A Reconsideration of the Theory of Value

By J. R. HICKS and R. G. D. ALLEN

Part I

By J. R. HICKS

THE pure theory of exchange value, after a period of intensive study by economists of the generation of Jevons and Marshall, has received comparatively little attention from their successors in the twentieth century. Apart from some very interesting inquiries into what may be called the dynamics of the subject, due to contemporary writers of the school of Vienna,[1] there has been only one major achievement in this field since 1900. That achievement was the work of Pareto, whose *Manuel* (and particularly its mathematical appendix) contains the most complete static theory of value which economic science has hitherto been able to produce.

Of all Pareto's contributions there is probably none that exceeds in importance his demonstration of the immeasurability of utility. To most earlier writers, to Marshall, to Walras, to Edgeworth, utility had been a quantity theoretically measurable; that is to say, a quantity which would be measurable if we had enough facts. Pareto definitely abandoned this, and replaced the concept of utility by the concept of a scale of preferences. It is not always observed that this change in concepts was not merely a change of view, a pure matter of methodology; it rested on a positive demonstration that the facts of observable conduct make a scale of preferences capable of theoretical construction (in the same sense as above) but they do not enable us to proceed from the scale of preference to a particular utility function.

The first signs of a break-down of the old conception of

[1] Schönfeld, *Grenznutzen und Wirtschaftsrechnung*; Hans Mayer, *Der Erkenntniswert der funktionellen Preistheorien* (*Wirtschaftstheorie der Gegenwart II*); Rosenstein-Rodan, "La Complementarità" (*Riforma Sociale*, May 1933).

utility had already made their appearance in Irving Fisher's *Mathematical Investigations into the Theory of Prices*. Fisher had pointed out (1) that the whole theory of equilibrium in a market depends on the assumption of directions of indifference, and does not involve anything more; (2) that with three or more commodities the directions of indifference may not be integrable, so that it is impossible to deduce any utility function from a knowledge of these directions.[1] This latter point also makes its appearance in Pareto; it led him to his celebrated but mysterious theory of "open cycles"; however, it is *not* with that point that we are at present concerned. It is with Pareto's more general and economically more significant contention, that even if it is possible to deduce a utility function from the directions of indifference, that utility function is to a very large extent indeterminate.

Though Pareto states this conclusion in the text of the *Manuel*,[2] he does not prove it there, and this has no doubt been responsible for the failure of many readers to see its significance. But a proof is given in the mathematical appendix;[3] it is not a difficult proof, and its sense can be set out in words quite easily.

Suppose, for the moment, that we have a utility function given; that is to say, we know, for the individual in question, how much utility he would derive from any given set of quantities of the goods on the market. Then we can deduce from this function (assuming always that he will prefer a higher to a lower utility) a scale of preferences; we can say, of any two sets, whether he will prefer one to the other, or whether they will be indifferent to him. If there are only two sorts of goods, this scale of preferences can be represented by a diagram of indifference curves.

It is thus possible to proceed from a utility function to a scale of preferences; but is it possible to proceed in the reverse direction? The answer is no; for the function with which we started is not the only function which will determine the scale in question. It is not only that there may be an indeterminate constant (this would not matter very much); but we can take as an "index" of utility any variable which has the same value all along an indifference curve, and which increases as we proceed from one indifference curve to a higher one. Thus we might take the function with which we started; or we might take double that function (but this would only mean a change in units); however, we might also take its

[1] *Mathematical Investigations*, p. 88. [2] *Manuel*, p. 159. [3] *Ibid.*, pp. 540-2.

square, or any variable having a more complex relation with the first, so long as the essential condition of increasing in preferred positions is preserved.

To take an arithmetical illustration: Successive positions might be numbered 1, 2, 3, 4, 5; or 1, 4, 9, 16, 25; or 1, 2, 4, 5, 7; or any increasing series we like to take. So far as the actual behaviour of any individual can possibly show, any such series would do absolutely as well as any other.

The methodological implications of this "ordinal" conception of utility have been discussed elsewhere;[1] they are very far-reaching indeed. By transforming the subjective theory of value into a general logic of choice, they extend its applicability over wide fields of human conduct. Two opportunities for the exercise of this new freedom seem of particular importance for the future of economics. One is the economic theory of the state, where the shackles of utilitarianism have always galled; the other is the theory of risk, where the application of the same logic seems fundamental to any progress in economic dynamics.

The present paper, however, is not concerned with these wide questions. Its task is the more pedestrian one of examining what adjustments in the statement of the marginal theory of value are made necessary by Pareto's discovery. As it happens, this task was not by any means completely carried through by Pareto himself. Much of his theory had already been constructed before he realised the immeasurability of utility, and he never really undertook the labour of reconstruction which his discovery had made necessary.

There are, however, two later writers whose work goes some way towards supplying this deficiency; they are W. E. Johnson and R. G. D. Allen. Johnson's work[2] does not appear to spring directly from Pareto; it is based rather upon Edgeworth; but it is much less dependent upon a "cardinal"

[1] Cf. Zawadski, *Les Mathématiques appliquées à l'économie politique*, ch. iii; Schönfeld, *Grenznutzen und Wirtschaftsrechnung*, Part I; Wicksteed, *Common Sense of Political Economy*, ch. v; Robbins, *Nature and Significance of Economic Science*, ch. vi.

Reference should also be made to Edgeworth's interesting remarks on Pareto's doctrine (*Papers*, vol. ii, pp. 472-6). It has become increasingly hard to accept Edgeworth's contention that the existence of theories of Public Finance and Industrial Conciliation depending on the measurability of utility ought to be regarded as an argument in favour of maintaining that assumption. For its abandonment need not imply the abandonment of these undoubtedly valuable doctrines; it serves instead as a stimulus to the construction of new theories of wider validity, into which the traditional teaching can subsequently be fitted as a special case, depending on the introduction of a particular ethical postulate.

