Creditor Rights, Inequality and Development in a Neoclassical Growth Model*

Benjamin Moll

University of Chicago

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Abstract

I study the implications of the limited enforceability of credit contracts for inequality and economic growth. I introduce limited enforcement into a deterministic neoclassical growth model. Two types of entrepreneurs differ in their initial wealth, ability and patience and each operate a private firm. The entrepreneurs can borrow and lend to each other but face enforcement constraints. This results in capital being misallocated across entrepreneurs. Three main conclusions are obtained from this model. First, capital misallocation disappears in the long run when entrepreneurs are equally patient. In contrast, when entrepreneurs' discount rates differ, capital misallocation persists asymptotically. Second, poor creditor rights magnify the effect of heterogeneity in ability on long run wealth inequality, because wealth accumulation functions as a substitute for poor creditor rights. Third, the interest rate is generally *lower* than in an economy without enforcement constraints, which has implications for the ability of the neoclassical growth model to explain sustained growth.

Introduction

The main purpose of this paper is to understand the implications of the limited enforceability of credit contracts for inequality and economic growth. Limited enforceability of contracts in general, and of credit contracts in particular, seems to be an important component of modern economies. Legal creditor rights, that is how easy it is to enforce a credit contract in court, differ

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substantially across countries and are important determinants of private credit (e.g. La Porta et al., 1997, 1998; Djankov, McLiesh and Shleifer, 2007). Poor creditor rights in turn imply that capital markets don't function smoothly which potentially results in the misallocation of capital.[1] A growing body of work has been interested in the effects of misallocation on aggregate output and Total Factor Productivity (e.g. Banerjee and Duflo, 2005; Restuccia and Rogerson, 2008; Bartelsman, Haltiwanger and Scarpetta, 2008; Hsieh and Klenow, 2009). However, as Banerjee and Moll (2009) argue one important question, that I will also try to address here, is why misallocation is in fact persistent.

Most work in this area concentrates on the effect that credit constraints and misallocation have on aggregate variables such as TFP and GDP, that is on inequality across countries. In my view, another interesting question is: What are the implications of credit constraints for inequality within a given country? Within-country inequality differs immensely across countries and there is (at least anecdotal) evidence that inequality is particularly high in developing countries. I want to argue in this paper that poor creditor rights are a potential reason for high within-country wealth inequality. As I will explain below, poor creditor rights can magnify the effect of heterogeneity in ability on long run wealth inequality, because wealth accumulation functions as a substitute for poor creditor rights.[2] Figure 1 presents some suggestive evidence that such a negative correlation between the quality of creditor rights and wealth inequality is in fact present in the data. To summarize, I ask the following question: How do poor creditor rights affect inequality and development?

To tackle this question, I introduce limited enforcement into a deterministic neoclassical growth model. Two types of entrepreneurs differ only in their initial wealth and each operate a private firm. The entrepreneurs can borrow and lend to each other but face enforcement constraints. As such the model is a reinterpretation of Kehoe and Perri (2002).[3] Limited enforcement is an endogenous credit market imperfection which results in capital misallocation. Because credit markets are imperfect, the representative agent framework is invalid. As such this paper presents an example of what Banerjee and Duflo (2005) term "non-aggregate growth theory". At the same time it incorporates elements from contract theory. Both contract theory and non-aggregate growth theory usually rely heavily on numerical simulations. The

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[1]See Banerjee and Duflo (2005) and Banerjee and Moll (2009) and the references cited therein for evidence on capital misallocation.

[2]A similar argument is put forth by Quadrini (2009), Cagetti and De Nardi (2006) and others. They argue that costly external finance can explain the high concentration of wealth among entrepreneurial households relative to non-entrepreneurial ones. My argument is different in that it considers wealth inequality *within* entrepreneurial households.

[3]Kehoe and Perri study the implications of limited enforcement for international business cycles in a two-country model. Their two countries correspond to my two types of entrepreneurs.
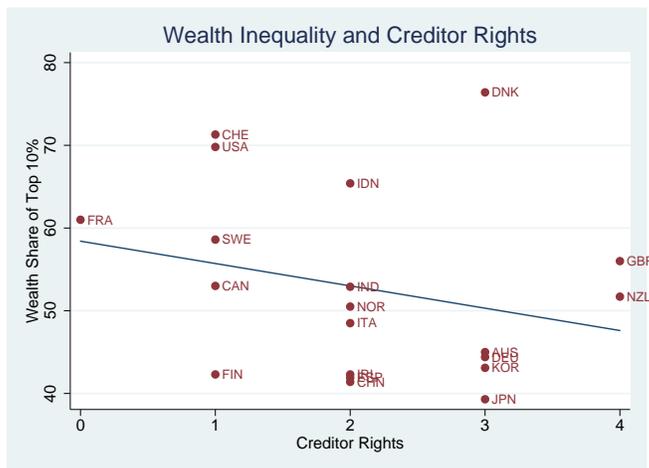
Figure 1: Wealth Inequality and Creditor Rights

Note: Data suggesting a possible negative relationship between the quality of creditor rights and the degree of wealth inequality. The data sources used for this regression are creditor rights data from Djankov, McLiesh and Shleifer (2007) and wealth inequality data from UN (2008).

main methodological contribution of this paper is then to provide some analytic results in a model that combines elements from both fields. These analytic results aid in understanding the economic mechanisms at work. I will briefly comment on the setup of the model and draw some connections to related theoretical literature. As already mentioned, the setup is very similar to Kehoe and Perri (2002) albeit with a different interpretation. In contrast, to their paper however, my model is deterministic. This delivers a majority of the analytic tractability. Ray (2002) and Acemoglu, Golosov and Tsyvinski (2008) are two other papers that feature deterministic setup with a very similar mathematical structure. They also stress the importance of discount rates. In contrast to each of them, my model is set up in continuous time. This delivers additional tractability.

Solving the model in continuous time discloses a simple connection between two branches of the literature on optimal contracts: The literature using the Lagrangian method developed by Marcet and Marimon (1999) and the promised value literature started by Spear and Srivastava (1987). Time-varying Pareto weights of Marcet and Marimon (1999) are simply the costates corresponding to promised value as a state.[4] The continous time setup also implies that the equilibrium of the economy is characterized by a system of ordinary differential equations. While the dimension of this system of ODEs is greater than two and cannot be analyzed with a standard phase diagram, the system is very similar in structure to the standard neoclassical growth model. In particular, these ODEs have a steady state around which they can be linearized. I prove that the linearized system of differential equations is saddle-path stable,

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[4]Marcet and Marimon (1999) already observed that a time-varying Pareto weight "acts as a costate variables". This statement is made more precise here.

using a result from a field of linear algebra known as "Inertia Theory". The results I use are from Datta (1999). For another example of their application to economics see Moll (2008).

Other related literature. In addition to the empirical papers already cited above, my paper is related to a broad theoretical literature on the macro-implications of credit market imperfections. Early examples are Banerjee and Newman (1993), Aghion and Bolton (1997), and Piketty (1997). Often these papers predict interesting phenomena such as persistent inequality and poverty traps. However, one main shortcoming of these analyses is that they are based on strong assumptions. Generations are assumed to live for a single time period and the evolution of wealth is determined by a warm-glow bequest motive that is not forward-looking. In contrast to such Solow-type overlapping-generations models of economic growth, modern macroeconomics typically assumes that agents are forward-looking and accumulate assets optimally as in the neoclassical growth model. Also, in the above papers usually either the interest rate and/or the identity of borrowers and lenders is exogenous. More recent contributions (Townsend and Ueda, 2006; Buera, Kaboski and Shin, 2009; Banerjee and Moll, 2009) address these problems. As already mentioned, one main contribution of this paper is to emphasize the forward-looking nature of both savings decisions and credit constraints while keeping the model relatively tractable.

The paper is organized in five sections. Section 1 presents the setup as a competitive equilibrium. This setup is then recast as a planning problem, first without enforcement constraints (unconstrained planning problem in section 2), then with enforcement constraints (constrained planning problem in section 3). Section provides a discussion of the model's main predictions. Section 5 concludes.

1   Setup

1.1   Preferences and Technology

Time is continuous. There are two types of entrepreneurs indexed by $i = 1, 2$, and for both types there is a continuum of equal mass.[5] Preferences for both types of entrepreneurs are given by

$$\int_0^\infty e^{-\rho_i t} u(c_i(t)) dt, \quad i = 1, 2. \tag{1}$$

Note that entrepreneurs potentially differ in their discount rates ρ_i. I assume that the function u is strictly increasing and strictly concave and satisfies standard Inada conditions. Each entrepreneur owns a private firm which uses k_i units of capital to produce $z_i f(k_i)$ units of

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[5]The mass of each type is normalized to one. Different masses for the two types of entrepreneurs could easily be incorporated but would not change any of the main results.

output. The intercept z_i is interpreted as entrepreneurial ability. I assume that the function f is strictly increasing and strictly concave and satisfies standard Inada conditions. Capital depreciates at the rate δ.

1.2 Market Structure and Equilibrium

Denote by $a_i(t)$ the wealth of a type i entrepreneur and by $r(t)$ the (endogenous) interest rate. Entrepreneurs can rent capital $k(t)$ in a rental market at a rental rate $R(t) = r(t) + \delta$. Then their wealth evolves according to

$$\dot{a}_i = z_i f(k_i) - (r + \delta)k_i + ra_i - c_i, \quad a_i(0) = a_{i0}. \tag{2}$$

Savings $\dot{a}_i$ equal profits – output minus rental costs – plus interest income minus consumption. The setup with a rental market is chosen solely for simplicity. Moll (2009) shows that it is equivalent to a setup in which entrepreneurs own and accumulate capital k_i and can trade in a risk-free bond.

In addition, at each point in time, entrepreneurs face borrowing or enforcement constraints. In particular, I assume that entrepreneurs who borrow capital can default and run away with a fraction $\phi \in [0, 1]$ of the capital they are using. There is some outside value $v_i(\phi k_i), i = 1, 2$ associated with defaulting.[6] I will consider different alternative default values below. For now, I only impose $u(0)/\rho_i \geq v_i(0), i = 1, 2$. The enforcement constraints are therefore:

$$\int_t^\infty e^{-\rho_i(\tau-t)} u(c_i(\tau)) d\tau \geq v_i(\phi k_i(t)), \quad i = 1, 2, \ t \geq 0. \tag{3}$$

Note that the standard growth model can be nested by letting $\phi = 0$ because then the constraints (3) never bind. It is also worth noting that enforcement constraints act as *borrowing constraints*. This is because (3) can always be satisfied by choosing k_i on the right hand side small enough.

