is certainly not exactly true in general. Further, the two variations are in the same direction (without necessarily being of equal magnitude) either if both income-elasticities are small or if they are large but differ by a small amount.[2] It is, how-

[1] Hicks, ECONOMICA, February 1934, p. 73.

[2] It is worth while distinguishing the two alternatives. It is only in the first alternative, where $E_\mu(x)$ and $E_\mu(y)$ are small, that we can say that the common sign of $\frac{\partial x}{\partial p_y}$ and $\frac{\partial y}{\partial p_x}$ is determined by the sign of σ_{xy}. In this case the two variations are positive if

ever, quite possible that $\frac{\partial x}{\partial p_y}$ and $\frac{\partial y}{\partial p_x}$ are different, not only in magnitude, but also in sign. The symmetry of the system does not necessarily extend to these two changes in demand.

3. *The case of independent goods.*—The three goods form an independent system if the marginal rate of substitution of one good for another depends only on the amounts possessed by the individual of these two goods, i.e. if R_x^y is a function of x and y, R_y^z a function of y and z and R_z^x a function of x and z. It follows at once that each marginal rate of substitution can be expressed as a product of two functions of a single variable, and that we can write:

$$\frac{1}{\phi_x(x)} = \frac{R_x^y}{\phi_y(y)} = \frac{R_x^z}{\phi_z(z)}.$$

The differential equation (8) is

$$\phi_x(x)dx + \phi_y(y)dy + \phi_z(z)dz = 0$$

and this is always integrable. The function index of utility is

$$u = F\{\Phi_x(x) + \Phi_y(y) + \Phi_z(z)\},$$

the general integral of the equation, where

$$\Phi_x(x) = \int \phi_x(x)dx;\ \Phi_y(y) = \int \phi_y(y)dy;\ \Phi_z(z) = \int \phi_z(z)dz.$$

The utility index is, in all its forms, a function of the sum of three functions of a single variable, and the goods X, Y and Z make independent contributions to the utility index. The case of independent goods is thus at least mathematically significant.

It is only in the independent goods case that we can derive anything corresponding to the marginal utility functions and curves of the traditional analysis. The marginal rate of substitution of Y for X is $R_y^x = \phi_x(x) : \phi_y(y)$, and this is a constant times $\phi_x(x)$ when only x varies. The same is true of the marginal rate of substitution of Z for X. There is, therefore, a single-variable function $\phi_x(x)$ which represents the marginal rate of substitution of any good for X for various amounts of X. It is only necessary to multiply the function by a constant depending on which good is substituted for X and on the (fixed) amount of this good possessed by the individual.

Let $E_x(\phi_x) = \frac{x}{\phi_x}\frac{d}{dx}\phi_x$ represent the elasticity of the function

X and Y are competitive and negative if X and Y are complementary. In the second alternative, though the variations are in the same direction, the direction does not necessarily correspond to the competitive or complementary relation between the pair of goods XY.

$\phi_x(x)$, i.e. the elasticity of the marginal rate of substitution of any good for X taken for variations in the amount possessed of X. In the same way we can obtain two other elasticities, $E_y(\phi_y)$ and $E_z(\phi_z)$. Each of these three elasticities is perfectly definite, being independent of units, of which good is substituted for the one named and of the amount of the substituted good that happens to be possessed. The elasticities correspond to the elasticities of the marginal utility functions or curves of Pareto's theory of value. If the signs of all $E_x(\phi_x)$, $E_y(\phi_y)$ and $E_z(\phi_z)$ are negative we have the case of " decreasing marginal utility " in Pareto's sense. It is, however, not necessary that this case should obtain for all sets of independent goods.

In the case of independent goods the following equilibrium values of the fundamental coefficients are obtained:

$$\sigma = \frac{1}{E_x E_y E_z} \frac{1}{K}; \; {}_{yz}\sigma_{yz} = - \frac{1-\kappa_x}{E_y E_z} \frac{1}{K - \frac{\kappa_x}{E_x}} \text{ etc.};$$

$$ {}_{xz}\sigma_{yz} = {}_{yz}\sigma_{xz} = \frac{1}{E_z} \text{ etc.}; \; \rho_x = E_y E_z \text{ etc.},$$

where $K = \frac{\kappa_x}{E_x} + \frac{\kappa_y}{E_y} + \frac{\kappa_z}{E_z}$ and E_x stands for $E_x(\phi_x)$, and so on.