[2] "The Pure Theory of Utility Curves," ECONOMIC JOURNAL, 1913.

conception of utility than any of theirs. It was further developed by Mr. Allen in a pair of articles written before our collaboration began;[1] the present paper is the result, first, of my own reflections about Mr. Allen's work, and secondly, of our collaboration in working out the details of a theory which shall be free of the inconsistencies detected in Pareto.[2]

What has now to be done is to take in turn a number of the main concepts which have been evolved by the subjective theory; to examine which of them are affected by the immeasurability of utility; and of those which have to be abandoned, to enquire what, if anything, can be put in their place. It is hoped in this way to assist in the construction of a theory of value in which all concepts that pretend to quantitative exactitude, can be rigidly and exactly defined.

I

1. *Marginal utility.* If total utility is not quantitatively definable, neither is marginal utility. But the theory of value does not need any precise definition of marginal utility. What it does need is only this; that when an individual's system of wants is given, and he possesses any given set of goods, X, Y, Z, . . . we should know his *marginal rate of substitution* between any two goods. The marginal rate of substitution of any good Y for any other good X is defined as the quantity of good Y which would just compensate him for the loss of a marginal unit of X. If he got less than this quantity of Y, he would be worse off than before the substitution took place; if he got more he would be better off; there must be some quantity which would leave him exactly as well off as before.

It will be evident to the reader that this *marginal rate of substitution* is nothing else than what we have been in the habit of calling the ratio of the marginal utility of X to that of Y; we might have called it the "relative marginal utility."

[1] "Nachfragefunktionen für Güter mit korreliertem Nutzen" (*Zeitschrift für Nationalökonomie*, Mar. 1934); "A Comparison between different definitions of complementary and competitive goods" (to appear in *Econometrica*).

[2] Our co-operation has been so close that it has been completely impossible to separate out his results from mine in any orderly presentation. It has therefore seemed best that I should present our whole theory in a non-mathematical form, while Mr. Allen follows it with a mathematical version. But this division does not of course correspond in any way to the actual process by which the theory was constructed. Mathematics and economics went hand in hand; nor would the reader find it easy to identify our respective shares by a consideration of the technique necessary to reach particular points.

My reasons for suggesting what is certain to be a rather tiresome change in terminology are these. If once we introduce marginal utilities, then, with the best will in the world, it is extraordinarily difficult to keep these two marginal utilities together; they have an almost irresistible tendency to wander apart. It would be possible to work out the whole of the following theory, using as our basic concept the ratio of the marginal utility of *X* to that of *Y*, the quantities possessed (or consumed) of all commodities being given; but we should have to keep a strong hold on ourselves, or we should soon be finding some indirect way of talking about one marginal utility by itself—or, what is equally indefensible, talking about the ratio of the marginal utility of *X* (when one set of quantities is possessed) to the marginal utility of *Y* (when the quantity possessed is in some way different).

A second reason may perhaps become clear in what follows. There does seem to be some advantage to be gained from concentrating our attention at this early stage on the essentially substitutional character of the concept.

If an individual is to be in equilibrium with respect to a system of market prices, his marginal rate of substitution between any two goods must equal the ratio of their prices. Otherwise he would clearly find an advantage in substituting some quantity of one for an equal value (at the market rate) of the other. This is the form in which we now have to write the law of proportionality between marginal utilities and prices.

When quantities of *X* and *Y* are represented on an indifference-diagram (quantities of all other goods possessed being therefore supposed given), the marginal rate of substitution between *X* and *Y* is measured by the slope of the indifference-curve which passes through the point at which the individual is situated. This depends simply upon the system of indifference-curves; given the indifference-map, we can read off directly the slope at any point; given the slopes at all points within a region, we can reconstruct the indifference-map for that region.[1]

[1] For more than two commodities the corresponding proposition is not necessarily true. For n goods, we have n-1 independent marginal rates of substitution (those of *X* and *Y*, *X* and *Z*, *X* and *W*, etc.; the rest can be deduced from these). But from these n-1 marginal rates of substitution it is only possible to construct an indifference-diagram (or what corresponds to an indifference-diagram in n-dimensional space) if some further conditions are satisfied (the integrability conditions). This proposition, which exercises a great fascination over the minds of mathematical economists, remains of doubtful economic significance. Some conclusions which are only valid if the integrability conditions are satisfied, will be given below.

2. *Diminishing marginal utility.* The principle of diminishing marginal utility must similarly give place to *increasing marginal rate of substitution.* Starting with given quantities of all the goods X, Y, Z, . . .; if we first replace a marginal unit of X by that quantity of Y which just makes up for it; and then replace a second marginal unit of X by that quantity of Y which just makes up for this second unit: the second quantity of Y must be greater than the first. In other words, the more we substitute Y for X, the greater will be the marginal rate of substitution of Y for X.

This condition is expressed on the indifference diagram by drawing the indifference-curves convex towards the axes. (The curves must of course always slope downwards if the goods are both positively desired.)

The replacement of *diminishing marginal utility* by this principle of *increasing marginal rate of substitution* is something more than a mere change in terminology. When we seek to translate the principle of *diminishing marginal utility* into definable terms, it does not appear at first sight evident that this is the condition we must use. And it is an interesting historical fact that when Pareto found himself confronted with this question, he first of all gave the condition that the indifference-curves must be convex to the origin, and then went on to add a further condition: that the marginal rate of substitution will increase, not only when Y is substituted for X, but also when the supply of Y is increased without any reduction in the supply of X. This condition looks as good a translation of *diminishing marginal utility* as the other, but (as Pareto ultimately realised[1]) it stands on an altogether different footing. Cases which do not satisfy this latter principle undoubtedly exist in plenty, and there is no particular difficulty in fitting them into a general theory.[2] Exceptions to the true principle of increasing marginal rate of substitution would be much more serious.

