Lecture 5: Investment and Capital Accumulation

Macroeconomics EC2B1
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1 Investment and Capital Accumulation in Partial Equilibrium

1.1 The basic model

There are two time periods, $t = 1, 2$. Firms have production functions $Y_t = A_t F(K_t)$. Capital in period 1, K_1, is fixed. Capital in period 2 can be changed through investment in period 1, I_1, and equals

$$K_2 = I_1 + (1-d)K_1$$

where d is the depreciation rate. Firm profits per period are

$$\Pi_1 = A_1 F(K_1) - I_1, \quad \Pi_2 = A_2 F(K_2) + (1-d)K_2.$$

That is, profits in period 1 are sales minus investment costs; profits in period 2 are sales plus the revenues from selling any undepreciated capital (recall that the world is over after $t = 2$ so therefore a firm will always want to sell its entire capital stock before dying). Firms maximize the present discounted value (PDV) of profits which is given by

$$W = \Pi_1 + \frac{\Pi_2}{1+r_1} \tag{1}$$

where r_1 is the real interest rate between $t = 1$ and $t = 2$. We will explain further below why firms would want to do this.

The firms' problem can therefore be summarized as

$$W = \max_{K_2} \left\{ A_1 F(K_1) + (1-d)K_1 - K_2 + \frac{A_2 F(K_2) + (1-d)K_2}{1+r_1} \right\} \tag{2}$$

The first-order condition is

$$1 = \frac{A_2 F'(K_2) + 1 - d}{1+r_1}$$

$$A_2 F'(K_2) = r_1 + d$$

In the lecture, we illustrated this equation graphically. Under the assumption that the production function is concave, $F''(K) < 0$, capital in period 2 can be shown to be increasing in productivity, A_2, and decreasing in the interest rate, r_1, and depreciation, d. The same comparative statics hold for investment, I_1. This follows since investment is given by $I_1 = K_2 - (1-d)K_1$ and K_1 is fixed.

1.2 Why would firms maximize the present discounted value of profits (1)?

In short, the answer is: because (a) the firms are owned by households who can save and borrow at interest rate r_1 and (b) also the firms can save and borrow at this same interest rate. The combination of these two assumptions means that households instruct the firms they own to maximize the present value of profits discounted at rate r_1. Here are the equations.

Assume firms are owned by a representative household with preferences $U(C_1) + \beta U(C_2)$. Given that households own the firms, two things happen: (a) households can tell firms what to do and (b) the households receive some dividend payments from the firms as income in the two periods, which denote by D_1 and D_2. As in Lecture 4, we could also assume that households earn some labor income. However, for simplicity and with an eye toward later parts of these lecture notes, we assume that they do not, i.e. labor income is zero.[1] Households can borrow and save in an asset b at an interest rate r_1. The households' budget constraints are

$$C_1 + b = D_1, \qquad C_2 = D_2 + (1 + r_1)b. \tag{3}$$

As before, the key assumption is that households can borrow or save as much as they want at interest rate r_1. Under this assumption, we can write a present-value budget constraint

$$C_1 + \frac{C_2}{1 + r_1} = D_1 + \frac{D_2}{1 + r_1} \tag{4}$$

Given this, it is clear that households who want to maximize utility will tell firms to maximize the present value of dividends $D_1 + D_2/(1 + r_1)$.

Next consider firms. The dividends paid by firms are given by

$$D_1 = \underbrace{A_1 F(K_1) - I_1}_{\Pi_1} - B, \quad D_2 = \underbrace{A_2 F(K_2) + (1 - d)K_2}_{\Pi_2} + (1 + r_1)B.$$

[1] It is easy to extend the argument in this subsection to the case with labor income or earnings. To this end, denote such earnings by E_1 and E_2 (in contrast to Lecture 4, I am now using E_t rather than Y_t to denote earnings because we already used Y_t for output). In this case the period budget constraints would be $C_1 + b = E_1 + D_1$ and $C_2 = E_2 + (1 + r_1)b + D_2$ and the present value budget constraint would be $C_1 + \frac{C_2}{1+r_1} = E_1 + \frac{E_2}{1+r_1} + D_1 + \frac{D_2}{1+r_1}$. All the other results of this section would go through in just the same way.

where B is firm saving. (In closed economy models we will typically assume that $b + B = 0$, i.e. that any firm borrowing comes from households – see Section 2.4 below.)

