Handling Non-Convexities:
Entrepreneurship and Financial Frictions

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1 Model Setup

This note describes a model of entrepreneurship and financial frictions, similar to that in Buera and Shin (2013) and Cagetti and De Nardi (2006). In particular, occupational choice in combination with financial friction introduces a non-convexity in individuals optimization problems. We also introduce a second additional non-convexity, namely a choice between operating two technologies. Achdou et al. (2014) consider a version with aggregate shocks and examine its business cycle implications s(here we focus on the model with idiosyncratic shocks only).

- Individuals' preferences
$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$
- Workers earn labor income wz^θ where $\theta \geq 0$
- Entrepreneurs choose technology, maximize profits
- Two technologies productive (p) and unproductive (u)

$$y_u = F_u(z, k, \ell) = zB_u k^\alpha \ell^\beta$$
$$y_p = F_p(z, k, \ell) = zB_p((k - f_k)^+)^\alpha((\ell - f_\ell)^+)^\beta$$

  - $B_p > B_u$, but per-period **overhead cost** f_k, f_ℓ
  - Notation: for any scalar x, $x^+ = \max\{x, 0\}$
  - F_p non-concave in k and ℓ
  - z: idiosyncratic shock, diffusion process

- Collateral constraints
$$k \leq \lambda a, \quad \lambda \geq 1.$$

- Income maximization: occupation and technology choice

$$M(a,z,A;w,r) = \max\{wz^\theta, \Pi_u(a,z,A;w,r), \Pi_p(a,z,A;w,r)\}$$

$$\Pi_j(a,z,A;w,r) = \max_{k\le\lambda a}\ F_j(z,A,k,\ell) - (r+\delta)k - w\ell, \quad j = p,u.$$

- Individuals solve

$$\begin{aligned}
\max_{\{c_t\}}\ & \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.} \\
& da_t = [M(a_t,z_t,A_t;w_t,r_t) + r_t a_t - c_t]dt \\
& dz_t = \mu(z_t)dt + \sigma(z_t)dW_t \\
& a_t \ge 0
\end{aligned}$$

- Optimal capital and labor choices corresponding to the productive technology

$$\begin{aligned}
k_p(a,z;w,r) &= \min\left\{\lambda a, (zAB_p)^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{r+\delta}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{\beta}{w}\right)^{\frac{\beta}{1-\alpha-\beta}} + f_k\right\} \\
\ell_p(a,z;w,r) &= \left(\frac{\beta z A B_p}{w}\right)^{\frac{1}{1-\beta}} k_p(a,z;w,r)^{\frac{\alpha}{1-\beta}} + f_\ell
\end{aligned}$$

  and a similar expression for optimal capital and labor choices corresponding to the unproductive technology.

- Representative firm

$$Y_c = F_c(A, K_c, L_c) = AB_c K_c^\eta L_c^{1-\eta},$$

2 Equilibrium Conditions

Individual optimization and evolution of distribution

$$\begin{aligned}
\rho v(a,z,t) = &\max_c\ u(c) + \partial_a v(a,z,t)[M(a,z;w(t),r(t)) + r(t)a - c] \\
&+ \partial_z v(a,z,t)\mu(z) + \frac{1}{2}\partial_{zz} v(a,z,t)\sigma^2(z) + \partial_t v(a,z,t)
\end{aligned} \tag{1}$$

$$\partial_t g(a,z,t) = -\partial_a[s(a,z,t)g(a,z,t)] - \partial_z[\mu(z)g(a,z,t)] + \frac{1}{2}\partial_{zz}[\sigma^2(z)g(a,z,t)], \tag{2}$$

$$s(a,z,t) = M(a,z;w(t),r(t)) + r(t)a - c(a,z,t) \tag{3}$$

Public firms

$$r(t) = \partial_K F_c(A, K_c(t), L_c(t)) - \delta, \quad w(t) = \partial_L F_c(A, K_c(t), L_c(t)) \tag{4}$$

Capital market clearing:

$$\begin{aligned} K_c(t) + & \int k_u(a,z;w(t),r(t)) \mathbf{1}_{\{\Pi_u > \max\{\Pi_p, wz^\theta\}\}} g(a,z,t) dadz \\ + & \int k_p(a,z;w(t),r(t)) \mathbf{1}_{\{\Pi_p > \max\{\Pi_u, wz^\theta\}\}} g(a,z,t) dadz \\ = & \int a g(a,z,t) dadz \end{aligned} \tag{5}$$

Labor market clearing:

$$\begin{aligned} L_c(t) + & \int \ell_u(a,z;w(t),r(t)) \mathbf{1}_{\{\Pi_u > \max\{\Pi_p, wz^\theta\}\}} g(a,z,t) dadz \\ + & \int \ell_p(a,z;w(t),r(t)) \mathbf{1}_{\{\Pi_p > \max\{\Pi_u, wz^\theta\}\}} g(a,z,t) dadz \\ = & \int z^\theta \mathbf{1}_{\{wz^\theta > \max\{\Pi_u, \Pi_p\}\}} g(a,z,t) dadz \end{aligned} \tag{6}$$

Given initial condition $g_0(a,z)$, the two PDEs (1), (2) together with (3) and the equilibrium conditions (4), (5) and (6) fully characterize equilibrium.

3 Numerical Solution

The algorithm for solving the HJB equation (1) and the Kolmogorov Forward equation (2) are nearly identical to that used for solving the Huggett model with a diffusion process described in section 5 here `http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf`. The algorithm is then a simple bisection algorithm on the equilibrium interest rate.