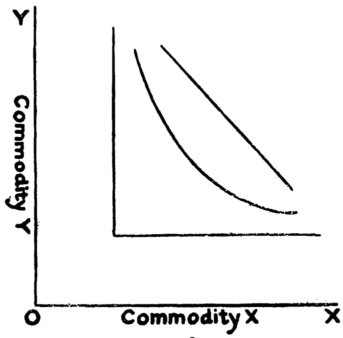
For it is certain that for a position to be one of stable equilibrium at given prices, the marginal rate of substitution *at that point* must be increasing. If it is not, then, even if the marginal rate of substitution equals the price ratio, so that the sale of one marginal unit of X would not give any appreciable advantage, nevertheless the sale of a larger quantity would be advantageous. Equilibrium would be unstable—the individual would be at a point of minimum, not maximum, satisfaction.

[1] *Manuel* (French edition), p. 573-4; cf. the earlier Italian edition, pp. 502-3 (of the 1919 reprint).
[2] See below, sec. ii, 1, of this paper.

The assumption that the principle of increasing marginal rate of substitution is universally true, thus means simply that any point, throughout the region we are considering, might be a point of equilibrium with appropriate prices. There must be some points at which it is true, or we could get no equilibrium at all. To assume it true universally is a serious assumption, but one which seems justifiable until significant facts are adduced which make it necessary for us to pay careful attention to exceptions.[1]

3. *Elasticity of substitution.* The replacement of *diminishing marginal utility* by *increasing marginal rate of substitution* has this further advantage: it becomes significant and useful to ask: " Increasing how rapidly ? " Economists whose theory was based on diminishing marginal utility have rarely had the courage to ask a corresponding question; and when they have done so they have not derived much advantage from it. But our conception is strictly quantitative; and the rate of increase of the marginal rate of substitution may be expected to play an important part in the development of theory.

It is obvious that the two main conditions under which indifference lines are drawn—(1) downward slope, since an increase in either commodity leads to a preferred position; (2) convexity to the origin, from the principle of *increasing marginal rate of substitution*—leave open a wide variety of different shapes which may be taken by the curves. They may vary from the one extreme of straight lines at an oblique angle to the axes (the case of perfect substitutes) to the other of pairs of perpendicular straight lines parallel to the axes (the case of goods which must be used in fixed proportions). Between these extremes any degree of curvature is possible.



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The curvature of the indifference-curve describes the same property as the " rate of increase of the marginal rate of substi-

[1] Exceptions would presumably take the form of " blind spots " on the indifference diagram—regions within which no stable equilibrium would be possible. These would also involve the possibility of cases of " Buridan's ass "; the consumer with given income, confronted with given prices, would still be unable to decide between a number of different distributions of expenditure.

tution." But to take either as our measure without correction for units would be impossible—the result would have as little significance as the uncorrected *slope* of a demand curve. A measure free from this objection fortunately now lies ready to our hand. It is the *elasticity of substitution*, when defined in a way analogous to that used by Mrs. Robinson and Mr. Lerner.[1]
Applied to this problem it becomes

$$\frac{\text{relative increase in the proportion possessed of the two commodities } (Y/X)}{\text{relative increase in the marginal rate of substitution of } Y \text{ for } X}$$

when a small amount of Y is substituted for X, in such a way as to compensate the consumer for his loss. (That is to say, it is taken along the indifference-curve.)

One of the advantages of this particular measure is that it is symmetrical; if we write X for Y, and Y for X, in the above, the result is unchanged. It is therefore a general measure of substitutibility; when the commodities are perfect substitutes, (so that the rate of increase of the marginal rate of substitution is zero), the elasticity of substitution becomes infinite; when they have to be used in fixed proportions (the other extreme) the elasticity of substitution is zero. Negative elasticities of substitution are, of course, ruled out by the principle of increasing marginal rate of substitution.

4. *Complementarity.* If (as appears from the above) any two goods are to be regarded as more or less substitutes, what becomes of the traditional doctrine that two goods may be either competitive or complementary? It will not be possible to give a full answer to this question until much later in this paper, but it is already possible to indicate why the traditional conception entirely fails to accommodate itself to our present construction. The definition of complementary (and competitive) goods given by Pareto and Edgeworth[2] (these seem to be the only major economists who have given an exact definition in terms of the general theory of wants) is completely dependent on the

[1] Robinson, *Economics of Imperfect Competition*, p. 256; Lerner, "Elasticity of Substitution" (*Review of Economic Studies*, Oct. 1933, pp. 68-70). The definition given in my *Theory of Wages*, though appropriate, under certain assumptions, to the theory of production, is not valid here.

[2] Pareto, *Manuel*, p. 268; Edgeworth, *Papers*, vol. i, p. 117.

notion of utility as a determinate function. On their view, complementary goods are such that an increase in the supply of one will increase the marginal utility of the other; competitive goods are such that the marginal utility of the other will be lowered. This test cannot be translated into terms of marginal rates of substitution; it becomes definitely ambiguous when account is taken of the immeasurability of utility. In the vast majority of cases, the goods will be complementary or competitive, *on this definition*, according to the particular arbitrary measure of utility we choose to take.[1]

For the moment, then, let us put complementarity aside.

[1] If the utility function could be uniquely defined as $\phi(x,y)$, then the Paretian test would be given by the sign of ϕ_{xy}. But if we adopt his own doctrine that any other function of ϕ, $F(\phi)$, could equally well be taken as the utility function, this test breaks down. For

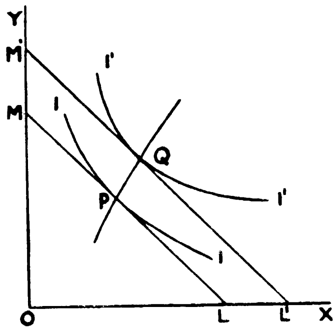
$$\frac{\partial^2}{\partial x \partial y} F(\phi) = F'(\phi) \cdot \phi_{xy} + F''(\phi) \cdot \phi_x \phi_y$$

and in general there is no reason why this should have the same sign as ϕ_{xy}, even though $F'(\phi)$ should be taken as positive.

