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Social optima in economies with heterogeneous agents

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ABSTRACT

This paper analyzes the problem of computing the social optimum in models with heterogeneous agents subject to idiosyncratic shocks. This is equivalent to a deterministic optimal control problem in which the state variable is the infinite-dimensional cross-sectional distribution. We show how, in continuous time, the problem can be broken down into two *finite*-dimensional partial differential equations: a dynamic programming equation and the law of motion of the distribution, and we introduce a new numerical algorithm to solve it. We illustrate this methodology with two examples: social optima in an Aiyagari economy with stochastic lifetimes and in a model of on-the-job search with learning.

1. Introduction

Models with heterogeneous agents have become a workhorse in macroeconomics since the seminal work of Bewley (1986), Hopenhayn (1992), Huggett (1993) and Aiyagari (1994).[2] The *competitive equilibrium* of these models has a common structure. Agents make choices taking as given some aggregate variables that depend on the entire distribution of individuals in the economy. Agents' choices together with idiosyncratic shocks in turn determine the evolution of this distribution. The equilibrium is characterized by the dynamic programming equation that describes the intertemporal problem of each agent, by the law of motion of the distribution and by the market clearing conditions that link individual choices to aggregate variables.

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[2] For a survey of this literature, see e.g. Heathcote et al. (2009).

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Less progress has been made in the analysis of the *social optimum* in models with heterogeneous agents. Defined as the allocation produced by a benevolent central planner who maximizes aggregate welfare, it is the natural benchmark against which to compare the competitive equilibrium. The planner decides the actions of each atomistic agent depending on the individual state and on the aggregate state of the economy. The challenge in models with nontrivial heterogeneity is that the aggregate state of the economy is an infinite-dimensional object.

In this paper we propose a novel methodology to analyze the social optimum in heterogeneous-agent models. The idea is that finding the social optimum is equivalent to solving a deterministic optimal control problem in a suitable infinite-dimensional Hilbert space.[3] The planner maximizes a social welfare functional, which aggregates utility across agents, subject to the law of motion of the distribution and to the market clearing conditions. As we work in continuous time the dynamics of the cross-sectional distribution are described by a partial differential equation known as the *Kolmogorov forward* (KF) or *Fokker–Planck* equation.[4] The necessary conditions can be obtained by computing the *Gateaux derivatives*, which generalize the concept of derivative to function spaces.

We show that, if a solution to the planning problem exists, it can be *decentralized* as an *auxiliary competitive equilibrium*. This equilibrium includes a modified dynamic programming equation – known in continuous time as *Hamilton–Jacobi–Bellman* (HJB) equation – the law of motion of the distribution and the market clearing conditions. The auxiliary equilibrium differs from the competitive equilibrium in that the HJB equation includes an extra term that captures the fact that individuals at different points in the space state affect the cross-sectional distribution and thereby the social planner's objective. This extra term contains the Lagrange multipliers associated to each of the market clearing conditions. The values of these Lagrange multipliers are themselves endogenous functions of the cross-sectional distribution and of the individual policies. If the Lagrange multipliers are zero the social optimum and the competitive equilibrium coincide.

We also introduce a numerical algorithm to compute the solution of social planning problems. The idea is to iterate on a sequence of auxiliary equilibria. Given an initial guess for the value of the Lagrange multipliers a competitive equilibrium is computed with the modified HJB equation. The Lagrange multipliers are then updated as a function of the equilibrium objects and the process is iteratively repeated until the multipliers converge. The solution of the auxiliary competitive equilibrium in each iteration is computed by the finite difference method presented in Achdou et al. (2017).

A variety of problems in economics can be analyzed using the methodology presented here. We provide two examples. First we analyze the normative properties of an incomplete-market economy *à la* Aiyagari (1994) with stochastic-lifetimes as in the "perpetual youth" models of Yaari (1965) and Blanchard (1985). Second we analyze the efficient allocation in a model of on-the-job search with learning, similar to Menzio and Shi (2011) or Li and Weng (2016).

In the Aiyagari economy we consider two optimality concepts. First we compute the *constrained-efficient* allocation, defined as the allocation in which the social planner decides the consumption of each agent while respecting all individual budget constraints. As discussed by Diamond (1967) and Dávila et al. (2012), this is a notion of efficiency that does not allow the planner to directly overcome the friction implied by missing markets. Dávila et al. (2012) compute the constrained-efficient allocation in a standard discrete-time Aiyagari model using calculus of variations and find that the competitive equilibrium is constrained inefficient as agents do not internalize the effect of their individual saving decisions on the aggregate capital stock. Second we compute the *first-best* allocation, in which the planner may redistribute among agents. In this case only the resource constraint is relevant.