The structure of the model here is almost precisely the same as outlined in section 6.1 of Banerjee and Duflo (2005). The only difference is that borrowing constraints are endogenous rather than exogenous and that I restrict the types of entrepreneurs to two.

Definition 1 *An* equilibrium with enforcement constraints *consists of a time path for the interest rate $r(t), t \geq 0$ and allocations $(a_i(t), k_i(t), c_i(t)), i = 1, 2, t \geq 0$ such that*

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[6]Note that the outside option is indexed by an entrepreneur's type i. I make this assumption not only because it is more general, but also because it is more natural: If entrepreneurs differ in their productivity z_i then one would also expect their outside options to differ; presumably, more productive entrepreneurs can put the stolen capital to better use and therefore have a higher outside option.

(i) Given the interest rate $r(t)$ and outside options $v_i(\phi k_i(t))$, for all a_{i0}, the allocations $(a_i(t), k_i(t), c_i(t))$ maximize (1) subject to (2) and (3).

(ii) the capital market clears for all t:

$$a_1(t) + a_2(t) = k_1(t) + k_2(t). \tag{4}$$

In such an equilibrium, any interesting role for capital markets must be due to some form of heterogeneity between the two types of entrepreneurs; if the two types of entrepreneurs were identical in every respect there would be no reason for them to trade in the capital market (or any market for that matter). The reader should therefore recall that there are *four* potential sources of heterogeneity between the two types of entrepreneurs: First, their initial wealth a_{i0}; second, their ability z_i, third, their discount rates ρ_i; and fourth, their outside options $v_i(\phi k_i)$.

This completes the entirely standard description of the economy. My ultimate goal is to analyze the dynamics of the capital stocks k_1 and k_2 under limited enforcement. It is, however, instructive to first have a closer look at a competitive equilibrium without any such imperfections, that is the case where $\phi = 0$. As is standard, such an equilibrium can be solved as a planning problem. This prelude to the actual problem of interest is contained in the next section, allowing me to both establish some useful benchmarks and introduce some notation in an entirely standard setting.

2   First-Best: Unconstrained Planning Problem

Consider the case $\phi = 0$ so that (3) never binds. The easiest way to characterize the competitive equilibrium is to solve a *planning problem*. This planning problem can be stated in a variety of ways. Anticipating the setup with an enforcement problem, the following one is most convenient:

$$V^u(w_0, k_0) = \max_{c_1, c_2, k_1, k_2} \int_0^\infty e^{-\rho_2 t} u(c_2) dt \quad \text{s.t.} \tag{U}$$

$$\int_0^\infty e^{-\rho_1 t} u(c_1) dt \geq w_0 \tag{5}$$

$$\dot{k} = z_1 f(k_1) + z_2 f(k_2) - \delta k - c_1 - c_2 \tag{6}$$

$$k_1 + k_2 \leq k, \quad k(0) = k_0. \tag{7}$$

Intuitively, the planner maximizes the present discounted value of utility of one type of agents – here type 2 without loss of generality – while delivering a value of at least w_0 to the other type –

6

here type 1. In so doing, the planner takes into account the constraints implied by the physical environment, that is the law of motion for aggregate capital k and the constraint that the sum of individual capital usage cannot exceed aggregate capital. In going from the competitive equilibrium to the planning problem described here, an open question is how the initial value of type 1 entrepreneurs is determined. For now, it suffices to note that w_0 is determined by the three sources of potential heterogeneity between the two types: their initial wealth a_{i0}, their ability z_i and their discount rates $\rho_i, i = 1, 2$.

The purpose of this section is to illustrate the dynamics of capital and consumption that are the solution to this unconstrained planning problem. Because this is just a prelude to the actual problem of interest, I derive these in a quick but relatively informal way. The unconstrained planning problem (U) is the special case with $\phi = 0$ of the constrained planning problem (C) in section 3 so that a formal derivation is contained there. Denote the Lagrange multiplier on the constraint (5) as α_0. Then the first terms of the Lagrangean for (U) are

$$\mathcal{L} = \int_0^\infty e^{-\rho_2 t} u(c_2(t)) dt + \alpha_0 \left[\int_0^\infty e^{-\rho_1 t} u(c_1(t)) dt - w_0 \right] + ... \tag{8}$$

The multiplier α_0 is therefore also a *Pareto weight*. To put this in a slightly different way, for any k_0 the function $V^u(\cdot, k_0)$ defines the Pareto frontier between types 1 and 2. By the envelope theorem, we have that $V_w^u(w_0, k_0) = -\alpha_0$ so that the Pareto weight α_0 is also the slope of the Pareto frontier. To proceed, further rewrite (8) as

$$\mathcal{L} = \int_0^\infty e^{-\rho_2 t} \left[\alpha(t) u(c_1(t)) + u(c_2(t)) \right] dt + ..., \quad \text{where} \quad \alpha(t) \equiv \alpha_0 e^{-(\rho_1 - \rho_2)t}.$$

The objective function is now the present discounted value of $\alpha(t) u(c_1(t)) + u(c_2(t))$ discounted by a common discount factor ρ_2. This is also the period payoff of a social planner who places a *time-varying Pareto weight* $\alpha(t)$ on type 1 agents. Note that if type 2 entrepreneurs are more patient $\rho_1 > \rho_2$, then the Pareto weight goes to zero asymptotically, so that the more patient type 2 entrepreneurs get all the surplus in the long run. Given this discussion, the following claim should not be surprising:

Claim 1 *The problem* (U) *can be solved in the following two stages. First, solve the two sharing rules:*[7]

$$U(c, \alpha) = \max_{c_1, c_2} \{ \alpha u(c_1) + u(c_2) \quad s.t. \quad c_1 + c_2 \leq c \} \tag{9}$$

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[7]For example with CRRA utility with parameter σ, we have $U(c, \alpha) = \left[\alpha^{1/\sigma} + 1 \right]^\sigma u(c)$.

$$F(k) = \max_{k_1, k_2} \{ z_1 f(k_1) + z_2 f(k_2) \quad s.t. \quad k_1 + k_2 \le k \} \tag{10}$$

Second, solve the system of three ordinary differential equations

$$\dot{\lambda} = \lambda[\rho_2 + \delta - F'(k)]$$
$$\dot{\alpha} = -[\rho_1 - \rho_2]\alpha \tag{11}$$
$$\dot{k} = F(k) - \delta k - U_c^{-1}(\lambda, \alpha)$$

where $U_c^{-1}(\cdot, \lambda)$ is the inverse of the marginal utility of consumption with respect to its first argument.[8]

This Claim will be verified formally in section 3 (see especially the system of ODEs (26)). Consider first the case where both types are equally patient, $\rho_1 = \rho_2$. In this case $\dot{\alpha} = 0$ or $\alpha(t) = \alpha_0$ for all t. The corresponding equation in (11) can then be dropped and we're left with a system of two differential equations in λ and k. These are precisely the same differential equations that characterize the solution to a standard neoclassical growth model; there is a λ is a co-state variable [9] In the case $\rho_1 \ne \rho_2$, the behavior of the system if identical with the difference that there is an additional equation to keep track of the Pareto weight α. As already noted the more patient type will generally end up with all the surplus in the long run. Note that – regardless of the evolution of the Pareto weight – as part of the solution of (10), marginal products of capital of both types are equalized at all points in time

$$z_1 f'(k_1) = z_2 f'(k_2). \tag{12}$$

That is, capital is allocated optimally across entrepreneurs. This will not necessarily be true anymore in the presence of enforcement constraints, $\phi > 0$. Anticipating the analysis of that case below, define by $k_i^u(k), i = 1, 2$ the amount of capital held by type i in the unconstrained planning problem if the aggregate capital stock is k, that is individual capital stocks satisfying (12). This unconstrained capital allocation will there be used to check whether enforcement constraints bind.

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[8] Any solution to (11) must also satisfy the boundary conditions $k(0) = k_0$, $\alpha(0) = \alpha_0$, $\lim_{t \to \infty} e^{-\rho_2 t} \lambda(t) k(t) = 0$. I impose an initial condition on the Pareto weight α_0 instead of type 1's value w_0; the two are equivalent. The last condition is a standard transversality condition.

[9] For the case $\rho_1 = \rho_2$, the system of ODEs (11) can be derived from the present value Hamiltonian $\mathcal{H}(k, \lambda) = U(c, \alpha) + \lambda[F(k) - \delta k - c]$. This derivation still works for the case $\rho_1 \ne \rho_2$ but is not entirely correct because the Pareto weight α is changing over time.

3   Limited Enforcement: Constrained Planning Problem

Now, consider the case $\phi > 0$ so that defaulting entrepreneurs get to keep a positive fraction of their operating capital stock. We can still solve a planning problem to describe the competitive equilibrium of this economy. In particular, one simply adds the constraints (3) to the constraint set of the planning problem (U).

The *constrained planning problem* is:

$$V(w_0, k_0) = \max_{c_1, c_2, k_1, k_2} \int_0^\infty e^{-\rho_2 t} u(c_2(t)) dt \quad \text{s.t.} \quad \int_0^\infty e^{-\rho_1 t} u(c_1(t)) dt \geq w_0 \tag{C}$$

$$\dot{k} = z_1 f(k_1) + z_2 f(k_2) - \delta k - c_1 - c_2, \quad k_1 + k_2 \leq k, \quad k(0) = k_0 \tag{13}$$

$$\int_t^\infty e^{-\rho_i(\tau - t)} u(c_i(\tau)) d\tau \geq v_i(\phi k_i(t)), \quad i = 1, 2, \ t \geq 0. \tag{14}$$

The planner simply solves the same problem as the unconstrained planning problem (U) but taking into account the enforcement constraints (3). To make the problem interesting, I impose the following restrictions on the default values $v_i(\phi k_i)$.