II

1. *The expenditure curve.* We have in the elasticity of substitution one of the fundamental concepts on which our further enquiries will be based; but it is not by itself an adequate foundation for a theory of value. For the elasticity of substitution refers only to one possible kind of change: that which takes place if one commodity is substituted for another, if, that is to say, the individual moves from one position to another on the same indifference-curve. But this kind of movement is not the only one of which we have to take account. When the conditions of the market change, the individual does not usually move along the same indifference-curve; he is usually made better off or worse off by the change, so that he moves from one indifference-line to another. We, therefore, need information, not only about the shapes of particular curves, but about the mutual relations of the curves.

Take any point P on a given indifference-map, and draw the tangent at P to the indifference-curve that passes through P. Now draw a series of straight lines parallel to that first tangent, and mark off on each line the point where it touches a curve of the system. (By the principle of increasing marginal rate of substitution there can for each line be only one such point.) Now join these points. The curve so formed I shall call an *expenditure-curve*. It follows from the same principle of increasing marginal rate of substitution, that this expenditure-curve can cut any indifference-curve in one point only, and that there can be only one expenditure-curve through any point. But through any point an expenditure-curve can be drawn.[1]



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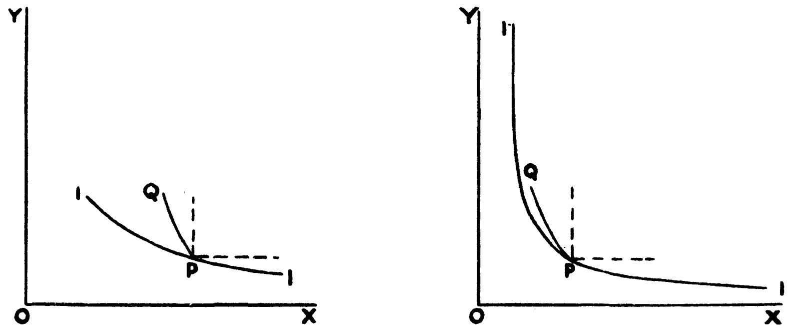
The significance of this construction should be clear. The point P is a position of equilibrium (income being spent wholly upon commodities X and Y), when the relative prices of X and

[1] The reason why these further elaborations are not necessary in the theory of production—at least in its elementary stages—is that the assumption of a homogeneous production function implies that all "expenditure-curves" are straight lines through the origin.

Y are as *OM/OL*, and when the income of the individual is *OL* (measured in terms of *X*) or *OM* (measured in terms of *Y*). The point *Q* is a position of equilibrium when the relative prices are the same (since the tangents are parallel) but income has increased (from *OL* to *OL′*, or *OM* to *OM′*). The expenditure-curve thus describes the way in which the consumption of the two commodities varies, when prices remain unchanged, but there is a change in total expenditure.

What is the relation between the expenditure-curve through *P* (or rather its slope at *P*) and the elasticity of substitution at *P*? Strictly, they describe different things, for while the latter is a characteristic of a single indifference-curve, the former describes the relationship of one indifference-curve to others. Expenditure-curves of all sorts of slopes are compatible with elasticities of substitution of all sorts of magnitudes.

There is, however, one limitation on this. Since no expenditure-curve can cut an indifference-curve more than once, the variety of possible slopes the expenditure-curve can show is a little more restricted when the elasticity of substitution is low than when it is high. For finite changes, at any rate, an expenditure-curve through *P* which slopes very much to the left, or very much downwards, becomes distinctly more probable the flatter the expenditure-curve at *P* is, though this probability is reduced if there are stretches of greater curvature



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(or lower elasticity of substitution) in the neighbourhood of *P*.[1] In the case of fixed proportions (elasticity of substitution

[1] Very abnormal expenditure-curves (downward or backward sloping) are undoubtedly most likely at the extremities of the indifference-curves ; for most indifference-curves become fairly flat as they approach the axes.

zero) the expenditure-curve must of course slope to the right and upwards.

Now if the expenditure-curve is positively inclined, this means that an increase in income will increase the consumption of both commodities (X and Y). If the expenditure-curve is downward-sloping, an increase in income will increase the consumption of X but diminish that of Y; if it is backward-sloping, X will be diminished and Y increased. These latter cases may arise whether or not the goods are easily capable of substitution, but they are distinctly less likely when the elasticity of substitution is low.

It is these cases which are ruled out by Pareto's condition, which we quoted above as a possible interpretation of *diminishing marginal utility*—the interpretation which we discarded. If, for example, the expenditure-curve is backward-sloping, this means that the point Q (where the higher indifference-curve has the same slope as the lower curve at P) lies to the left of P; and since (by *our* principle of increasing marginal rate of substitution) the slope of any indifference-curve must increase from right to left, or diminish from left to right, the higher indifference-curve must have a smaller slope at the point vertically above P than the lower indifference-curve has at P. The marginal rate of substitution therefore diminishes when Y is increased and X is left unchanged.

Pareto's condition would thus limit us to positively-inclined expenditure-curves; but there is no particular reason why we should limit ourselves to cases which satisfy this condition.[1] Negatively-inclined expenditure-curves do occur; they are found whenever one of the commodities is an "inferior" good, which is most largely consumed at relatively low levels of income, being replaced (or partially replaced) by goods of higher quality when income increases.

