

# Supplement to Lecture 3: Derivation of CES Special Cases

Macroeconomics, EC2B1

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In the lecture notes we defined the CES production function

$$Y = \left( \alpha^{\frac{1}{\sigma}} (A_K K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} (A_N N)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \quad \sigma = \text{elasticity of substitution}$$

and asserted that there are three special cases

1. Case  $\sigma = 1$ : Cobb-Douglas

$$Y = \left( \frac{A_K K}{\alpha} \right)^{\alpha} \left( \frac{A_N N}{1-\alpha} \right)^{1-\alpha}$$

2. Case  $\sigma \rightarrow \infty$ : perfect substitutes

$$Y = A_K K + A_N N$$

3. Case  $\sigma = 0$ : perfect complements, fixed proportions, or “Leontief”

$$Y = \min \left\{ \frac{A_K K}{\alpha}, \frac{A_N N}{1-\alpha} \right\}$$

This supplement to the lecture notes provides the corresponding derivations.

## 1 Case $\sigma = 1$ : Cobb-Douglas

In order to make computations easier, we can define  $\rho = \frac{1}{\sigma} - 1 \Leftrightarrow \sigma = \frac{1}{1+\rho}$ :

$$Y = \left( \alpha^{1+\rho} (A_K K)^{-\rho} + (1-\alpha)^{1+\rho} (A_N N)^{-\rho} \right)^{-\frac{1}{\rho}}$$

Therefore studying the case of  $\sigma = 1$  is the same as  $\rho = 0$ . Taking the limit as  $\rho \rightarrow 0$ :

$$Y = \exp \left\{ \lim_{\rho \rightarrow 0} -\frac{1}{\rho} \ln \left( \alpha^{1+\rho} (A_K K)^{-\rho} + (1-\alpha)^{1+\rho} (A_N N)^{-\rho} \right) \right\}$$

Taking the limit yields  $\frac{0}{0}$ . Therefore we use L'Hôpital's rule:

$$Y = \exp \left\{ \lim_{\rho \rightarrow 0} - \frac{\alpha^{1+\rho} \ln(\alpha)(A_K K)^{-\rho} - \alpha^{1+\rho}(A_K K)^{-\rho} \ln(A_K K) + (1-\alpha)^{1+\rho} \ln(1-\alpha)(A_N N)^{-\rho} - (1-\alpha)^{1+\rho}(A_N N)^{-\rho} \ln(A_N N)}{\alpha^{1+\rho}(A_K K)^{-\rho} + (1-\alpha)^{1+\rho}(A_N N)^{-\rho}} \right\}$$

Taking the limit and using a bit of algebra:

$$Y = \exp \left\{ - \frac{\alpha \ln(\alpha) - \alpha \ln(A_K K) + (1-\alpha) \ln(1-\alpha) - (1-\alpha) \ln(A_N N)}{\alpha + (1-\alpha)} \right\} = \left( \frac{A_K K}{\alpha} \right)^\alpha \left( \frac{A_N N}{1-\alpha} \right)^{1-\alpha}.$$

## 2 Case $\sigma \rightarrow \infty$ : perfect substitutes

Again, taking the limit:

$$Y = \lim_{\sigma \rightarrow \infty} \left( \alpha^{\frac{1}{\sigma}} (A_K K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} (A_N N)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

$$Y = \left( \lim_{\sigma \rightarrow \infty} \left[ \alpha^{\frac{1}{\sigma}} (A_K K)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} (A_N N)^{\frac{\sigma-1}{\sigma}} \right] \right)^{\lim_{\sigma \rightarrow \infty} \frac{\sigma}{\sigma-1}}$$

$$Y = A_K K + A_N N$$

## 3 Case $\sigma = 0$ : perfect complements, fixed proportions, or “Leontief”

Going back to the alternative representation of the CES as a function of  $\rho$  and taking the limit as  $\rho \rightarrow \infty$ :

$$Y = \exp \left\{ \lim_{\rho \rightarrow \infty} - \frac{1}{\rho} \ln \left( \alpha^{1+\rho} (A_K K)^{-\rho} + (1-\alpha)^{1+\rho} (A_N N)^{-\rho} \right) \right\}$$

This time taking the limit  $\rho \rightarrow \infty$  yields  $\frac{\infty}{\infty}$ . Therefore again using L'Hôpital's rule:

$$Y = \exp \left\{ \lim_{\rho \rightarrow \infty} - \frac{\alpha^{1+\rho} \ln(\alpha)(A_K K)^{-\rho} - \alpha^{1+\rho}(A_K K)^{-\rho} \ln(A_K K) + (1-\alpha)^{1+\rho} \ln(1-\alpha)(A_N N)^{-\rho} - (1-\alpha)^{1+\rho}(A_N N)^{-\rho} \ln(A_N N)}{\alpha^{1+\rho}(A_K K)^{-\rho} + (1-\alpha)^{1+\rho}(A_N N)^{-\rho}} \right\}$$

Let  $x = \min \left\{ \frac{A_K K}{\alpha}, \frac{A_N N}{1-\alpha} \right\}$ ,  $\theta_K = \frac{A_K K}{\alpha x}$  and  $\theta_N = \frac{A_N N}{(1-\alpha)x}$ . Take the previous expression and divide the numerator and denominator by  $x^{-\rho}$ . Simplifying, we get:

$$Y = \exp \left\{ \lim_{\rho \rightarrow \infty} - \frac{\alpha \ln(\alpha) \theta_K^{-\rho} - \alpha \ln(A_K K) \theta_K^{-\rho} + (1-\alpha) \ln(1-\alpha) \theta_N^{-\rho} - (1-\alpha) \ln(A_N N) \theta_N^{-\rho}}{\alpha \theta_K^{-\rho} + (1-\alpha) \theta_N^{-\rho}} \right\}$$

Using the definitions of  $x$ ,  $\theta_K$  and  $\theta_N$  we have  $1 = \min \{\theta_K, \theta_N\}$ . Therefore we must have either  $\theta_K = 1 \Leftrightarrow x = \frac{A_K K}{\alpha}$  or  $\theta_N = 1 \Leftrightarrow x = \frac{A_N N}{1-\alpha}$ , since one of  $\theta_K$  or  $\theta_N$  has to be the smaller of the

two. It also follows that  $\theta_i \geq 1, \forall i = \{K, N\}$ , given that one of  $\theta_K$  or  $\theta_N$  has to be the smaller of the two. We have that:

$$\lim_{\rho \rightarrow \infty} \theta_i^{-\rho} = \begin{cases} 1 & \text{if } \theta_i = 1 \\ 0 & \text{if } \theta_i > 1 \end{cases}$$

Applying this result to the limit of interest:

$$\begin{aligned} Y &= \begin{cases} \exp \left\{ -\frac{\alpha \ln(\alpha) - \alpha \ln(A_K K)}{\alpha} \right\} & \text{if } \theta_K = 1 \\ \exp \left\{ -\frac{(1-\alpha) \ln(1-\alpha) - (1-\alpha) \ln(A_N N)}{1-\alpha} \right\} & \text{if } \theta_N = 1 \end{cases} \\ &= \begin{cases} \frac{A_K K}{\alpha} & \text{if } x = \frac{A_K K}{\alpha} \\ \frac{A_N N}{\alpha} & \text{if } x = \frac{A_N N}{\alpha} \end{cases} \\ &= x = \min \left\{ \frac{A_K K}{\alpha}, \frac{A_N N}{1-\alpha} \right\} \end{aligned}$$