Assumption 1 (Feasibility) *Default values $v_i(\cdot), i = 1, 2$ are such that $V^u[v_1(\phi k_1^u(k)), k] \geq v_2(\phi k_2^u(k))$ for all k.*

This assumption guarantees that for every value of the aggregate capital stock k, there is some value for type 1's continuation value such that both enforcement constraints (3) are slack. This is illustrated in Figure 2: there is some region on the (unconstrained) Pareto frontier $V^u(w, k)$ such that both constraints are slack. Consider for example the symmetric case $\rho_1 = \rho_2$, $z_1 = z_2$, $v_1(\cdot) = v_2(\cdot)$. Then it is easy to see that Assumption 1 is satisfied if the consequence of default is being cast into autarky forever.

Problem C is in general a non-convex optimization problem implying first-order conditions are only necessary but not sufficient. This is because the functions $v_i(\phi k_i)$ are concave so that the constraint (14) are not convex. The following assumption is sufficient to make the constraint set of (C) convex, as show in the corresponding Lemma.

Assumption 2 (Convexity) $\frac{z_i f'(k)}{v_i'(\phi k)\phi}$ *is decreasing in k for all k and for $i = 1, 2$.*

Lemma 1 *Under assumption 2, the constraint set of the constrained planning problem (C) is convex.*

(All proofs are in the Appendix.) Assumption 2 is for example satisfied if outside options are linear in capital, $v_i(k_i) = k_i$, or more generally if the outside options are "not too concave"
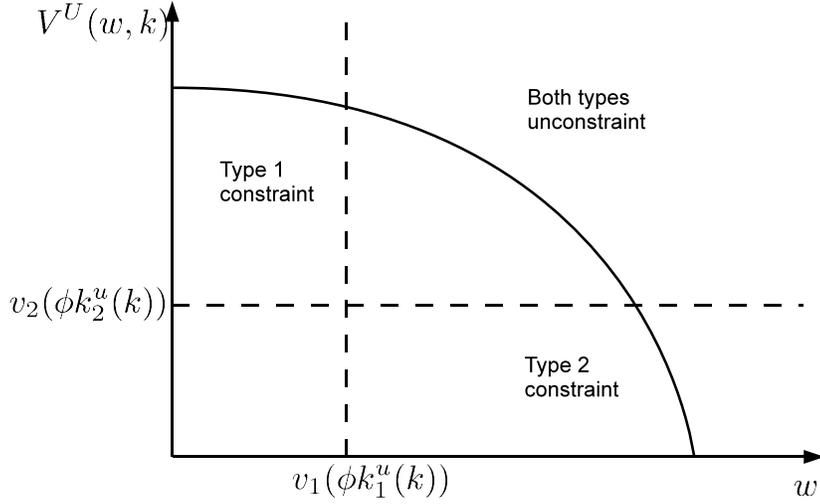
Figure 2

Note:

and ϕ is small. This assumption will be imposed in some of the results below, but not all. As just noted, it guarantees that first-order conditions are both necessary and sufficient. This is useful in some of the results, but not required in others. There is, however, another useful consequence of assumption 2:

Lemma 2 *Under assumption* 2, *the Pareto frontier* $V(w_0, k_0)$ *is concave in* w_0 *for all* k_0.

This Lemma follows almost immediately from Lemma 1. It captures the standard intuition that the set of attainable utilities is convex so that the Pareto frontier is concave.

3.1   The Optimal Contract

To solve the problem, it is convenient to define the continuation value of type 1 entrepreneurs

$$w(t) \equiv \int_t^\infty e^{-\rho_1(\tau - t)} u(c_1(\tau)) d\tau \tag{15}$$

Clearly, the enforcement constraints reduces to

$$w(t) \geq v_1(\phi k_1(t)), \quad V(w(t), k(t)), \quad t \geq 0. \tag{16}$$

10

Differentiating (15) with respect to time results in a law of motion for this continuation value[10]

$$\dot{w} = \rho_1 w - u(c_1). \tag{17}$$

I follow the branch of the literature that uses continuation values as state variables (Spear and Srivastava, 1987) and include w in the state space. The law of motion (17) is the continuous time version of the "promise-keeping constraint". The problem (C) can then be written in recursive using the continuation value of type 1 as a state variable, together with the aggregate capital stock k. The value to type 2, $V(w, k)$, must satisfy

$$\rho_2 V(w, k) = \max_{c_1, c_2, k_1, k_2, \dot{k}, \dot{w}} u(c_2) + V_w(w, k)\dot{w} + V_k(w, k)\dot{k} \quad \text{s.t.} \tag{B}$$

$$\dot{w} = \rho_1 w - u(c_1) \tag{18}$$

$$\dot{k} = z_1 f(k_1) + z_2 f(k_2) - \delta k - c_1 - c_2 \tag{19}$$

$$k = k_1 + k_2 \tag{20}$$

$$w \geq v(\phi k_1), \quad V(w, k) \geq v(\phi k_2) \tag{21}$$

This Hamiltona-Jacobi-Bellman equation is similar to the one studied by Hopenhayn and Werning (2009) and by Kocherlakota (1996) in discrete time. The thought experiment is as follows: a social planner promises a value w to type 1. Taking this promise as given in the form of (18), the planner seeks to maximize the amount of utility type 2 receives. The planner determines the evolution of the aggregate capital stock, (19), and how to split that capital and consumption between the two types, taking into account the enforcement constraints of the two types, (21).

3.2   Optimality Conditions

The optimality conditions are derived in the usual way. Use the following Lagrange multipliers: α on (18); λ on (19); q on (20); and μ_1 and μ_2 on the corresponding enforcement constraints (21). Further define where $r \equiv q/\lambda - \delta$ (as discussed in section 4.3 below r, turns out to have the interpretation of an interest rate). The first-order conditions are

$$V_w(w, k) = -\alpha, \quad V_k(k, w) = \lambda \tag{22}$$

$$\alpha u'(c_1) = u'(c_2) = \lambda \tag{23}$$

$$z_i f'(k_i) - \mu_i \frac{\phi v_i'(\phi k_i)}{\lambda} = r + \delta, \quad i = 1, 2. \tag{24}$$

[10]Rearranging (17) as $\rho_1 w = u(c_1) + \dot{w}$ makes clear that this is simply the continuous time Bellman equation for type 1.

The following complementary slackness condition also holds.

$$\mu_1[w - v_1(\phi k_1)] = 0, \quad \mu_2[V(w, k) - v_2(\phi k_2)] = 0 \tag{25}$$

As in the analysis of the unconstrained planning problem in section 2, the Lagrange multiplier $\alpha = -V_w(w, k)$ in (22) is the slope of the Pareto frontier, or – equivalently and as is apparent from (23) – the Pareto weight on agent 1. The equations (24) and (25) determine how the aggregate capital stock k is split between the two types. If both types' enforcement constraints are slack, then $\mu_i = 0, i = 1, 2$ and (24) collapses to the familiar condition of equalized marginal products (12). If in contrast, say, type 1's constraint binds, then his marginal product of capital will be higher than that of type 2, $z_1 f'(k_1) > z_2 f'(k_2)$ (note that from Assumption 1 it is impossible for both constraints to bind simultaneously). Condition (24) and (25) are crucial and will be discussed in more detail below.

Using (22), the envelope conditions of the Bellman equation (B) can be written in terms of laws of motions for the Lagrange multipliers[11]. I state those here together with the laws of motion (18) and (19) for reasons that will become clear shortly:

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\begin{aligned}
\dot{\lambda} &= \lambda[\rho_2 - r] \\
\dot{\alpha} &= \alpha[\rho_2 - \rho_1 - \mu_2] + \mu_1 \\
\dot{k} &= z_1 f(k_1) + z_2 f(k_2) - \delta k - c_1 - c_2 \\
\dot{w} &= \rho_1 w - u(c_1).
\end{aligned}
\tag{26}
$$

This system of differential equations together with the first order conditions (23) and (24) and the complementary slackness condition (25) summarizes the dynamics of the equilibrium with limited enforcement.[12] Note the similarity of this system of equations to the system of ODEs (11). In particular, (11) can be obtained from (26) as the special case in which $\phi = 0$ (thereby also verifying Claim 1 in the previous section). In that case the enforcement constraints never bind so that the multipliers, $\mu_i = 0, i = 1, 2$. Because the constraints never bind, one does not need to keep track of type 1's continuation value w and can therefore discard the law of motion for w.

An easy connection can be drawn to the time-varying Pareto-weights of Marcet and Marimon

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[11]Consider for example the first equation. It is derived from noting that the envelope condition is $\rho_2 V_k = V_{wk}\dot{w} + V_{kk}\dot{k} - \lambda\delta + q$ and using the fact that $V_k = \lambda$ so that $V_{kw}\dot{w} + V_{kk}\dot{k} = \dot{\lambda}$. A similar derivation applies to the second equation.

[12]Boundary conditions are $k(0) = k_0$, $w(0) = w_0$, $\lim_{t\to\infty} e^{-\rho_2 t}\lambda(t)k(t) = 0$, $\lim_{t\to\infty} e^{-\rho_2 t}\alpha(t)w(t) = 0$. The last two conditions are transversality conditions that come from the fact that λ and α play the role of co-state variables.

(1999) and Kehoe and Perri (2002), using the law of motion for α in (26) and the first-order condition for consumption (23). These are restated here as

$$\frac{u'(c_2(t))}{u'(c_1(t))} = \alpha(t), \quad \dot{\alpha}(t) = [\rho_2 - \rho_1 - \mu_2(t)]\alpha(t) + \mu_1(t). \tag{27}$$

The interpretation of (27) is clear: $\alpha(t)$ is entrepreneur 1's Pareto weight, and this Pareto weight is varying over time as in Marcet and Marimon (1999) and Kehoe and Perri (2002). Suppose for the moment that entrepreneurs are equally patient, $\rho_1 = \rho_2$. Then, this Pareto weight is the initial Pareto weight plus Lagrange multipliers accumulated over the past.[13] Embedded in equation (27) is the effect that limited enforcement has on the allocation of consumption across entrepreneurs. Suppose for the moment that the utility function $u(\cdot)$ is of the CRRA type with parameter σ. Continue to assume $\rho_1 = \rho_2$ and differentiate (27) with respect to time

$$\sigma \left[\frac{\dot{c}_1}{c_1} - \frac{\dot{c}_2}{c_2} \right] = -\mu_2 + \frac{\mu_1}{\alpha}$$

The right-hand side is positive whenever type 1's enforcement constraint binds and negative whenever type 2's binds. In that case

$$\frac{\dot{c}_1}{c_1} > \frac{\dot{c}_2}{c_2}.$$

That is entrepreneurs with binding constraints experience higher consumption growth. This result is typical in the literature on optimal contracts: In effect, the planner "bribes" the entrepreneurs not to default by assigning them a higher continuation value. This also implies that incentives will generally "back-loaded" as for example in Thomas and Worrall (1988) and Kocherlakota (1996). The fact that an agent's Pareto weight tends to increase when his constraint binds, will be the driving force in many of the following results. Expression (27) will therefore play a prominent role in the remainder of the paper.