The most convenient measure for that property expressed by the expenditure-curve is simply the elasticity of demand for X (or Y) *in terms of income*. (The two are interdependent.) We shall find it convenient in this paper to use the conception of elasticity of demand in several senses additional to that given it by Marshall. Strictly speaking, the individual's demand for any commodity depends, not only on the price of that commodity, but also on the prices of all other commodities purchased, *and on his income*. A change in any one of these

[1] A theory limited by this condition (and by this alone), would not be appreciably simpler than a more general theory; and it would certainly fail to cover all the facts.

variables may affect the demand for X; and we can measure the dependence of demand on any of these variables by an elasticity. (Of course, many of these elasticities will usually be negligible.)[1]

The income-elasticity of demand for X therefore

$$= \frac{\text{relative increase in demand for } X}{\text{relative increase in income}}$$

when income is increased by a small amount, but the prices of all goods remain the same.

If there are only two goods purchased (the case to which our expenditure-curve directly refers), then a negative income-elasticity of demand for X means that the expenditure-curve is backward-sloping. A zero elasticity gives a vertical expenditure-curve. An elasticity of unity indicates that the consumption of each good increases in the same proportion as income, so that the slope of the expenditure-curve becomes the same as that of the line OP. If the expenditure-curve is downward-sloping, the income-elasticity of Y must be negative, and consequently the income-elasticity of X must be greater than $\frac{1}{k_x}$, where k_x is the proportion of income initially spent on X.[2]

The conception of income-elasticity of demand is obviously applicable, however many are the goods on which income is spent.

2. *Constant marginal utility*. A simple application of the preceding argument is the translation of Marshall's "constant marginal utility" into exactly definable terms. If the marginal utility of commodity Y is constant, the marginal rate of substitution between X and Y must depend on X only. If the quantity of X is given, the marginal rate of substitution (or the slope of the indifference-curve) is given, too; the tangents to the indifference-curves at all points with the same abscissa must be parallel.

Since the expenditure-curve is drawn through points of

[1] Cf. Lange, "Die allgemeine Interdependenz der Wirtschaftsgrössen und die Isolierungsmethode" (*Zeitschrift für Nationalökonomie* 1932).

[2] This latter proposition follows at once from the condition that $k_x \times$ income-elasticity of demand for $X + k_y \times$ income-elasticity of demand for $Y = 1$; for the small increase in income is supposed to be spent wholly upon X and Y. A similar proposition holds for any number of commodities.

parallel tangency, the expenditure-curve must be vertical, and the income-elasticity of demand for X must be zero. This property is again capable of extension to any number of goods. If the marginal utility of any one commodity out of many is constant, the income-elasticities of all the rest will be zero.[1]

III

1. *The demand-curve.* The two indices we have now developed, the elasticity of substitution and the income-elasticity of demand, describe the two most important characteristics of the individual's scale of preferences in the immediate neighbourhood of the position where he happens to find himself. They are the analytical tools which we may now proceed to apply; and the first object of analysis must inevitably be the ordinary demand-curve.

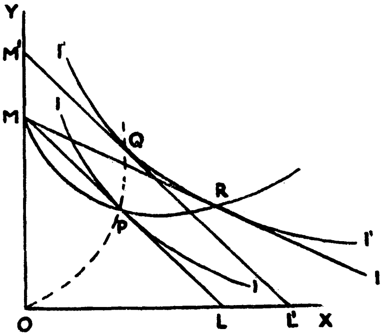
Here we may conveniently begin with a geometrical treatment, concentrating in consequence on the case where income is spent on two goods only—the case most amenable to the geometrical method.

Income is now to be taken as fixed, and the price of Y as fixed: but the price of X is variable. The possibilities of expenditure open to him are thus given by straight lines joining M ($OM =$ income measured in terms of Y) to points on OX which vary as the price changes. Each price of X will determine a line LM (OL increasing as the price falls); and the point of equilibrium corresponding to each price will be given by the point where the line LM touches an indifference-curve. Joining these points, we get a demand-curve.[2]

Now it is obvious (again from the convexity of the indifference-curves) that any single indifference-curve must be touched by a line through M at a point to the right

[1] When restated in accordance with this, the argument of Marshall V. 2. (that notable incursion into the dynamic theory of value) remains of course perfectly valid. If the article on which interest is concentrated is only one among many, only a small part of the increase in income due to an early favourable bargain is likely to be spent on that particular article; so that the demand curve for further units is unlikely to be much affected by such market aberrations. That is essentially all Marshall's argument comes to.

[2] Strictly speaking, demand-and-supply curve reversed. Supposing the individual to start with a given amount OM of Y, we might subtract each ordinate of the above curve from OM, and get a demand-and-supply (or offer) curve of the ordinary type.



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of that where it is touched by a line parallel to *LM* and above it. Therefore, as we move on to higher indifference curves, the demand-curve through *P* must lie to the right of the expenditure-curve through *P*; that is to say, the slope of the demand-curve must be less than the slope of the expenditure-curve.

Further, it is fairly evident from the diagram that the difference between these slopes—the extent to which *R* will be pushed to the right of *Q*—will depend upon the curvature of the indifference-curves, that is to say, upon the elasticity of substitution. The greater the elasticity of substitution—the flatter, therefore, the indifference-curves—the greater will be the divergence between the expenditure-curve and the demand-curve.

The increase in demand for a commodity *X*, which results from a fall in its price, depends therefore partly upon the income-elasticity of demand for *X*, and partly upon the elasticity of substitution between *X* and *Y*. We can in fact look upon the increase in demand as consisting of two parts, one of which is due to the increase in real income which a fall in the price of *X* entails, the other to the opportunity of substituting *X* for other goods which results from the fall in the *relative* price of *X*.

The relative importance of these two components depends fairly obviously upon the proportion of income initially spent on *X*. The larger that proportion, the greater will be the increase in real income resulting from a given fall in the price of *X*; and this will increase the importance of the income-elasticity relatively to the elasticity of substitution.

These geometrical and verbal reasonings hardly enable us to proceed to a formula for the elasticity of demand for *X* (in the ordinary sense, elasticity with respect to the price of *X*). But they are exactly corroborated by the algebraic analysis which will be given by Mr. Allen.[1] It is there rigorously proved that with two commodities:

[1] See Part II of this article, sect. I, 3 (*b*).