Note that, just like households, firms can borrow or save as much as they want at interest rate r_1. With these assumptions we have that the PDV of dividends (the thing households care about) equals

$$D_1 + \frac{D_2}{1+r_1} = \Pi_1 - B + \frac{\Pi_2 + (1+r_1)B}{1+r_1} = \Pi_1 + \frac{\Pi_2}{1+r_1},$$

i.e. the PDV of dividends equals the PDV of profits as defined in (1). Given that households instruct firms to maximize the PDV of dividend payments, firms will maximize this PDV of profits. It is also worth noting that the household budget constraint (4) becomes

$$C_1 + \frac{C_2}{1+r_1} = W, \qquad W = \Pi_1 + \frac{\Pi_2}{1+r_1}.$$

2 Dynamic General Equilibrium with Capital Accumulation

Thus far, we have studied households' savings behavior and firms' capital accumulation decisions. These notes show how to put these together in general equilibrium.

To make some progress, it will turn out to be useful to work with special functional forms (namely, linear production functions and utility functions with a constant intertemporal elasticity of substitution – see below). However, it should be clear that the construction of an equilibrium for this economy follows more general steps that would also be valid with general utility and production function satisfying standard assumptions.

The formulation of a dynamic general equilibrium model presented here will also be useful later in the course when we will study the "New Keynesian model", more specifically a simple formulation due to Mankiw and Weinzierl (2011).[2] As you will see, this New Keynesian model will be the same model as the one studied here except that we will modify one assumption: prices will be sticky instead of flexible like here. The dynamic general equilibrium model outlined below will therefore be an important building block for future topics.

2.1 Firms

Firms have production functions $Y_t = A_t K_t$. Capital evolves as

$$K_2 = I_1 + (1-d)K_1$$

[2] Title "An Exploration of Optimal Stabilization Policy."

but $d = 1$. For simplicity, we assume full depreciation, $d = 1$ so that, $K_2 = I_1$. Firm profits are then $\Pi_1 = Y_1 - I_1, \Pi_2 = Y_2$. Firms maximize the present value of profits which is then given by

$$W = \max_{K_2} \left\{ A_1 K_1 - K_2 + \frac{A_2 K_2}{1 + r_1} \right\} \tag{5}$$

2.2 Households

The household has utility function

$$U(C_1) + \beta U(C_2) \tag{6}$$

It will be useful to specialize this utility to

$$U(C_t) = \frac{C_t^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} \tag{7}$$

where the parameter σ is called the "elasticity of intertemporal substitution (IES)". More on this momentarily. Households maximize utility subject to the budget constraint

$$C_1 + \frac{C_2}{1 + r_1} = W \tag{8}$$

where W is the present value of the firms (5) owned by households and r_1 is the real interest rate between periods one and two. The corresponding Euler equation is

$$\frac{U'(C_1)}{\beta U'(C_2)} = \frac{C_1^{-1/\sigma}}{\beta C_2^{-1/\sigma}} = 1 + r_1$$

or

$$\frac{C_2}{C_1} = [\beta(1 + r_1)]^{\sigma} \tag{9}$$

The intertemporal elasticity of substitution, σ, therefore governs the responsiveness of the growth rate of consumption to changes in the interest rate and discount factor. For instance, a low IES means that households dislike intertemporal substitution, i.e. want to smooth consumption which is reflected in C_2/C_1 not being very responsive to $\beta(1 + r_1)$. Log-utility is the special case with $\sigma = 1$.[3]

2.3 Equilibrium

The resource constraints of the economy are

$$C_1 + I_1 = Y_1, \qquad C_2 = Y_2 \tag{10}$$

Claim: The equilibrium of this economy takes the form:

$$\begin{aligned}
C_1 &= \frac{\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2}{1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2} A_1 K_1 \\
C_2 &= \frac{A_2}{1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2} A_1 K_1 \\
I_1 &= \frac{1}{1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2} A_1 K_1 \\
Y_1 &= A_1 K_1 \\
Y_2 &= \frac{A_2}{1+\left(\frac{1}{\beta A_2}\right)^{\sigma} A_2} A_1 K_1 \\
1 + r_1 &= A_2
\end{aligned} \tag{11}$$