3.3 Steady State Misallocation

The *steady state* of the economy (k^*, w^*) is defined by setting all time derivatives equal to zero in (32).[14]

Proposition 1 *There is a continuum of steady states* (k^*, w^*). *In any steady state*

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[13] Equation (27) is the continuous time analogue of equation (7) in Kehoe and Perri (2002). The same differential equation for α could have been derived directly from the Lagrangean for (C) by use of an integration by parts argument.

[14] One could define the steady state to be the vector $(\lambda^*, \alpha^* k^*, w^*)$ but I here choose to denote it in terms of k and w only because those are the state variables. Instead λ and α play the role of co-states. In any case, if k and w are constant this immediately implies that so must be $\lambda = V_k(w, k)$ and $\alpha = -V_w(w, k)$.

(1) When $\rho_1 = \rho_2$, there is no capital misallocation in steady state. The steady state capital stock k^* is first-best and satisfies

$$F'(k^*) = \rho + \delta, \tag{28}$$

where $F(k)$ is the aggregate production function defined in (10).

(2) When $\rho_1 \neq \rho_2$, there is capital misallocation in steady state.

The intuition for this result is straightforward. Consider first part 1. That there is no capital misallocation in steady state follows almost directly from the law of motion of the time varying Pareto weight (27). If entrepreneurs are equally patient, $\rho_1 = \rho_2$, this changes whenever an enforcement constraint binds. For the economy to be in steady state, the Pareto weight must be constant; but this immediately implies that constraints cannot bind in steady state so that capital is allocated efficiently. Because marginal products of capital are equalized the production side of the economy can be represented by the aggregate production function (10). As stated in expression(28), the marginal product of this aggregate production function equals $\rho + \delta$ in steady state. Note that this is exactly the same equation solved by the steady state capital stock in a standard neoclassical growth model. Finally, consider the statement that there is a continuum of steady states, that is the steady state is not unique. This is illustrated in Figure 3. I have just argued that the steady state capital stock k^* solves the expression (28). It is
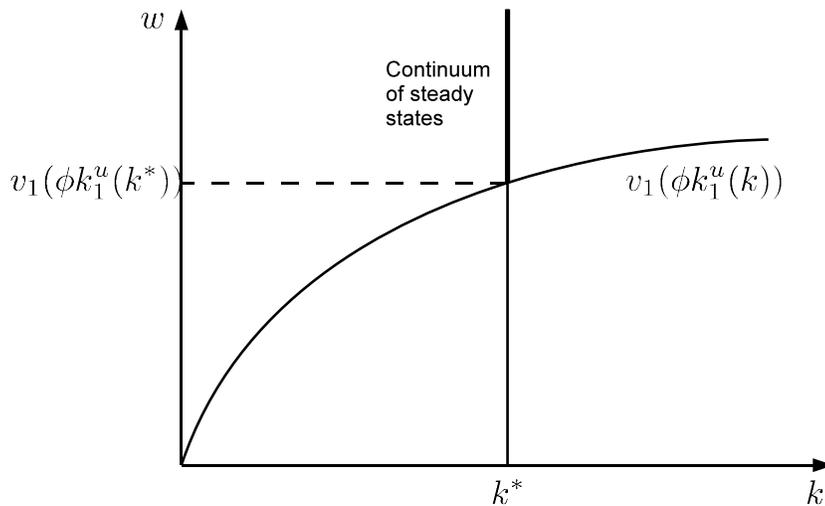

Figure 3: State Space $K \times W$

Note: .

therefore unique. However, as illustrated in the Figure, the steady state continuation value w^*

is not unique. This can again be seen from the fact that enforcement constraints don't bind in steady state, implying that (k^*, w^*) satisfy

$$w^* \geq v_1(\phi k_1^u(k^*)) \quad \text{and} \quad V^u(w^*, k^*) \geq v_2(\phi k_2^u(k^*)).$$

From assumption 1, there are multiple values w^* that are consistent with these inequalities. Any such value is consistent with the economy being in steady state.

Next, consider part 2 of the proposition. If entrepreneurs' discount rates differ, $\rho_1 \neq \rho_2$, the logic of the first part of the Proposition breaks down. Consider again the law of motion for the time-varying Pareto weight (27). There are now two reasons for the Pareto weight to change over time: differences in the discount rates, and binding enforcement constraints. In fact, the Pareto weight can only be constant if one of the constraints binds so as to offset the drift of the Pareto weight due to differential discount rates.

3.4 Dynamics of Misallocation

Proposition 1 stated that there is no misallocation in steady state when entrepreneurs are equally patient. This begs the question whether this steady state outcome is actually achieved over time. This is a harder question so that more can only be said under stronger assumptions.[15]

For the remainder of this section I therefore impose Assumption 2. From Lemma 2 we know that the Pareto frontier $V(w, k)$ is then concave in w. This is helpful because from the first-order condition $V_w(w, k) = -\alpha$, there is then a monotone relationship between α and w. I further impose a symmetry assumption

Assumption 3 (Symmetry) $z_1 = z_2$, and $v_1(\cdot) = v_2(\cdot)$.

This assumption says that the two types of agents are identical except possibly in their discount rates $\rho_i, i = 1, 2$ and their initial values, w_0 and $V(w_0, k_0)$. Note that under this assumption, it is always optimal to split aggregate capital equally between the two types $k_i^u(k) = k/2, i = 1, 2$. The following statement about the dynamics of misallocation can be made.

Proposition 2 *Let Assumptions 2 and 3 be satisfied. Then the steady state outcome in Proposition 1 is stable, in the sense that*

(1) When $\rho_1 = \rho_2$, capital misallocation disappears asymptotically. That is (12) holds as $t \to \infty$.

---

[15]Note that this is also a *different* question. Proposition 1 only says that if $\rho_1 = \rho_2$ and the economy starts out at a steady state with no misallocation, it will stay there. It does not say whether the economy would converge to such an outcome over time. This is akin to the distinction between the question of existence of a steady state and its stability (for example, in a standard neoclassical growth model).

(2) When $\rho_1 \neq \rho_2$, capital misallocation persists even asymptotically.

Most of the intuition for this result can again be obtained from examining equation (27). This equation states that the Pareto weight of a type increases whenever his enforcement constraint binds. To fix ideas, consider the case where type 1's enforcement constraint binds and where *capital is fixed* $k(t) = \bar{k}$ for all t (for whatever reason). In that case, there is a one-to-one mapping between the Pareto weight and promised value $\alpha(t) = V_w(w(t), \bar{k})$ which implies that w increases whenever α. From the law of motion for the Pareto weight α, we can also see that the function $\alpha(t)$ – and therefore also $w(t)$ – is weakly increasing over time. It therefore must be true that at some date t, $w(t) \geq v(\phi\bar{k}/2)$, at least asymptotically as $t \to \infty$.

A problem stands in the way: capital $k(t)$ is typically changing as well. This has two implications. First, because the Pareto frontier $V(w, k)$ depends on k, promised value w may be decreasing even though the slope $\alpha = -V_w(w, k)$ is increasing. Second, the outside options change with the aggregate capital stock. These two observations imply there may be reversals in the binding pattern of the constraints. That is, the constraint of type 1 binds first, then no constraint binds, and then the constraint of type 2 does. This would imply that $\alpha(t)$ may not be monotone over time any more, invalidating the main argument above. It is the role of assumption 3 to guarantee that these problems do not occur. That being said, one could also make other assumptions than symmetry to guarantee this.

If entrepreneurs' discount rates differ, $\rho_1 \neq \rho_2$, the logic of the first part of the Proposition breaks down. This result can best be understood by recalling the unconstrained planning problem in section 2. There, the different discount rates had the effect that the more patient entrepreneur ended up with all the surplus in the long-run. This was captured by the fact that the Pareto weight $\alpha(t)$ varied over time. In particular, if type 1 entrepreneurs were the less patient ones, $\rho_1 > \rho_2$, then their Pareto weight converged to zero, $\alpha(t) \to 0$. It is now almost immediate that this outcome does not survive the addition of enforcement constraints. In the long-run, the enforcement constraint of the less patient type 1 entrepreneurs must always bind, which implies that capital misallocation persists.

The result in proposition 2 was cast in terms of a planning problem. It can also be understood in terms of the decentralized competitive equilibrium which was the starting point of my analysis (section 1). In a decentralized equilibrium, entrepreneurs accumulate wealth over time. That capital misallocation disappears in the long-run (if entrepreneurs are equally patient) is a result similar to that in **?** and Banerjee and Moll (2009). The intuition is the same as there: being credit constrained creates big incentives to save in the form of a high marginal product of capital which implies a high marginal rate of substitution between consumption today and consumption tomorrow. Therefore, individuals use wealth accumulation as a substitute for

poorly functioning credit markets. This logic breaks down, however, if entrepreneurs discount at differential rates. In that case, the marginal rate of substitution also depends on the discount rate so that an constrained entrepreneur might actually have lower incentives to save than a constrained one.