Price-elasticity of demand for X

$$= k_x \times \text{income-elasticity of demand for } X + (1 - k_x) \times \text{elasticity of substitution between } X \text{ and } Y$$

(where k_x is the proportion of income spent upon X).

The price-elasticity of demand is thus not an independent index; it is reducible to the two primary characteristics which we described above.

2. *Extension to more than two goods.* Our formula has this further convenience, that it is capable of extension, with the slightest possible amendment, to the much more important case where more than two goods are consumed. We have only to write, instead of " elasticity of substitution between X and Y," " elasticity of substitution between X and all the other goods taken together." For the rest, the formula remains unchanged.[1]

The sense of this extension can be interpreted as follows: Since it is only the price of X which varies, while the prices of Y, Z . . . remain unchanged, these latter goods remain freely substitutible for each other at fixed ratios given by their relative prices. They behave, therefore, just like perfect substitutes, and a collection of perfect substitutes can be regarded as a single commodity. But the single composite " commodity," which is thus formed by Y, Z, . . . taken together must be regarded as similar to a commodity with a wide variety of *uses*; the substitution among themselves of Y, Z, . . . is of precisely the same character as the reshuffling of quantities of the second commodity among different uses which might very well take place, even if there were only two commodities altogether.

Now it is fairly clear that with two commodities only, the elasticity of substitution between X and Y is likely to be greater, other things being equal, if Y has a wide variety of uses than it will be if Y is very specialised—this is evidently one of the main influences affecting the elasticity of substitution. Applying this to the " many commodities " case, it follows that the elasticity of substitution between X and the " composite commodity " is likely to be greater the more various the components of the latter are, i.e. the smaller are their mutual elasticities of substitution.[2]

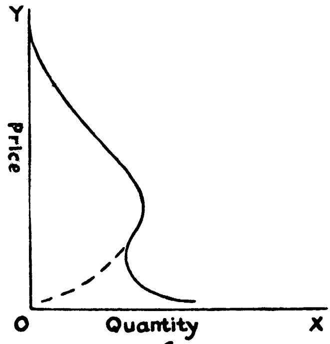
[1] See Part II, sect. II, 4 (*b*).

[2] The elasticity of substitution between X and YZ thus varies inversely with the elasticity of substitution between Y and Z.

Consequently, the elasticity of demand for any commodity is likely to be greater the more various are the objects of consumption with which it is in competition.

3. *The "rising" demand-curve.* Our analysis has now provided us with an exact definition of the conditions on which the elasticity of an individual's demand for a particular commodity X must depend. Since, of the two terms of which our formula is composed, the second must be positive, but the first is not restricted in sign, a highly inelastic demand is possible either (1) if both terms are positive and very small, or (2) if the first term is negative. Now, as we have seen, when X is only one good among several, it is unlikely that the elasticity of substitution between X and the other goods together will be very small, so that a highly inelastic demand is less probable in case (1) than in case (2). In that second case, where X is an inferior good, the elasticity of demand will clearly be smaller, the higher the proportion of income spent on X.

When the income-elasticity is negative, there is no absolute reason why we should be limited to positive price-elasticities of demand, i.e. to downward-sloping demand-curves. If X is a good very decidedly "inferior," so that its income-elasticity is negative and fairly large; if k_x is also large, so that a large proportion of income is spent on X; if, finally, the elasticity of substitution between X and other goods is moderately small; then the first (negative) term in our formula may outweigh the second (positive) one.



6

This possibility can easily be recognised as the celebrated Giffen case referred to by Marshall,[1] when the consumption of bread may actually be reduced by a fall in its price. Our analysis shows that it is perfectly consistent with the principle of *increasing marginal rate of substitution*; but it is only possible at low levels of income, when a large proportion of expenditure is devoted to this "inferior" commodity, and when, among the small number of other objects consumed, there are none that are at all easily substitutible for the first. As the standard of living rises, and

[1] Marshall, *Principles*, 8th edition, p. 132.

expenditure becomes increasingly diversified it is a situation which becomes increasingly improbable.[1]

IV

Complementarity

1. It is perfectly consistent with the theory we have so far elaborated, to suppose that all goods are more or less related in consumption; yet we have made no use of the conception of *complementary and competitive goods*. We have not used it, because we had no need to use it; we had not yet come to the problem where it is relevant.

Substitution, indeed, comes into the theory of value from the start. Any two goods are substitutes—more or less. But complementarity, in the strict sense in which we shall define it, is not a possible property of two goods; it only has sense when the goods in question are at least three.

We have already examined the reaction of a fall in the price of one good on the quantity demanded of that good; and we have discovered that our analysis was applicable, however many other goods are simultaneously consumed. We have now to enquire how such a fall in price reacts on the demand for one particular good out of these other goods.

The same principle which we have previously applied will obviously hold here. The change in the demand for Y resulting from a fall in the price of X will again consist of two parts. (1) There will be the change in demand for Y resulting from the increase in real income; (2) there will be the change in demand resulting from the substitution of X for the rest, owing to the fall in its relative price.

Of these two components, the first will normally be positive, but will be negative if Y is an "inferior" good. The second will depend on how far the substitution in favour of X takes place at the expense of Y rather than of the other goods (Z). If Y and Z are more or less on the same footing in the scale of wants, so that they are sacrificed fairly equally, then the second

[1] The demand curve on our diagram (p. 21) must first descend from M as the price of X falls (its elasticity $>$ 1). After a time it may rise again (become inelastic); and then—only then—it may curl back towards the Y-axis.