The equilibrium values of the twelve indices of the individual's complex of preferences are:

(1) the elasticities of substitution are

$$\frac{\sigma}{{}_{yz}\sigma_{yz}} = - \frac{1}{1-\kappa_x} \frac{1}{E_x} \left(1 - \frac{\kappa_x}{E_x} \frac{1}{K}\right).$$

and two similar expressions, all being positive

(2) the elasticities of complementarity are

$$\sigma_{xy} = \frac{1}{E_x E_y} \frac{1}{K} = \sigma E_z.$$

and two similar expressions

(3) the coefficients of income-variation are

$$\rho_x = E_y E_z; \; \rho_y = E_x E_z \text{ and } \rho_z = E_x E_y,$$

and so $\quad E_\mu(x) = \sigma E_y E_z; E_\mu(y) = \sigma E_x E_z$ and $E_\mu(z) = \sigma E_x E_y$.

The indices are all determined by the three elasticities E_x, E_y and E_z. There must, therefore, be a number of relations between them. As in the general integrability case the elasticities of substitution can be expressed in terms of the elasticities of complementarity, or conversely, by means of equations (18). In the independent goods case, however, there is a further set of relations between the indices. The coefficients of income-variation, and hence the income-elasticities of demand, can be expressed in terms of the elasticities of complementarity also. We have:

$$E_{\mu}(x) = \frac{\sigma_{xy}\sigma_{xz}}{\sigma}\ ;\ E_{\mu}(y) = \frac{\sigma_{xy}\sigma_{yz}}{\sigma} \text{ and } E_{\mu}(z) = \frac{\sigma_{xz}\sigma_{yz}}{\sigma} \quad \ldots(19).$$

As in the integrability case at least two of σ_{xy}, σ_{yz} and σ_{xz} must be negative. Since $\sigma_{xy} = \sigma E_z$, $\sigma_{yz} = \sigma E_x$ and $\sigma_{xz} = \sigma E_y$ (σ being positive), at least two of E_x, E_y and E_z must also be negative. There are thus only two possibilities; *either* all E_x, E_y and E_z are negative, *or* one of them is positive and the other two negative.

(1) When E_x, E_y and E_z are all negative we have the case of "decreasing marginal utility." From the expressions above all the elasticities of substitution are positive, all the elasticities of complementarity are negative, and all the coefficients of income-variation are positive. It follows that all the income-elasticities of demand are positive. Further, from the results (16), $E_{px}(x)$, $E_{py}(y)$ and $E_{pz}(z)$, i.e. the demand elasticities with respect to the price of the good concerned, are also all positive. The three goods compete with each other in pairs and there is no possibility of "exceptional" behaviour of any kind.

(2) When E_x is positive and E_y and E_z are negative we have the case where one good has "increasing marginal utility." Since σ must be positive there is one restriction in this case:

$$\frac{\kappa_x}{E_x} > \frac{\kappa_y}{-E_y} + \frac{\kappa_z}{-E_z}.$$

The elasticity of complementarity of the pair YZ is now positive, and the other two elasticities are negative. Of the income-elasticities of demand only $E_{\mu}(x)$ is positive and the other two are negative. In this case both Y and Z are inferior to X, and they complement each other while competing separately with X. The p_x-elasticities of demand are all definite in sign from (16), $E_{px}(x)$ being positive and the other two negative. A fall in the price of the superior good X increases the demand for the good at the expense of a decrease in the demand for each of the inferior goods. The other price-elasticities can be of either sign and, in particular, rising demand curves for Y and Z are possible.

The independent goods case includes a case of perfectly "normal" relationship between X, Y and Z. But it also includes a case where a pair of goods is inferior to, and complement each other against, the third good. It is, therefore, not sufficient to take the goods as an independent set if it is desired to give a simple analytical treatment (as a first approximation) of the case of three goods "normally" related. One additional condition is necessary for this, the condition that the elasticities of the marginal rates of substitution are all negative.[1]

To return to the case of two goods for a moment, the only definition of two independent goods that can be given is that the marginal rate of substitution divides into two functions of a single variable: $R_x^y = \frac{\phi_y(y)}{\phi_x(x)}$. There are again two possibilities. *Either* $E_x(\phi_x)$ and $E_y(\phi_y)$ are both negative, in which case the income-elasticities of demand are both positive. *Or* $E_x(\phi_x)$ and $E_y(\phi_y)$ are of opposite signs, in which case the income-elasticities of demand are also of opposite signs. But these two possibilities cover all the cases that *can* arise when there are only two goods. The case of two independent goods is thus no more restricted (in general terms) than the complete case, and any relationship between two goods can be represented, at least approximately for small variations, by the independent relationship. The two-goods case can be treated perfectly well by assuming independence from the beginning.[2] This is not

[1] Only *one* condition is required here since two of the three elasticities are negative in any case.

