Supplement to Lecture 3: Derivation of CES Special Cases

Macroeconomics, EC2B1

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In the lecture notes we defined the CES production function

$$Y = \left(\alpha^{\frac{1}{\sigma}}(A_K K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}(A_N N)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \quad \sigma = \text{elasticity of substitution}$$

and asserted that there are three special cases

1. Case $\sigma = 1$: Cobb-Douglas

$$Y = \left(\frac{A_K K}{\alpha}\right)^{\alpha}\left(\frac{A_N N}{1-\alpha}\right)^{1-\alpha}$$

2. Case $\sigma \to \infty$: perfect substitutes

$$Y = A_K K + A_N N$$

3. Case $\sigma = 0$: perfect complements, fixed proportions, or "Leontief"

$$Y = \min\left\{\frac{A_K K}{\alpha}, \frac{A_N N}{1-\alpha}\right\}$$

This supplement to the lecture notes provides the corresponding derivations.

1 Case $\sigma = 1$: Cobb-Douglas

In order to make computations easier, we can define $\rho = \frac{1}{\sigma} - 1 \Leftrightarrow \sigma = \frac{1}{1+\rho}$:

$$Y = \left(\alpha^{1+\rho}(A_K K)^{-\rho} + (1-\alpha)^{1+\rho}(A_N N)^{-\rho}\right)^{-\frac{1}{\rho}}$$

Therefore studying the case of $\sigma = 1$ is the same as $\rho = 0$. Taking the limit as $\rho \to 0$:

$$Y = \exp\left\{\lim_{\rho\to 0} -\frac{1}{\rho}\ln\left(\alpha^{1+\rho}(A_K K)^{-\rho} + (1-\alpha)^{1+\rho}(A_N N)^{-\rho}\right)\right\}$$

Taking the limit yields $\frac{0}{0}$. Therefore we use L'Hôpital's rule:

$$Y = \exp\left\{\lim_{\rho\to 0} -\frac{\alpha^{1+\rho}\ln(\alpha)(A_K K)^{-\rho} - \alpha^{1+\rho}(A_K K)^{-\rho}\ln(A_K K) + (1-\alpha)^{1+\rho}\ln(1-\alpha)(A_N N)^{-\rho} - (1-\alpha)^{1+\rho}(A_N N)^{-\rho}\ln(A_N N)}{\alpha^{1+\rho}(A_K K)^{-\rho} + (1-\alpha)^{1+\rho}(A_N N)^{-\rho}}\right\}$$

Taking the limit and using a bit of algebra:

$$Y = \exp\left\{-\frac{\alpha\ln(\alpha) - \alpha\ln(A_K K) + (1-\alpha)\ln(1-\alpha) - (1-\alpha)\ln(A_N N)}{\alpha + (1-\alpha)}\right\} = \left(\frac{A_K K}{\alpha}\right)^{\alpha}\left(\frac{A_N N}{1-\alpha}\right)^{1-\alpha}.$$

2 Case $\sigma \to \infty$: perfect substitutes

Again, taking the limit:

$$Y = \lim_{\sigma\to\infty}\left(\alpha^{\frac{1}{\sigma}}(A_K K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}(A_N N)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

$$Y = \left(\lim_{\sigma\to\infty}\left[\alpha^{\frac{1}{\sigma}}(A_K K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}}(A_N N)^{\frac{\sigma-1}{\sigma}}\right]\right)^{\lim_{\sigma\to\infty}\frac{\sigma}{\sigma-1}}$$

$$Y = A_K K + A_N N$$

3 Case $\sigma = 0$: perfect complements, fixed proportions, or "Leontief"

Going back to the alternative representation of the CES as a function of ρ and taking the limit as $\rho \to \infty$:

$$Y = \exp\left\{\lim_{\rho\to\infty} -\frac{1}{\rho}\ln\left(\alpha^{1+\rho}(A_K K)^{-\rho} + (1-\alpha)^{1+\rho}(A_N N)^{-\rho}\right)\right\}$$

This time taking the limit $\rho \to \infty$ yields $\frac{\infty}{\infty}$. Therefore again using L'Hôpital's rule:

$$Y = \exp\left\{\lim_{\rho\to\infty} -\frac{\alpha^{1+\rho}\ln(\alpha)(A_K K)^{-\rho} - \alpha^{1+\rho}(A_K K)^{-\rho}\ln(A_K K) + (1-\alpha)^{1+\rho}\ln(1-\alpha)(A_N N)^{-\rho} - (1-\alpha)^{1+\rho}(A_N N)^{-\rho}\ln(A_N N)}{\alpha^{1+\rho}(A_K K)^{-\rho} + (1-\alpha)^{1+\rho}(A_N N)^{-\rho}}\right\}$$

Let $x = \min\left\{\frac{A_K K}{\alpha}, \frac{A_N N}{1-\alpha}\right\}$, $\theta_K = \frac{A_K K}{\alpha x}$ and $\theta_N = \frac{A_N N}{(1-\alpha)x}$. Take the previous expression and divide the numerator and denominator by $x^{-\rho}$. Simplifying, we get:

$$Y = \exp\left\{\lim_{\rho\to\infty} -\frac{\alpha\ln(\alpha)\theta_K^{-\rho} - \alpha\ln(A_K K)\theta_K^{-\rho} + (1-\alpha)\ln(1-\alpha)\theta_N^{-\rho} - (1-\alpha)\ln(A_N N)\theta_N^{-\rho}}{\alpha\theta_N^{-\rho} + (1-\alpha)\theta_N^{-\rho}}\right\}$$

Using the definitions of x, θ_K and θ_N we have $1 = \min\{\theta_K, \theta_N\}$. Therefore we must have either $\theta_K = 1 \Leftrightarrow x = \frac{A_K K}{\alpha}$ or $\theta_N = 1 \Leftrightarrow x = \frac{A_N N}{1-\alpha}$, since one of θ_K or θ_N has to be the smaller of the two. It also follows that $\theta_i \geq 1, \forall i = \{K, N\}$, given that one of θ_K or θ_N has to be the smaller of the two. We have that:

$$\lim_{\rho \to \infty} \theta_i^{-\rho} = \begin{cases} 1 & \text{if } \theta_i = 1 \\ 0 & \text{if } \theta_i > 1 \end{cases}$$

Applying this result to the limit of interest:

$$\begin{aligned} Y &= \begin{cases} \exp\left\{-\frac{\alpha \ln(\alpha) - \alpha \ln(A_K K)}{\alpha}\right\} & \text{if } \theta_K = 1 \\ \exp\left\{-\frac{(1-\alpha)\ln(1-\alpha) - (1-\alpha)\ln(A_N N)}{1-\alpha}\right\} & \text{if } \theta_N = 1 \end{cases} \\ &= \begin{cases} \frac{A_K K}{\alpha} & \text{if } x = \frac{A_K K}{\alpha} \\ \frac{A_N N}{\alpha} & \text{if } x = \frac{A_N N}{\alpha} \end{cases} \\ &= x = \min\left\{\frac{A_K K}{\alpha}, \frac{A_N N}{1-\alpha}\right\} \end{aligned}$$