On a price-quantity diagram the resultant curve would look like Fig. 6, page 68. But it might conceivably continue to curve back to the price-axis (dotted line).

component will evidently be negative; and such negativeness we clearly ought to regard as the normal case. When the second component is negative, we shall say that *Y is competitive with X against Z.*

On the other hand, it is possible that the substitution in favour of *X* may not, as in this case, take place partly at the expense of *Y*, partly at the expense of *Z*. It may carry with it a simultaneous substitution of *Y* for *Z*; so that the whole effect of the substitution in favour of *X* is that the consumption of *X and Y* is increased, but that of *Z* is diminished—of course, more than in the preceding case. If this is so, we shall say that *Y is complementary with X against Z*; and here the *second component* of the preceding paragraphs will be positive.

For three goods, we may thus distinguish three possible cases; either *Y* and *Z* are both competitive with *X* (against *Z* and *Y* respectively); or *Y* is complementary and *Z* competitive; or *Z* is complementary and *Y* competitive. It is impossible for both *Y* and *Z* to be complementary with *X*, since this would infringe the principle of increasing marginal rate of substitution.

For more than three goods, the possibilities are obviously extended; but it remains impossible for all of the other n-1 goods to be complementary with any one good. It is possible, however, for all the remaining n-1 goods to be competitive with the first.

2. The definition of complementarity just given, although it indicates the most important property of complementary (or competitive) goods, is, as a definition, not altogether satisfactory. For there is implied in it the assumption that when *X* is substituted for *Y* and *Z*, the ratio of the prices of *Y* and *Z* remains unchanged, and it is only the price of *X* relatively to these prices which varies. (Any change in the *YZ* price-ratio would of course affect the quantities substituted.) Since there is implicit in our definition this assumption about price-ratios, we have not succeeded in defining complementarity (as we ought to do) purely in terms of the individual's preference-scale; we are making a reference to the market which is better avoided.

Since there is an indefinite number of ways in which two goods *Y* and *Z* can be substituted for a marginal unit of *X*, it is best to concentrate our attention on that case which is the watershed between competitiveness and complementarity—the case when substitution in favour of *X* tends to leave the amount consumed of Y unchanged. Suppose then that X is substituted for Z, but Y remains unchanged. This simple substitution will affect not only the marginal rate of substitution between X and Z (in the way previously analysed), it will also affect the marginal rate of substitution between Y and Z. Since the quantity of Z is being diminished, the "normal" effect will be to shift the marginal rate of substitution between Y and Z in favour of Z, or against Y[1]; Y is then competitive with X against Z. But if (as is possible) the marginal rate of substitution is shifted in the opposite direction (in favour of Y), then Y is complementary with X.

This second definition is really nothing more than a restatement of the first, and their equivalence is readily shown. If the marginal rate of substitution is shifted against Y, then (if the price ratio between Y and Z remains unchanged), there will be a tendency to substitute Z for Y, i.e. some Y will be sacrificed. If the marginal rate of substitution is shifted in favour of Y, then not only is X substituted for Z (as we are already supposing to be the case), but Y increases at the expense of Z as well.[2]

3. The test of complementarity or competitiveness is thus established: the change in the marginal rate of substitution between Y and Z which follows on a marginal substitution of X for Z. Not only does the direction of this change indicate *whether* the goods are complementary or competitive, but also the degree of change (when properly adjusted) can be used as a *measure* of complementarity. But the definition of this "elasticity of complementarity," and the detailed analysis which it makes possible, are so complex, that we must content ourselves here with a mere statement of results, whose proof must be left over to Mr. Allen's mathematical version.[3]

It is there shown that:

(*a*) the elasticity of demand for Y relatively to the price of X
$= k_x \times$ (Income-elasticity of demand for Y + Elasticity of complementarity of Y with X against Z).

(The elasticity of complementarity is, of course, positive or negative, according as the goods are complementary or competitive.)

[1] That is to say, it will increase the amount of Y needed to replace a marginal unit of Z. [2] See Part II, sect. II, 4, 5. [3] See Part II, sect. II and sect III, 1, 2.

(*b*) If the integrability conditions are satisfied (and as a general rule we may probably take it that they are), so that *a* utility function could be formed, though not one utility function only; then it is true that the elasticity of complementarity of Y with X against Z *equals* the elasticity of complementarity of X with Y against Z. In general, therefore, it is quite correct to talk about X and Y being complementary (or competitive) with respect to Z, without having recourse to the more elaborate terminology we have hitherto employed.[1]

(*c*) From this it can be shown in precisely what way complementarity will be reflected in demand relations. For the elasticity of demand for X relatively to the price of Y $= k_y \times$ (Income-elasticity of demand for $X +$ elasticity of complementarity of XY against Z).

Whether or not this will have the same sign as the elasticity of demand for Y relatively to the price of X thus depends on the difference between the income-elasticities of the two commodities. If this difference is small, the two cross-elasticities of demand will generally have the same sign; but they may not if the difference between the income-elasticities is considerable. If, for example, one of the income-elasticities is positive and the other negative, then we may get reactions of price on demand which go in completely opposite directions—unless the goods are sufficiently complementary (or sufficiently competitive) for this variation to be swamped by the complementarity term.[2]

(*d*) Since the elasticity of complementarity of Y with X against Z measures the extent to which a substitution of X for YZ takes place at the expense of Y (when the relative prices of Y and Z are unchanged); and since the elasticity of complementarity of Z with X against Y measures the extent to which the same substitution takes place at the expense of Z; there must be a relation between these two elasticities and the elasticity of substitution of X with YZ taken to-

[1] Subject to the same condition, it follows that of three goods, X, Y, Z, only one pair at most can be complementary.

[2] The valuable investigation into this problem by Professor Henry Schultz ("Interrelations of Demand," *Journal of Political Economy*, August 1933) is limited by the assumption of "constant marginal utility of money" (i.e. of our third good Z). This comes to the same thing as neglecting the income-elasticities of demand, which the present analysis shows to be highly significant for the problem, as they may easily be of comparable magnitude with the (symmetrical) complementarity term.

gether. In the general case, where there are six elasticities of complementarity and three elasticities of substitution (*X* for *YZ*, etc.) we get three equations connecting them, and could thus write the elasticities of substitution in terms of the elasticities of complementarity.