The results in Proposition 2 does *not* say that the steady state in Proposition 1 is globally stable. The discussion here is however suggestive of this property of the model, at least for the case $\rho_1 = \rho_2$. This is illustrated in Figure 4. The figure shows the dynamics of the system of
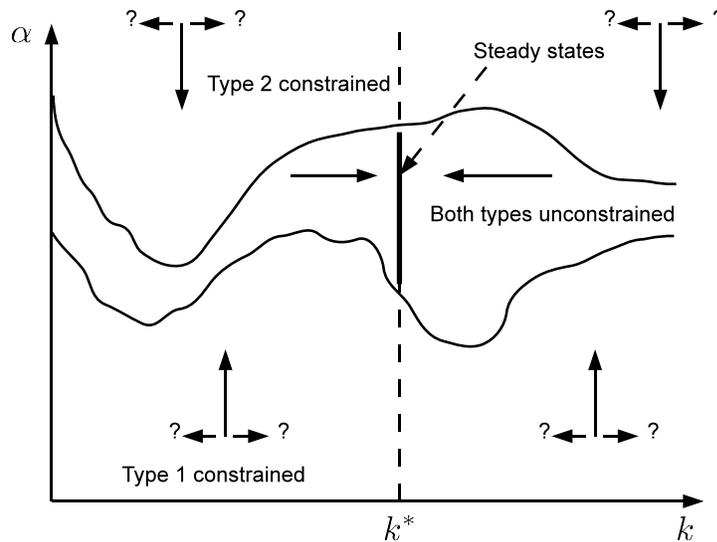

Figure 4: Global Stability in the Case $\rho_1 = \rho_2$

Note:

ODEs (26) in (α, k) space (this is the state space of Marcet and Marimon (1999)). This space can be divided into three different regions. For low values of the Pareto weight α, the constraint of type 1 binds; for high values of α the constraint of type 2 binds; for intermediate values they are both slack.[16] The arrows in the figure indicate what we know about the dynamics of the system (26) in any particular region. Consider first the unconstrained region for intermediate α. In this region, $\dot{\alpha} = 0$ (I continue to assume $\rho_1 = \rho_2$). Further, the dynamics for capital are *the same as in a neoclassical growth model*, that is capital converges monotonically towards the steady state k^*.[17] Next, consider the region in which type 1 is constrained (low α). In this case,

---

[16]That there is a region in which both constraints are slack is guaranteed by assumption 1. That one can look at the binding pattern of constraints in (α, k) rather than (w, k)-space is guaranteed by assumption 2 which implies that $V(w, k)$ is concave and therefore that $\alpha = -V_w(w, k)$ is an increasing function of w. That the boundaries between the sets are continuous also assumes that $V_w(w, k)$ is continous in k.

[17]To see this formally, note that in the unconstrained region of the state space (w, k), we have that the value functions of the constrained planning problem (C) coincides with that of the unconstrained problem (U),

17

I am not able to say anything about the dynamics of k. However, I know that $\dot{\alpha} = \mu_1 > 0$, that is α is strictly increasing. This implies that there is a strong pressure for α to adjust in the direction of alleviating the enforcement constraint of type 1 over time. An analogous logic holds for the region in which type 2 is constrained. Examining the arrows in Figure 4, there is therefore a strong force for the system to converge to a steady state without misallocation (see Proposition 1).

4 Model Predictions: Creditor Rights, Inequality and Development

I set out with the goal of better understanding the interaction between limited enforceability of credit contracts, inequality and economic growth. I here provide some simulations that highlight the predictions of the model for these issues. How the simulations are computed is outlined in section 5. I assume in the present section that $\rho_1 = \rho_2$, so that there is no capital misallocation in the long-run.[18] I use the following standard functional forms and parameter values: CRRA utility with parameter $\sigma = 2$; Cobb-Douglas production with capital share $\gamma = 0.3$; Discount and depreciation rates $\rho_1 = \rho_2 = \delta = 0.05$; The value of default is given by being cast into autarky forever; I assume that type 1 is the more able type, $z_1 = 1, z_2 = 0.5$ I vary the parameter capturing the quality of creditor rights ϕ, and the initial condition (k_0, w_0).

4.1 Creditor Rights and Growth

I have already shown that capital misallocation disappears asymptotically if entrepreneurs have the same discount rates, $\rho_1 = \rho_2$ so that limited enforcement does not matter for aggregate outcomes in the long run. However, one might ask how the transition path to the steady state is affected. To answer this question, consider the following experiment. Two countries are identical in every respect except their quality of creditor rights ϕ. In particular, they are characterized by the same initial conditions for aggregate capital and inequality (k_0, w_0), but one has a higher ϕ (worse creditor rights) than the other. I have in mind here, say, Argentina and Chile in the 1970s. I assume that one contry has perfect credit markets, $\phi = 0$ whereas the other is characterized by an intermediate level of creditor rights, $\phi = 0.5$. Initial conditions are $(k_0, w_0) = (0.8k^*, 0.7w^*)$. Figure 5 plots the time paths for the aggregate capital stocks, for individual marginal products of capital, for aggregate production and total factor

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$V(w, k) = V^u(w, k)$. Consequently, the policy functions must coincide as well.

[18]In light of Propositions 1 and 2 the more general case is obviously interesting and I plan to add the corresponding simulations in a future version.

productivity.[19] Consider first the paths of the aggregate capital stock for the two economies.
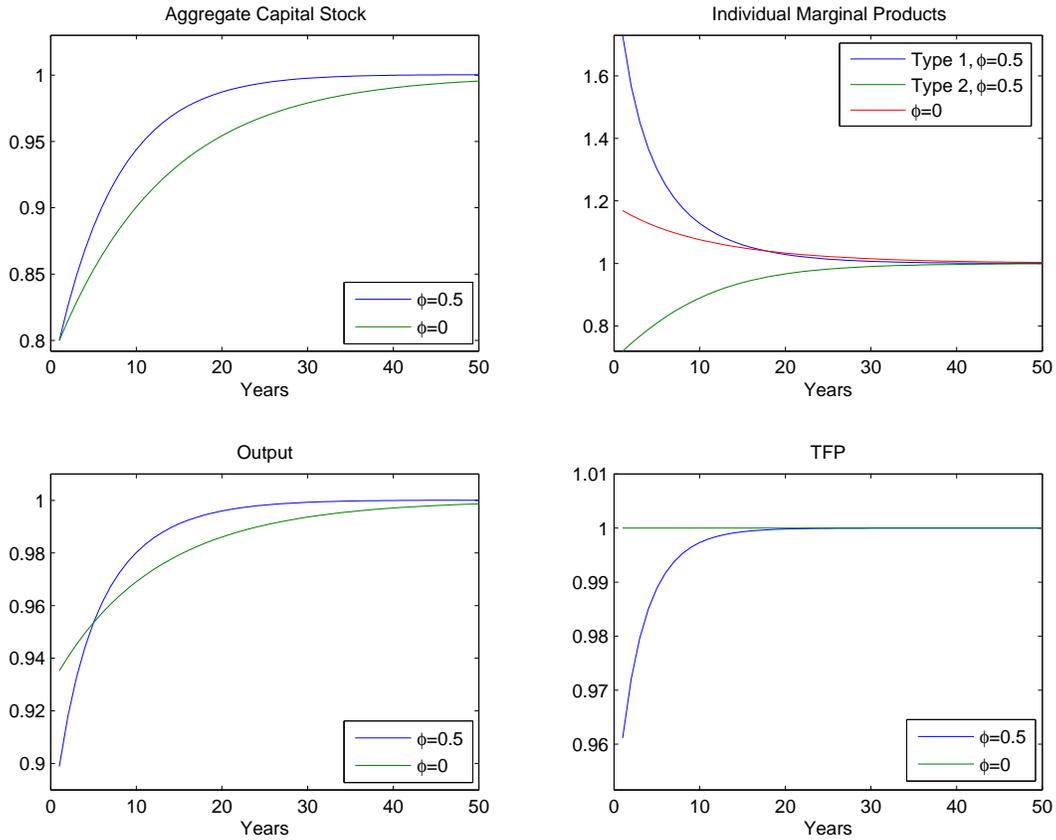

Figure 5: Different Creditor Rights

Note: .

Limited enforcement has an effect on aggregate savings. Perhaps somewhat surprisingly, capital accumulation is higher in the country with worse creditor rights, that is there is a positive *savings effect*. This savings effect can go either way however, and depends crucially on the specification of the outside option. For example, with a linear outside option $v_i(\phi k) = \phi k$, it goes the other way. Consider, next the inidividual marginal products of capital. The presence of limited enforcement implies that marginal products are not equalized so that there is capital misallocation. More formally, from (24) a binding enforcement constraint implies that

$$z_1 f'(k_1) > z_2 f'(k_2)$$

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[19]Total factor productivity is simply defined as the residual y/k^γ where $y = z_1 k_1^\gamma + z_2 k_2^\gamma$ and normalized by first-best TFP y^u/k^γ where $y^u = z_1(k_1^u)^\gamma + z_2(k_2^u)^\gamma$.

which can be observed in the second panel of Figure 5. This misallocation shows up as low output or equivalently as low TFP (panel 4). I term this the *TFP effect* of limited enforcement.[20] I have proved in proposition 4 that this misallocation disappears over time so that marginal products are equalized eventually. This result is very much in the spirit of Banerjee and Moll (2009) and the references therein. Finally, consider GDP in the two economies. The effect on GDP is the combination of the *savings* and *TFP* effects. Because the former is positive while the latter is negative, the overall effect is ambiguous. A similar conclusion holds when considering GDP *growth* as opposed to GDP *levels*: because of the two counteracting effects, the overall impact of bad creditor rights on growth is again ambiguous.[21]

4.2   Creditor Rights and Inequality

Figure 1 in the introduction provided some suggestive evidence that wealth inequality is higher in countries with worse creditor rights. I want to argue in this section that poor creditor rights magnify the effect of heterogeneity in ability on long run wealth inequality in the present model, providing a potential rationale for the relation observed in Figure 1. Compare again two countries: one with perfect credit markets, $\phi = 0$, and one with an intermediate quality of credit rights, $\phi = 0.5$. In both countries, entrepreneurs are equally wealthy at time zero but type 1 has higher ability $z_1 > z_2$. The aggregate capital stock is in steady state $k_0 = k^*$. Figure 6 depicts the evolution of (consumption) inequality. In the country with perfect credit markets, $\phi = 0$ there is a complete separation between ownership and production. Individual consumptions are equal, $c_1(t) = c_2(t)$ for all t. In contrast, with poor creditor rights there is a fanning out of inequality. The consumption of the more able type 1s increases and that of the less able type 2's decreases. As discussed above, capital misallocation disappears asymptotically because entrepreneurs use wealth accumulation as a substitute for poor creditor rights. The same effect is also responsible for the fanning out of inequality observed in Figure 6. With perfect credit markets, the more able type 1 entrepreneurs, simply borrow some capital from the less able type 2 entrepreneurs. This is not possible with poor creditor rights. Instead, the more able entrepreneurs accumulate wealth in order to be able to exploit their superior

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[20]In the present example, the TFP loss is always below four percent. The size of the TFP loss depends on the number of entrepreneurs in the economy. Consider the upper bound on the TFP loss which is given by the loss incurred from giving the all capital to one entrepreneur and none to others. It is $1 - k^\gamma/[n(k/n)^\gamma] = 1 - (1/n)^{1-\gamma}$, which is increasing in n.