Derivation: The Euler equation (9) together with the budget constraint (8) imply that consumption demands are

$$C_1 = \frac{\left(\frac{1}{\beta(1+r_1)}\right)^{\sigma}(1+r_1)}{1+\left(\frac{1}{\beta(1+r_1)}\right)^{\sigma}(1+r_1)} W, \quad C_2 = \frac{1+r_1}{1+\left(\frac{1}{\beta(1+r_1)}\right)^{\sigma}(1+r_1)} W \tag{12}$$

[3] This can be seen directly from (7) by taking the limit as $\sigma \to 1$. Since $U(C)$ evaluated at $\sigma = 1$ gives $0/0$ we need to use l'Hopital's rule. I also find it easier to take the equivalent limit as $1/\sigma \to 1$. First write

$$\frac{C^{1-1/\sigma}-1}{1-1/\sigma} = \frac{e^{(1-1/\sigma)\log C}-1}{1-1/\sigma}$$

Next take the limit:

$$\lim_{\sigma\to 1} \frac{C^{1-1/\sigma}-1}{1-1/\sigma} = \lim_{1/\sigma\to 1} \frac{\frac{\partial}{\partial(1/\sigma)}\left(e^{(1-1/\sigma)\log C}-1\right)}{\frac{\partial}{\partial(1/\sigma)}(1-1/\sigma)} = \lim_{1/\sigma\to 1} \frac{-\log C e^{(1-1/\sigma)\log C}}{-1} = \log C.$$

The first order condition from the firm's profits maximization problem (5) gives an expression for the real interest rate $1+r_1 = A_2$. Hence the value of a firm is $W = A_1 K_1$. Substituting into (12) gives the expressions for C_1 and C_2. The expression for $I_1 = K_2$ is found from $C_2 = Y_2$ and hence $K_2 = C_2/A_2$.

Note in particular that the solution takes the form of a constant saving rate $s(A_2)$ out of current output, Y_1, i.e. we can write

$$I_1 = s(A_2)Y_1, \quad C_1 = [1 - s(A_2)]Y_1, \quad s(A_2) = \frac{1}{1 + \left(\frac{1}{\beta A_2}\right)^{\sigma} A_2} \tag{13}$$

A Recession: Consider the effect of a drop in current productivity A_1 on variables on the economy. As can be seen from (11), clearly all of $(C_1, C_2, I_1, Y_1, Y_2)$ fall when A_1 falls.

Next, consider the effect of a drop in future productivity A_2 on the economy. As can be seen from (11) or equivalently (13), the effect on I_1 and C_1 is generally ambiguous and depends on whether the IES, σ, is smaller or larger than one. We have that

$$\sigma < 1 \quad \Rightarrow \quad s'(A_2) < 0 \quad \Rightarrow \frac{\partial I_1}{\partial A_2} < 0, \quad \frac{\partial C_1}{\partial A_2} > 0$$

So if people dislike substituting intertemporally (low σ), they decrease their consumption in period one, C_1, in response to a drop in future productivity. Another way of saying this is that there is an income and a substitution effect. The income effect is that people are now poorer so will consume less. The substitution effect is that capital is less productive tomorrow so people want to save less and hence consume more. With $\sigma < 1$ the income effect dominates. (also see the discussion in Mankiw-Weinzierl, p.10). We will take this case to be the canonical case. Note that output in the second period, Y_2, decreases in a recession regardless of the value of σ. To see this write Y_2 in (11) as

$$Y_2 = \frac{1}{\frac{1}{A_2} + \left(\frac{1}{\beta A_2}\right)^{\sigma}} A_1 K_1.$$

In response to a decrease in A_2, the denominator increases regardless of the value of σ and hence Y_2 decreases.

2.4 Comment on Market Clearing Conditions: Credit Market

As stated in the lecture slides, there really is a third market and correponding market clearing condition, namely a credit market in which households and firms borrow/lend from/to each other at interest rate r_1. This is closely related to the discussion in Section 1.2 explaining why firms maximize the PDV of profits. The reason that this market clearing condition does not make an appearance among the market clearing conditions in equation (10) above is that we can drop this market clearing condition due to Walras' Law. Some of you may nevertheless find it useful to see the corresponding equations spelled out. This is what we do here. If the discussion in Section 2 thus far made sense to you, you can skip this subsection.

To explain the credit market, adopt of the notation of Section 1.2: households have period budget constraints (3) where b is household saving and D_1 and D_2 are firm dividend payments as before. As before $b > 0$ denotes saving and $b < 0$ denotes borrowing.