3.1 Algorithm for Steady State

Use a bisection algorithm for $r_{\min} \leq r \leq r_{\max}$. Given an initial guesses r_0 for $\ell = 0, 1, 2, ...$ follow

1. given r_ℓ, find $\xi_\ell = K_c/L_c$ from (4)
2. given ξ_ℓ find w from (4)

3. given r_ℓ and w_ℓ, solve the HJB equation
4. find $K_{c,\ell}$ from (5)
5. Compute $L_{c,\ell} = K_{c,\ell}/\xi_\ell$ and compute "excess labor demand"

$$\begin{aligned} D_\ell =& L_c(t) + \int \ell_u(a,z;w(t),r(t)) \mathbf{1}_{\{\Pi_u > \max\{\Pi_p, wz^\theta\}\}} g(a,z,t) dadz \\ &+ \int \ell_p(a,z;w(t),r(t)) \mathbf{1}_{\{\Pi_p > \max\{\Pi_u, wz^\theta\}\}} g(a,z,t) dadz - \int z^\theta \mathbf{1}_{\{wz^\theta > \max\{\Pi_u, \Pi_p\}\}} g(a,z,t) dadz \end{aligned}$$

6. Update r_ℓ: if $D_\ell > 0$, choose $r_{\ell+1} < r_\ell$ and vice versa.

4 Results

Figure 1 plots the saving and consumption policy functions. The policy functions can be non-monotonic. Figure 2 plots the wealth distribution. The wealth distribution has a fat right tail as in Cagetti and De Nardi (2006).

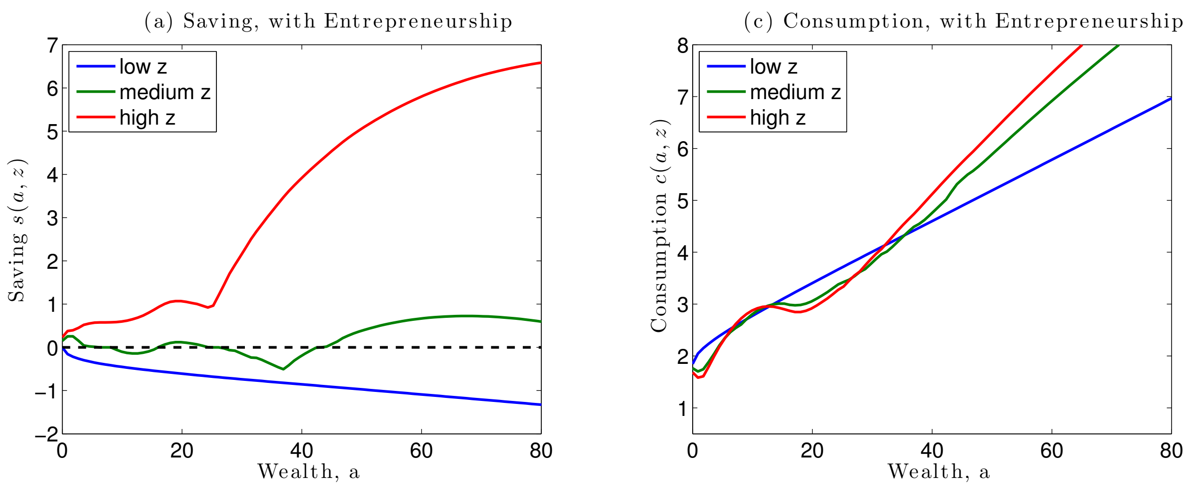

Figure 1: Saving and Consumption Policy Functions

References

Achdou, Yves, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. 2014. "Wealth Distribution and the Business Cycle: The Rold of Private Firms." Princeton University Working Papers.

Buera, Francisco J., and Yongseok Shin. 2013. "Financial Frictions and the Persistence of History: A Quantitative Exploration." *Journal of Political Economy*, Forthcoming.

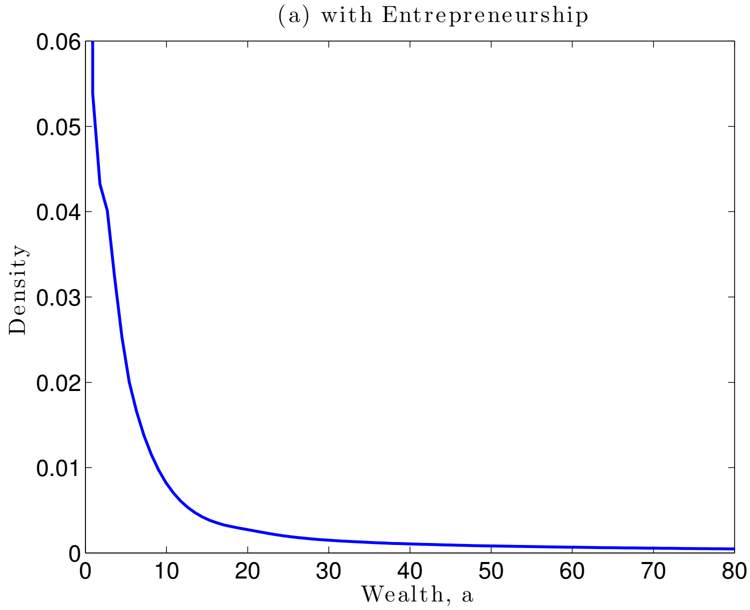

Figure 2: Wealth Distribution

Cagetti, Marco, and Mariacristina De Nardi. 2006. "Entrepreneurship, Frictions, and Wealth." *Journal of Political Economy*, 114(5): 835–870.