[21]One might also be interested in the relation between initial inequality and the time path of aggregate output. One measure of initial inequality is type 1's promised value w_0. Consider then two countries that have the same ϕ and k_0 but different k_0. The time paths will be qualitatively identical to the ones in Figure 5. In particular, the effect of initial inequality on output levels and growth is ambiguous. This complex relationship is in line with the argument made by Banerjee and Duflo (2005) that "estimating the relationship between inequality and growth in a cross-country dataset [...] has, at best, very limited use."
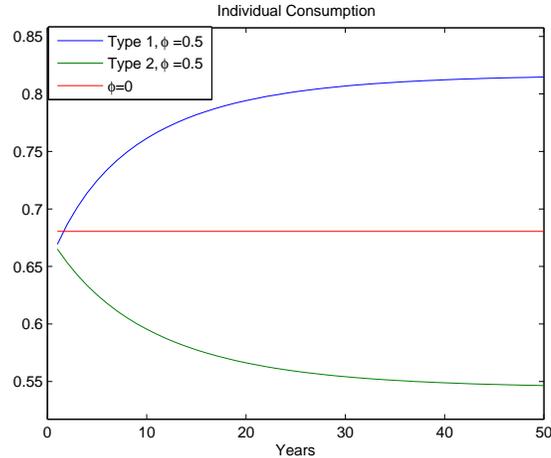
Figure 6

Note: .

production technologies.

4.3 Low Interest Rates

Hausmann, Rodrik and Velasco (2005) suggest that interest rate data provide a "growth diagnostic" in that binding credit constraints can be inferred from it. I examine whether this is true in the present model. The interest rate in the economy can be backed out from the planning problem as

$$r = \rho - \frac{\dot{\lambda}}{\lambda},$$

where $\lambda = u'(c_2)$ is the marginal utility of the unconstrained type 2 entrepreneurs. This strategy of backing out the interest rate from a planning problem follows Alvarez and Jermann (2000). It states that the interest rate is given by the continuous time analogue of the marginal rate of substitution across time for the unconstrained type 2 entrepreneurs. From (26), we then have that

$$r = z_2 f'(k_2) - \delta,$$

where I continue to assume that $\rho_1 = \rho_2$ and that type 1 entrepreneurs are constrained. Figure 7 plots the time path of the interest rate. This time path is shaped by the same TFP and savings effects as in Figure 5 above. Consider first the case where the savings effect is absent so that aggregate capital accumulation is the same as in the unconstrained neoclassical growth model. In that case, the interest rate is *always lower* compared to an economy without enforcement constraints. This follows immediately from the fact that the interest rate is determined by the marginal product of *unconstrained* entrepreneurs who get a bigger share of the aggregate
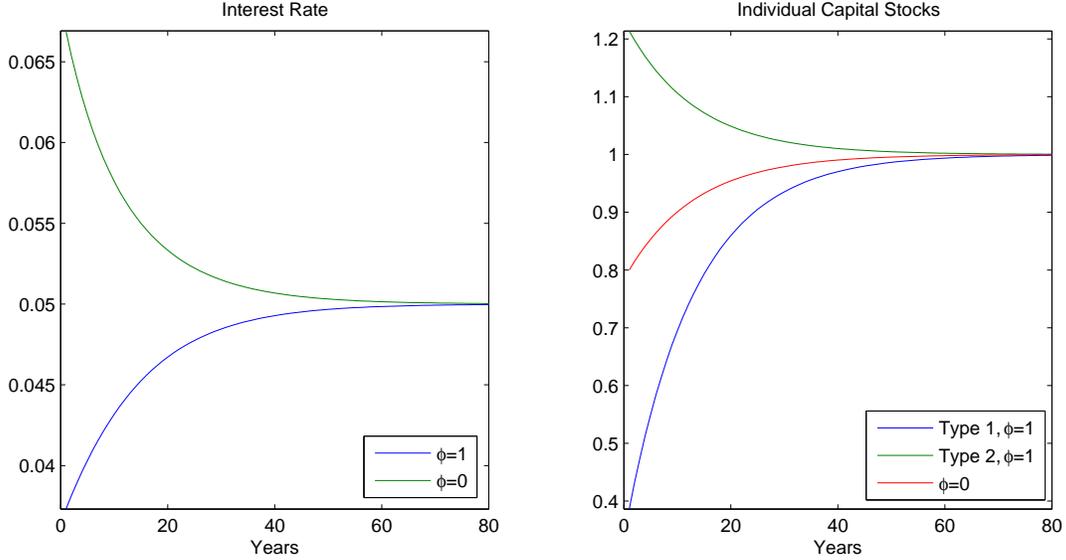
21

Figure 7: Interest Rate

Note: .

capital stock in the presence of enforcement constraints. In general equilibrium, the flip-side of the constrained marginal product being relatively high, is that the unconstrained marginal product is relatively low. Similar results about the interest rate are obtained in Kehoe and Levine (1993) and Alvarez and Jermann (2000). Another way of putting this is that

$$r = z_2 f'(k_2) - \delta < F'(k) - \delta, \tag{29}$$

where $F(k)$ is the aggregate production function defined in (10). In the absence of enforcement constraints, the interest rate would equal the aggregate marginal product $F'(k) - \delta$. Enforcement constraints break this link.

However, this logic ignores the presence of a savings effect: the aggregate capital stock might be higher which might overturn the TFP effect. This might result in the interest rate being higher in an economy with enforcement constraints. Overall, the effect of poor creditor rights on the interest rate is ambiguous. In that sense, the level of the interest rate is uninformative about the presence of credit constraints in an economy.

That enforcement constraints break the link between the interest rate and the aggregate marginal product of capital as in (29) is also relevant with respect to an argument put forth by King and Rebelo (1993): If one wants to explain sustained growth by transitional dynamics of the standard neoclassical growth model, one generates extremely counterfactual implications for the time path of the interest rate. For example, King and Rebelo (1993) argue that if the neoclassical growth model were to explain the postwar growth experience of Japan, the interest

22

rate in 1950 should have been around 500 percent. As can be seen in Figure 7, it is theoretically possible that both the capital stock and the interest rate approach the steady state *from below*, offering a way out of problem raised by King and Rebelo (1993).

5 Local Stability and Computation: Some Technical Results

More can be said about the behavior of the dynamic system (26) for the case where individuals are equally patient $\rho_1 = \rho_2(= \rho)$. In particular, I here show that the dynamics of the economy are very similar to that of the standard neoclassical growth model and that steady states are locally stable. I impose

Assumption 4 *Type 1's enforcement constraint binds at date zero, $w_0 < v_1(\phi k_1^u(k))$.*

The case where no constraint binds is generally uninteresting because dynamics will be the same as in the neoclassical growth model; the case where type 2's constraint binds is symmetric.

5.1 Effective Uniqueness of Steady State

The system of ODEs is relatively easy to analyze, if the economy is always in the constraint region of the state space for all $t \geq 0$.[22] I first provide conditions that guarantee that this is the case. Since I only analyze the local stability of the system of ODEs (26), these conditions only need to be valid in a neighborhood of the steady states. The following assumption implies that the economy will always be in the constraint region of the state space.

Assumption 5 *The slope of the unconstrained Pareto frontier at the point $w = v_1(\phi k_1^u(k))$, $V_w^u[v_1(\phi k_1^u(k)), k]$ is independent of k in a neighborhood of the steady state k^*.*

Lemma 3 *Let Assumptions 2, 4 and 5 be satisfied and let $\rho_1 = \rho_2$. Then the enforcement constraint (3) binds for all $t \geq 0$, in a neighborhood of the steady state. That is, for (k_0, w_0) sufficiently close to (k^*, w^*)*

$$w(t) \leq v_1[\phi k_1^u(k(t))] \quad t \geq 0.$$

This Lemma establishes that under certain assumptions, the economy is always in the constrained region of the state space. It can then be seen from Figure 3 that the steady state must lie at the boundary of the half-line $w^* \geq v_1(\phi k_1^u(k^*))$. That is, it satisfies $w^* = v_1(\phi k_1^u(k^*))$ so that constraint (3) is "barely binding". For all practical purposes, then the steady state is unique.

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[22]It is also easy to analyze if the economy is always in the unconstrained region. The dynamics are then identical with the ones of the standard neoclassical growth model. I therefore concentrate on the constraint case.

5.2   Local Stability

I next define two useful constructs which I term "pseudo-aggregate production function" and "pseudo-aggregate utility function". While the representative agent framework is invalid, these constructs will play exactly the same role as the aggregate production function (10) and the aggregate utility function (9) in the unconstrained planning problem.[23] The "pseudo-aggregate production function" is

$$F(k, w) = \max_{k_1, k_2} z_1 f(k_1) + z_2 f(k_2) \quad \text{s.t.} \quad k_1 + k_2 \leq k, \quad w \geq v_1(\phi k_1) \tag{30}$$

If we define the Lagrange multipliers on the first and second constraint to be R and μ/λ respectively, then the first order and complementary slackness conditions coincide with (24) and (25). Similarly, the "pseudo-aggregate utility function" is

$$U(c, x) = \max_{c_1, c_2} u(c_2) \quad \text{s.t.} \quad c_1 + c_2 \leq c, \quad u(c_1) \geq x \tag{31}$$

Now, denote by λ and α the Lagrange multipliers on the first and second constraint respectively. Then the first order condition is (23).