But when the integrability conditions are satisfied, and the six elasticities of complementarity therefore reduced to three, we can also use these three equations to give us the elasticities of complementarity in terms of the elasticities of substitution.[1] Hence we can derive the following propositions:

(1) *XY* are more likely to be complementary with regard to *Z*, the lower is the elasticity of substitution between *X* and *Y*, relatively to those between *X* and *Z*, and *Y* and *Z*.

(2) *XY* are more likely to be complementary with respect to *Z*, the larger is the proportion of total income spent upon *Z*, and therefore the smaller the proportion spent on *X* and *Y* together.

(3) If the elasticity of substitution between *X* and *Y* is zero, they must be complementary with respect to any third good less closely related ; if the elasticity of substitution between them is infinite, they must be competitive.[2] If the elasticities are equal, they must be competitive.

[1]See Part II, sect. III, 2.

[2] In this sense, therefore, and in this sense alone, is it possible to say that competitive goods are easily substitutible ; complementary goods not easily substitutible. This statement, so agreeable to common sense, turns out to be correct—so long as we speak in relative, not absolute, terms.

V

Independence

1. According to the Edgeworth-Pareto definition of complementarity (based on the reaction of the marginal utility of one commodity to a change in the quantity of the other), it was natural to regard the case intermediate between complementarity and competitiveness (where the effect on the marginal utility is zero) as a case of " independent goods." This definition must be abandoned for the same reason as we have abandoned their definition of complementarity.[1]

Nor is it in any way appropriate to regard the watershed between complementarity and competitiveness (on our definition)[2] as a case of independence. For, if, as would happen at our watershed, the marginal rate of substitution between Y and Z is unaffected by compensating changes of X and Z, this does not mean that the goods are in any useful sense " independent "—there subsists a very complex relation between them.

But there does exist another property to which the term *independence* can much more usefully be applied. If the marginal rate of substitution between Y and Z is unaffected by the quantity of X possessed, then we may say that YZ is *independent* of X.[3] If this condition holds, then it is clear that a substitution of X for Z can exert an influence on the marginal rate of substitution between Y and Z in only one way. The increase in X has no influence at all; it is only the decrease in Z which is effective. But that decrease in Z may still affect the marginal rate of substitution between Y and Z in either direction. For although the normal effect will undoubtedly be to move the marginal rate in favour of Z, nevertheless, if the relationship between Y and Z is such that, were they to be consumed in isolation, Y would be an *inferior* good, then the marginal rate of substitution would be shifted in favour of Y.[4]

It is therefore possible for YZ to be independent of X, and at the same time XY may be either competitive or complementary against Z.

[1] It was a feeling of disquiet about this definition in the mind of Dr. Rosenstein-Rodan which first led me to a consideration of the whole problem of this paper.

[2] When the elasticity of complementarity is zero.

[3] Similarly, for more than three goods, if the marginal rate of substitution between any pair depends on the quantities of these goods alone, it may be said to be " independent."

[4] See above, sect. II, 1.

2. To say that YZ is independent of X, is a very different matter from saying that X, Y, Z, are independent goods. Can we give any meaning to the latter statement?

There is only this. If YZ is independent of X, and XZ is independent of Y, then XY may also be independent of Z.[1] If this is the case, then X, Y, Z are clearly independent in a wider sense, which approximates more closely to the older definition.[2] (This can be extended to any number of goods, which will be independent if the marginal rate of substitution between any pair of them depends on the amounts of those goods alone.)

Independent goods may be either complementary or competitive; but it follows from the preceding section that X and Y can only be complementary if Y is an *inferior* good. Further, since the integrability conditions must always be satisfied for independent goods, a substitution of Y for Z must also move the marginal rate of substitution between X and Z in favour of X, i.e. X must be an *inferior* good too. There are thus two possible cases of independent goods:

(1) where all pairs are competitive, and all the income-elasticities positive.

(2) where one pair is complementary, and two income-elasticities negative.

It will be shown in the mathematical analysis that this relation between complementarity and income-elasticity (in the case of independent goods) refers not only to sign, but to magnitude as well. If three goods are independent, then their income-elasticities depend on their complementarities.[3] It would not be difficult to demonstrate this on a three-dimensional indifference-diagram, where it would emerge in the following form: that in this case, given one indifference-surface, all the other indifference-surfaces of the system could be deduced.

This property, however, does not reproduce itself in the case of two commodities. The independence condition[4] is not then sufficient to enable us to deduce other indifference-curves from one curve alone. And so, in the two-commodity

[1] It will be if the integrability condition is satisfied.

[2] The marginal rate of substitution between any pair XY must then be of the form $f(x)/g(y)$, where x,y, are the quantities possessed. (This can be used as a definition of independence in the case of two goods.) In cases of complementarity and inferiority, $f'(x)$ or $g'(y)$ may be positive, i.e. we must avoid being entrapped again in the law of diminishing marginal utility!

[3] See Part II, sect. III, 3.

[4] See note 2.

case, but only in this case, it is always possible, however the goods are in fact related, to find a pair of independent utility functions which will give us an indifference-map, closely approximating, over a small region, to the true map. This suggests a method of mathematical economic analysis which is much simpler than the quite general analysis followed in this paper, and which will, for small variations, give a close approximation to correct results.

Substantially, that method is the method of Marshall; it is one which has rendered great services to economics, even when *rationale* was not fully understood. But it is a method which is applicable, in strictness, only to the case of two commodities; for more than two commodities it loses its generality altogether.[1]

[1] The second part of this article, by Mr. R. G. D. Allen, will be published in ECONOMICA, No. 2 (May, 1934).