Again analogous to Section 1.2, firm dividend payments are given by

$$D_1 = \underbrace{A_1 K_1 - K_2}_{\Pi_1} - B, \quad D_2 = \underbrace{A_2 K_2}_{\Pi_2} + (1 + r_1)B. \tag{14}$$

where B denotes firm saving with $B < 0$ denoting borrowing. As in Section 1.2 these equations imply that (a) firms maximize the PDV of profits (5) and (b) the household budget constraint becomes (8).

With this notation, the credit market clearing condition is

$$b + B = 0, \tag{15}$$

i.e. since the economy is a closed system, the total amount of borrowing and lending has to add up to zero (same logic as in Lecture 4). This market clearing condition should really be counted as a third market clearing condition, in addition to the two goods market clearing conditions (10).

However, as expected from Walras' Law, the budget constraints together with the market clearing conditions (10) imply the credit market clearing condition (15) and hence we can drop the latter from the analysis. Here is the derivation: adding the household budget constraint for $t = 1$ from (3), $C_1 + b = D_1$ and the equation for dividends for $t = 1$ from (14), $D_1 = Y_1 - I_1 - B$ we have

$$C_1 + b = Y_1 - I_1 - B$$

From (10), the market clearing condition at $t = 1$ is $C_1 + I_1 = Y_1$ and so we immediately obtain the credit market clearing condition (15), i.e. you have just rediscovered Walras' Law.

The typical situation in an equilibrium will be $b > 0$ and $B = -b < 0$, i.e. firms borrow from households to finance their investment.

2.5 Oil Shocks as Productivity Shocks

One natural question is: what on earth is a negative productivity shock? I here argue that this idea should not be taken too literally; and that probably the best example of a negative productivity shock is an oil shock as during the 1973 and 1979 oil crises. I here elaborate on this point.

Consider the following simple extension of the model above. Instead of just capital, firms also use oil, O_t, in production

$$\tilde{Y}_t = \tilde{A}_t K_t^\alpha O_t^{1-\alpha}.$$

Within each period, firms maximize output net of oil expenditure

$$Y_t = \max_{O_t} \ \tilde{A}_t K_t^\alpha O_t^{1-\alpha} - p_t O_t \tag{16}$$

where p_t is the price of oil. In what follows we will show that there is a one-to-one mapping between a negative shock to productivity, $\tilde{A}_t$, and an oil shock, that is a positive shock to p_t.

The first-order condition is

$$(1-\alpha)\tilde{A}_t K_t^\alpha O_t^{-\alpha} = p_t$$

Therefore

$$O_t = \left(\frac{(1-\alpha)\tilde{A}_t}{p_t}\right)^{1/\alpha} K_t$$

Plugging back into (16)

$$\begin{aligned} Y_t &= \tilde{A}_t K_t^\alpha \left(\frac{(1-\alpha)\tilde{A}_t}{p_t}\right)^{(1-\alpha)/\alpha} K_t^{1-\alpha} - p_t \left(\frac{(1-\alpha)\tilde{A}_t}{p_t}\right)^{1/\alpha} K_t \\ &= \tilde{A}_t^{1/\alpha} K_t \left[\left(\frac{1-\alpha}{p_t}\right)^{(1-\alpha)/\alpha} - (1-\alpha)\left(\frac{1-\alpha}{p_t}\right)^{(1-\alpha)/\alpha}\right] \\ &= \alpha \tilde{A}_t^{1/\alpha} \left(\frac{1-\alpha}{p_t}\right)^{(1-\alpha)/\alpha} K_t \end{aligned} \tag{17}$$

We can therefore write output net of oil expenditures as

$$Y_t = A_t K_t, \quad \text{where} \quad A_t \equiv \alpha \tilde{A}_t^{1/\alpha} \left(\frac{1-\alpha}{p_t}\right)^{(1-\alpha)/\alpha}$$

is "effective productivity". It can be seen from this expression that an increase in the price of oil, p_t, looks exactly like a decrease in productivity $\tilde{A}_t$: both lead to a decrease in "effective productivity", A_t. The model in sections 1 to 3 with production technology, $Y_t = A_t K_t$ can then simply be viewed as a "reduced form" of the more elaborate model with oil shocks presented here. Therefore, an oil shock is a very good example of a negative aggregate productivity shock.