Lemma 4 *The pseudo-aggregate utility function $U(c, x)$ is strictly concave. The pseudo-aggregate production function is strictly concave under assumption 2.*

In what follows it turns out to be more convenient to work with

$$\psi \equiv -\alpha.$$

Just as in the unconstrained planning problem, the (pseudo-) aggregate production and utility functions can be used to simplify (26). In particular, using the envelope theorem in (30), we see that

$$F_k(k, w) = R, \quad F_w(k, w) = \mu/\lambda$$

Similarly, applying the envelope theorem in (31), we have that

$$U_c(c, x) = \lambda, \quad U_x(c, x) = \psi \quad \text{or} \quad \partial U(c, x) = (\lambda, \psi)$$

---

[23]I add the qualifier "pseudo" because these are clearly not an aggregate production/utility functions in the usual sense. In particular, they depend on entrepreneur 1's utility and continuation value.

where $\partial U : R^2 \to R^2$ is the gradient of U. Because U is concave, ∂U has an inverse. Then

$$(c, x) = (\partial U)^{-1}(\lambda, \psi) \quad \text{or} \quad (c, x) = [c(\lambda, \psi), x(\lambda, \psi)]$$

Using these properties of the (pseudo-) aggregate production and utility functions, (26) can be rewritten as a system of autonomous differential equations in (λ, ψ, k, w) only.

$$\begin{aligned}
\dot{\lambda} &= \lambda[\rho + \delta - F_k(k, w)] \\
\dot{\psi} &= -\lambda F_w(k, w) \\
\dot{k} &= F(k, w) - \delta k - c(\lambda, \psi) \\
\dot{w} &= \rho w - x(\lambda, \psi)
\end{aligned} \tag{32}$$

These four ODEs and boundary conditions together with the solutions to the pseudo-aggregate production and utility functions (30) and (31) completely characterize the competitive equilibrium with enforcement constraints.[24] Linearizing (32) around the steady state $(w^*, k^*) = (\phi v_1(k_1^u(k^*)), k^*)$ (which is effectively unique given the argument in section 5.1) yields

$$\dot{x} = Ax, \quad A = \begin{bmatrix} 0 & 0 & \partial\dot{\lambda}/\partial k & \partial\dot{\lambda}/\partial w \\ 0 & 0 & \partial\dot{\psi}/\partial k & \partial\dot{\psi}/\partial w \\ \partial\dot{k}/\partial\lambda & \partial\dot{k}/\partial\psi & \rho & 0 \\ \partial\dot{w}/\partial\lambda & \partial\dot{w}/\partial\psi & 0 & \rho \end{bmatrix}, \quad x \equiv \begin{bmatrix} \lambda - \lambda^* \\ \psi - \psi^* \\ k - k^* \\ w - w^* \end{bmatrix}. \tag{33}$$

This can be written more compactly as

$$A = \begin{bmatrix} 0 & X \\ Y & \rho I \end{bmatrix}$$

where 0 is a 2×2 matrix of zeros and I is the 2×2 identity matrix. Denote the 2×2 Hessian matrices of the aggregate production function and aggregate utility function by $\partial^2 F(k, w)$ and $\partial^2 U(c, x)$ respectively. It is easy to show that X and Y are simply given by

$$X = -\lambda^* \partial^2 F(k^*, w^*), \quad Y = -[\partial^2 U(c^*, x^*)]^{-1}$$

As show in Lemma 4, $U(c, x)$ and $F(k, w)$ are strictly concave. This immediately implies that both X and Y are positive definite matrices.

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[24]The boundary conditions are $k(0) = k_0$, $w(0) = w_0$, $\lim_{t \to \infty} e^{-\rho t}\lambda(t)k(t) = 0$, $\lim_{t \to \infty} e^{-\rho t}\psi(t)w(t) = 0$.

The stability proof makes use of some results from a field of linear algebra called "Inertia Theory". The following results are taken from Datta (1999).[25] I'm interested in applying the theory to the 4×4 matrix A. However, the theory applies to matrices of any dimension. I choose to present it in its most general form.

Definition 2 *The* inertia *of a matrix A, denoted by $In(A)$, is the triplet $(\pi(A), \nu(A), \delta(A))$ where $\pi(A), \nu(A)$ and $\delta(A)$ are, respectively, the number of eigenvalues of A with positive, negative, and zero real parts, counting multiplicities.*

A linear system of $2n$ differential equations such as (32), is saddle path stable if exactly half of the eigenvalues of A have negative real parts. Saddle path stability is therefore equivalent to the statement

$$In(A) = (n, n, 0)$$

The following theorem is very useful in establishing sufficient conditions under which indeed $In(A) = (n, n, 0)$.[26] In the following, $M \geq 0$ means that the matrix M is positive semidefinite.

Proposition 3 *Let $\delta(A) = 0$, and let W be a nonsingular symmetric matrix such that*

$$WA + A^T W = M \geq 0 \tag{34}$$

Then $In(A) = In(W)$.

This Theorem is a generalization of the Lyapunov Stability Theorem for matrices (see Theorem 3.2. in Datta (1999)). It is used in the same way, and its advantages and disadvantages are similar. In particular, the exercise boils down to finding a matrix W whose inertia we can easily determine. If we are not able to find such a matrix, the theorem is worthless.

Proposition 4 $In(A) = (2, 2, 0)$ *and hence the steady state $(\lambda^*, \psi^*, k^*, w^*)$ is locally stable.*

The linearized system (33) can be used to compute solutions to the constrained planning problem (C). In the Appendix I derive in more details the entries of the matrix A.

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[25]Benhabib and Nishimura (1981) use a similar result by Wielandt (1973) in a stability proof for n-sector growth.

[26]It is natural to ask whether corresponding techniques are also available for systems of *difference* equations. Datta (1999) also defines a unit circle inertia as the triplet of the number of eigenvalues outside, inside and on the unit circle. Theorem 4.5. in this paper is the analogue of Theorem 3 for the unit circle inertia.

6 Conclusion

I have presented a deterministic neoclassical growth model in which the presence of limited enforcement inhibits the efficient working of capital markets, and used this framework to study the implications of the quality of creditor rights for inequality and economic growth. Binding enforcement constraints imply that capital is misallocated across entrepreneurs which shows up as low aggregate TFP and GDP. I provided a characterization of the evolution of this distortion over time. When entrepreneurs are equally patient capital misallocation disappears in the long run. In contrast, when entrepreneurs' discount rates differ, capital misallocation persists asymptotically. Because they result in an inefficient allocation of capital, poor creditor rights also break the link between the interest rate and the aggregate marginal product of capital, in particular typically *lowering* the interest rate. This was relevant because this disconnect offers a way out of the problem raised by King and Rebelo (1993) that wanting to explain sustained growth with a neoclassical growth model generates extremely counterfactual implications for the interest rate.

Poor creditor rights also magnify the effect of heterogeneity in ability on long run (consumption) inequality. A high ability entrepreneur with low initial wealth (as measured by promised utility) initially has a binding enforcement constraint. To alleviate this constraint, the social planner shifts more and more consumption towards him until his constraint ceases to bind. This implies that more more able entrepreneurs generally end up with more consumption than they would in the absence of enforcement constraints. Observe that the same mechanism that is good for aggregate activity, here increases inequality. More generally, my model departed from the representative agent framework and therefore has a non-trivial interplay between the evolution of inequality and the evolution of aggregates. Both are determined endogenously and simultaneously. One can easily come up with configurations in which inequality increases while GDP does. The reverse is also possible. When reading the empirical literature examining the relationship between inequality and growth, one wonders: Why do so many authors obtain such widely different results? The complex relationship generated by a simple model such as the one presented here provides a possible answer.

Appendix

A   Proofs

A.1   Proof of Lemma 1

I follow Aguiar and Amador (2009, footnote 16) in making the following change of variables in problem (C). Let $V_i = v_i(\phi k_i)$ be my choice variables instead of k_i, and define $K_i(V_i)$ to be the inverse function of $v(\phi k_i)$.[27] Similarly, make utility itself the choice variable and let $c(u_i)$ denote the inverse utility function, that is, the consumption required to deliver utility u_i. Problem (C) becomes

$$V(w_0, k_0) = \max_{\{u_1(t), u_2(t), v_1(t), v_2(t)\}} \int_0^\infty e^{-\rho_2 t} u_2(t) dt \quad \text{s.t.} \quad \int_0^\infty e^{-\rho_1 t} u_1(t) dt \geq w_0, \qquad (35)$$

$$\dot{k} = z_1 f[K_1(V_1)] + z_2 f[K_2(V_2)] - \delta k - c(u_1) - c(u_2), \qquad (36)$$

$$k - K_1(V_1) - K_2(V_2) \geq 0, \quad k(0) = k_0, \qquad (37)$$

$$\int_t^\infty e^{-\rho_i(\tau - t)} u_i(\tau) d\tau \geq V_i(t), \ i = 1, 2, \ t \geq 0. \qquad (38)$$

The objective function and constraints (35) and (38) are linear in choice variables. Since $K_i(V_i)$ is convex for $i = 1, 2$, (37) is concave. Finally, $f[K_i(V_i)]$ is concave by assumption 2 so that (36) is also convex.□

A.2   Proof of Lemma 2

Use the notation of the proof of Lemma 1. Define the utility possibility set

$$\mathcal{W}(k_0) = \left\{ (w_1, w_2) \mid w_i \leq \int_0^\infty e^{-\rho_i t} u_i(t) dt, \quad (36) \text{ to } (38) \text{ hold} \right\}$$

Because (36) to (38) are convex, and $\int_0^\infty e^{-\rho_i t} u_i(t) dt$ is linear, the utility possibility set $\mathcal{W}(k_0)$ is convex. The Pareto frontier $V(w_0, k_0)$ is the upper envelope of $\mathcal{W}(k_0)$ which is then concave.□

A.3   Proof of Proposition 1

Part 1: Setting $\dot{\alpha} = 0$, since $\rho_1 = \rho_2 = \rho$ it must be that in steady state $0 = -\alpha^* \mu_2^* + \mu_1^*$. By assumption 1, only one of the constraints can bind. But then it must be that the other doesn't bind either, $\mu_i^* = 0, i = 1, 2$. This immediately implies that

$$z_1 f'(k_1^*) = z_2 f'(k_2^*) = \rho + \delta$$

so that $k^* = k_1^* + k_2^*$ is the steady state capital stock of the standard neoclassical growth model.