[2] See Hicks, ECONOMICA, February 1934, pp. 75-6. In the case of two independent goods X and Y, the two fundamental elasticities are $E_x(\phi_x)$ and $E_y(\phi_y)$. The former is the elasticity (X variable) of the marginal rate of substitution of Y for X, or (broadly) the elasticity of the marginal utility of X. The other elasticity is similarly interpreted. All demand elasticities are expressed in terms of E_x and E_y; in particular

$$E_\mu(x) = -\sigma E_y \text{ and } E_\mu(y) = -\sigma E_x,$$

where $\sigma = -\frac{1}{E_xE_y} \frac{1}{\frac{\kappa_x}{E_x} + \frac{\kappa_y}{E_y}}$, the elasticity of substitution between X and Y.

These remarks throw some light upon the meaning of Professor Frisch's "money flexibility" (*New Methods of Measuring Marginal Utility*, 1932). Professor Frisch, in effect, takes X as one particular commodity (say sugar) and Y as the group of all other commodities. Then E_y is the elasticity of the marginal rate of substitution of sugar for all other commodities when the expenditure on the latter varies. This is the elasticity of the marginal utility of all other commodities, i.e. of money income. Hence, Professor Frisch's money flexibility is E_y. It follows, for example, that income-elasticity of demand for sugar is the product of the elasticity of substitution between sugar and all other commodities and the numerical value of money flexibility. The question arises whether the value of E_y is independent of the choice of X. If this is the case, as Professor true in the case of more than two goods, and here the case of independence is definitely more restricted than the general case (even if integrability is assumed), and must be treated as such.

4. *Notes on the " degrees of freedom " of the system.*—Three cases of the relationship between three or more goods have been considered:

(*a*) The general case.
(*b*) The case where the integrability condition is satisfied and a utility function index exists.
(*c*) The case of independent goods.

It is proposed to add a number of concluding remarks on the way in which the number of " degrees of freedom " of the system is decreased as we proceed from the general to the more particular cases. The meaning of the term " degrees of freedom " will be apparent from the nature of these remarks.

(*a*) In the general case it is found that three relations (17) exist between the elasticities of substitution and of complementarity and one relation (14) between the coefficients of income-variation. The restrictions (9) are only inequalities and do not affect the independence of the indices. There are thus *eight independent indices of the individual's complex of preferences* and these can be taken as the six elasticities of complementarity and two of the coefficients of income-variation.

The variation of individual demand for changes in income or in the market prices is described by twelve income and price-elasticities of demand. By the relations (14) and (17) one income-elasticity and three price-elasticities depend on the others. There are thus eight independent elasticities of demand and the eight independent indices account for these. The general case, therefore, has eight degrees of freedom.

(*b*) The integrability case introduces symmetry into the system. The six elasticities of complementarity are reduced to three, and there are now only *five independent indices of the individual's complex of preferences.* These can be taken, for example, as the three elasticities of complementarity and two coefficients of income-variation. There is a reduction of three in the number of degrees of freedom in the system. On the demand side this must be paralleled by three relations between the eight independent elasticities of demand. These relations

Frisch claims, the income-elasticity of demand for any commodity is a constant multiple of the elasticity of substitution between this and all other commodities.

are provided by the symmetry of the system and are, from the equations (16),

$$\frac{1}{\kappa_x} E_{px}(y) - \frac{1}{\kappa_y} E_{py}(x) = E_\mu(y) - E_\mu(x)$$

and two similar relations. As we found, these relations provide a means of comparing $E_{px}(y)$ and $E_{py}(x)$, and so on.

The fact that there are five degrees of freedom in the system is also shown by the number of second-order derivatives of $\phi(x,y,z)$. There are six of these derivatives and one is not independent since the integrability condition must be satisfied. The remaining five are independent and describe the system.

(*c*) The further particular case of a set of three independent goods introduces two further restrictions. From the equations (19), it follows that the two independent coefficients of income-variation are expressible in terms σ_{xy}, σ_{yz} and σ_{xz}, and are no longer independent. There are now only *three independent indices of the individual's complex of preferences*. These can be taken as σ_{xy}, σ_{yz} and σ_{xz}. Alternatively, since all indices can be expressed in terms of the definite elasticities $E_x(\phi_x)$, $E_y(\phi_y)$ and $E_z(\phi_z)$, these can be taken to represent the three independent indices. There are thus three degrees of freedom in the system.

On the demand side, in addition to the relations of the integrability case, there are two relations connecting the income-elasticities and the price-elasticities. Hence, of the elasticities of demand, only three are independent (e.g. the p_x-elasticities), and these are accounted for by the three independent indices. This is checked by the fact that there are only three non-zero second-order derivatives of the function $\phi(x,y,z) = \Phi_x(x) + \Phi_y(y) + \Phi_z(z)$.

Finally, the independent goods case with one additional condition can be used to describe a case of three goods related in a perfectly " normal " way. Since the additional condition takes the form of an inequality there is no further reduction in the number of degrees of freedom. The case of independent goods " normally " related is still described by three independent indices and still displays three degrees of freedom.