Given k^*, any w^* such that $w^* \geq v_1(\phi k_1^*)$ and $V(w^*, k^*) \geq v_2(\phi k_2^*)$ satisfies $\mu_i^* = 0, i = 1, 2$. Now pick any such w^*. c_1^* is pinned down by $w^* = u(c_1^*)/\rho_1$. Further, $c_2^* = c^* - c_1^*$ where $c^* = z_1 f(k_1^*) + z_2 f(k_2^*) - \delta k^*$. Given c_2^*, $\lambda^* = u'(c_2^*)$ and $\alpha^* = u'(c_1^*)/u'(c_2^*)$. Summarizing, k^* is

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[27]V_i is simply the value of the outside option the entrepreneur would get from defaulting with any given capital stock.

unique but there are multiple steady state values of promised utility w^* (see Figure 3). Given a choice of w^*, (λ^*, α^*) are unique as well.

Part 2: Without loss of generality, consider the case $\rho_1 > \rho_2$. Then $\mu_1^* = [\rho_1 - \rho_2]\alpha^* > 0$. Since only one type's enforcement constraint can bind, $\mu_2^* = 0$ so that

$$z_1 f'(k_1^*) > z_2 f'(k_2^*) = \rho_2 + \delta.$$

That is, marginal products aren't equalized so that capital is misallocated.□

A.4   Proof of Proposition 2

I first present two Lemmas, which aid in establishing the proposition below. Their purpose is to say more about the binding pattern of enforcement constraints.

Lemma 5 *Assume $\rho_1 = \rho_2$ and further let assumptions 3 and 2 be satisfied. Then there will be no reversals in the binding pattern of the constraint. That is, if for some t type 1's constraint binds, $\mu_1(t) > 0$, then type 2's constraint can never bind, $\mu_2(t) = 0$ for all t (and vice versa).*

Proof Step 1: Under the assumptions, all heterogeneity between the two agents is summarized by the state (k, w), and the problem is otherwise symmetric. This is illustrated in Figure 8. It follows from assumption 1 that any (w, k) such that $V(w, k) = w$ must have both enforcement constraints slack so that $V(w, k) = V^u(w, k)$. Furthermore, by symmetry if $V[w(s), k(s)] = w(s)$ for some s, then $V[w(t), k(t)] = w(t)$ for all $t > s$. This is because then both types are exactly identical at date s and will remain so.
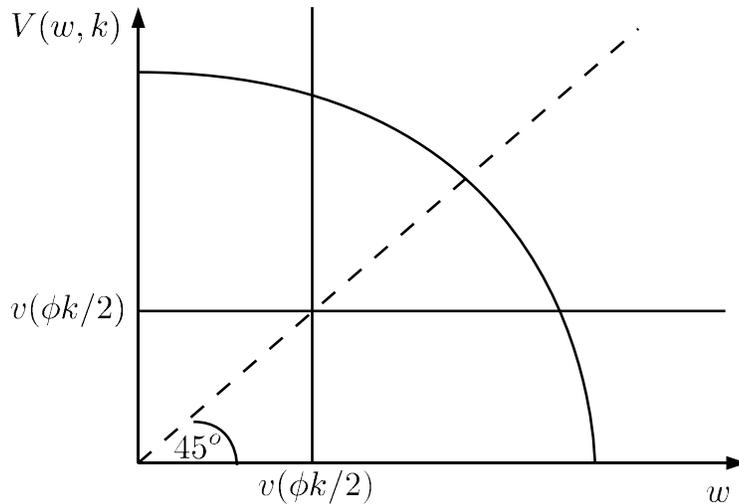

Figure 8

Note:

Step 2: I next show that if $\mu_1(t) > 0$, then $\mu_2(\tau) = 0$ for all $\tau > t$. By way of contradiction, suppose that $\mu_1(t) > 0$ but $\mu_2(\tau) > 0$ for some $\tau > t$. Because $\mu_1(t) > 0$, we know that

$w(t) < V(w(t), k(t))$. Similarly, $w(\tau) > V(w(\tau), k(\tau))$. Because $w(t)$ is continuous in t, for some $t < s < \tau$, $V(w(s), k(s)) = w(s)$. But this is a contradiction to what was said in Step 1.

Proof of Proposition 2.

Part 1: Without loss of generality consider the case where $V(w_0, k_0) > w_0$. From 5, then $\mu_1(t) \geq 0$ and $\mu_2(t) = 0$ for all t. When $\rho_1 = \rho_2$, this immediately implies that the function $\alpha(t)$ is nondecreasing. Therefore, $\lim_{t\to\infty} \alpha(t) = \sup_t \alpha(t)$. There are only two possible cases:

Case 1: $\sup_t \alpha(t) = \alpha^* < \infty$. This is only possible if $\mu(t) \to 0$ as $t \to \infty$, implying the desired result.

Case 2: $\sup_t \alpha(t) = \infty$. This implies $u'(c_1(t))/u'(c_2(t)) \to 0$ as $t \to \infty$. This is only possible if either $c_1(t) \to \infty$ or $c_2(t) \to 0$. But then

$$\lim_{t\to\infty} w(t) > \lim_{t\to\infty} V(w(t), k(t)),$$

which is a contradition because by following similar steps as in Lemma 5, one can show that $V(w_0, k_0) > w_0$ implies $V(w(t), k(t)) \geq w(t)$ for all t.

Part 2: Suppose that $\alpha(t)$ converges to some α^*. Then $\dot{\alpha}(t) \to 0$ and from (27), either $\mu_1(t) \to \mu_1^* > 0$ or $\mu_2(t) \to \mu_2^* > 0$. Next, suppose that $\{\alpha(t)\}_{t=0}^\infty$ does not converge, i.e. it oscillates. Then $\limsup \dot{\alpha}(t) = \liminf \dot{\alpha}(t) = 0$ so that, again from (27), $\limsup \mu_i(t) > 0$ for one of $i = 1, 2$. Consequently distortions persist asymptotically.

A.5   Proof of Lemma 4

Consider $U(c, x)$. Substituting in for the constraints

$$U(c, x) = u[c - u^{-1}(x)]$$

Pick any $\theta \in (0, 1)$ and any c, c', x, x'. Define $c^\theta = \theta c + (1 - \theta)c'$, and similarly for x^θ. I want to show that $U(c^\theta, x^\theta) > \theta U(c, x) + (1 - \theta)U(c', x')$. This follows because

$$u[c^\theta - u^{-1}(x^\theta)] > u[c^\theta - (\theta u^{-1}(x) + (1 - \theta)u^{-1}(x'))] > \theta u[c - u^{-1}(x)] + (1 - \theta)u[c' - u^{-1}(x')],$$

where the first inequality follows from the fact that the inverse of a concave function is convex. $\square$

A.6   Proof of Lemma 3

The proof works with the Marcet and Marimon (1999) state space (α, k). By assumption 5, $\underline{\alpha} = -V_w[v_1(\phi k_1^u(k)), k]$ is independent of k. By assumption 2, the enforcement constraint binds if $\alpha < \underline{\alpha}$ and is slack otherwise. Figure 9 is a phase diagram in (α, k) space under assumption 5. The plane is divided into two parts depending on whether $\alpha \gtrless \underline{\alpha}$. Define $C \equiv \{(\alpha, k) \in S : \alpha < \underline{\alpha}\}$. The enforcement constraint (3) binds if and only if $(\alpha, k) \in C$. The theorem is proven if I can show that an economy starting out in C, will never leave that region. The arrows indicate what we know about the dynamics of (α, k). For $\alpha < \underline{\alpha}$, we know that $\dot{\alpha} = \mu_1 > 0$, but the dynamics for k are unclear. For the part of the state space where $alpha \geq \underline{\alpha}$, the dynamics of the system are those of the unconstrained benchmark in section 1.3, and we know that the aggregate capital stock converges to its steady state value k^* while the
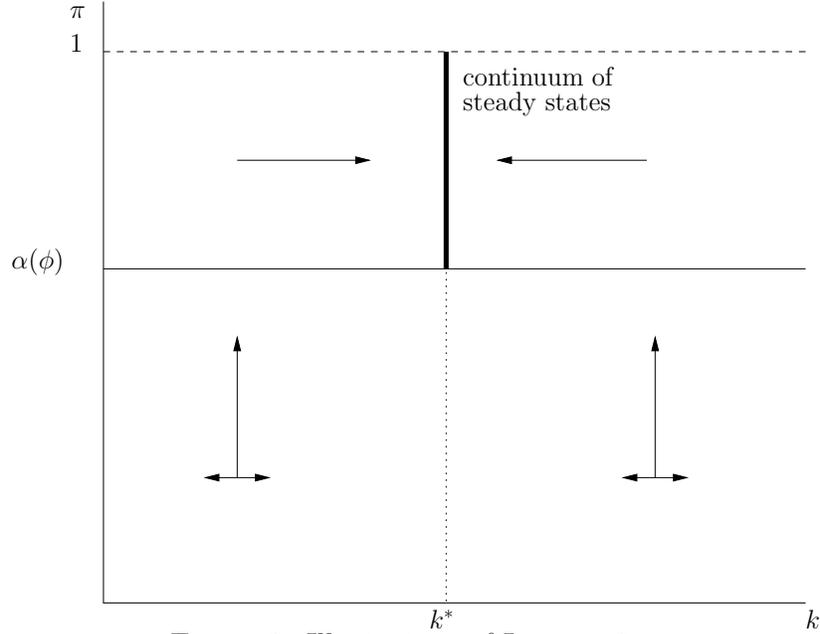
Figure 9: Illustration of Lemma 3

Note: .

Pareto weight α is constant. It can easily be seen from the figure 9 that an economy starting out in C will never leave that region.$\square$

A.7 Proof of Proposition 3

See Theorem 4.4. in Datta (1999). The theorem is originally due to Carlson and Schneider (1963).

A.8 Proof of Proposition 4

First, recall that X and Y are both positive definite matrices. Define

$$W = \begin{bmatrix} -Y & 0 \\ 0 & X \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ 0 & 2\rho X \end{bmatrix}$$

We have that

$$WA + A^T W = M \geq 0$$

Applying Theorem 3 we see that

$$In(A) = In(W) = (n, n, 0)$$

where the second equality follows because W is block diagonal and its eigenvalues are those of $-Y$ and X.

A.9 Computation

TO BE DONE.

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