Two opposite forces operate in the competitive equilibrium. On the one hand it is well known, at least since the original Blanchard (1985) paper, that stochastic lifetimes generate capital *underaccumulation* as agents save less than in the counterfactual case with infinite lifetimes. On the other hand, incomplete markets à la Aiyagari–Bewley–Hugget generate capital *overaccumulation* due to precautionary savings motives. It is not clear a priori which of the two forces will prevail in our setting.

We present a number of new theoretical results characterizing the steady-state policies and the stationary distribution. In the constrained-efficient allocation consumption is constant for agents with large asset holdings: they all consume the same amount independently of their wealth or productivity. This contrasts with the competitive equilibrium, in which consumption is proportional to wealth for agents with large asset holdings, and with the first-best allocation, in which consumption is constant for *all* agents. Aggregate variables in the first best coincide with those in the infinite-horizon representative-agent model. In the three cases the distribution asymptotically follows a power law, albeit with different tail exponents.[5]

We then calibrate the economy as in the original Aiyagari (1994) paper and numerically compute the steady-state social optima. We find that the interest rate in the constrained-efficient allocation is smaller than the subjective discount rate whereas in the competitive equilibrium the interest rate is larger. In the first-best allocation the interest rate equals the discount rate. As aggregate capital is inversely proportional to the interest rate, capital in the constrained-efficient allocation

[3] A Hilbert space is a particular function space possessing an inner product. It is the natural generalization of the Euclidean space to function spaces. For instance, the space of square-integrable functions, L^2, is a Hilbert space. In contrast, L^p spaces for $p \neq 2$ are not Hilbert spaces.

[4] There is a long tradition of continuous-time macroeconomic models with heterogeneous agents. Some examples are Jovanovic (1979), Luttmer (2007), Alvarez and Shimer (2011) or Rocheteau et al. (2015). See Achdou et al. (2014) for a recent survey.

[5] The emergence of power laws in the wealth distribution has been analyzed in a number of previous papers, such as Wold and Whittle (1957), Benhabib and Bisin (2007), Benhabib et al. (2011, 2015), Piketty and Zucman (2015), Jones (2015), Acemoglu and Robinson (2015) and Gabaix et al. (2016), among others.

is higher than in the first best, which in turn is higher than in the competitive equilibrium. The competitive equilibrium thus displays capital underaccumulation, both compared to the first best and to the constrained-efficient allocation. These results contrast starkly with the case of infinite lifetimes analyzed in Aiyagari (1994) and Dávila et al. (2012) in which the competitive equilibrium always has more capital than the first best. In the case of the original Aiyagari calibration considered here, Dávila et al. (2012) find that the constrained-efficient allocation displays ever-increasing wealth inequality and thus no stationary allocation can be computed. Finally, we compute the consumption-equivalent welfare gains in both the first-best and the constrained-efficient allocation and find them to be very close to each other and of first order with respect to the competitive equilibrium.

The second example we consider is a model of on-the-job search in which transitions of workers between unemployment and employment and across employers are driven by heterogeneity in the quality of firm-worker matches. Each firm-worker pair gradually learns its unknown match quality based on cumulative output using Bayes' rule as in Jovanovic (1984), Moscarini (2005) or Papageorgiou (2014). Search is directed in the sense that a worker knows the conditions offered by different firms before choosing where to apply for a job. In the efficient allocation a benevolent planner decides when and where each worker should search a job based on the posterior belief about the match quality.

Menzio and Shi (2011) and Li and Weng (2016) find that the socially efficient solution is *block recursive*, meaning that individual value and policy functions do not depend on the distribution of workers across employment states.[6] They also find that the competitive equilibrium is efficient in the sense that it decentralizes the social planner's allocation. We show how these results are a particular case of the general theory presented here. The absence of aggregate variables or market clearing conditions in the model implies that the Lagrange multipliers in the planner's HJB equation are zero. As discussed above, when the Lagrange multipliers are zero the competitive equilibrium is optimal.

The assumption of *directed search* is key to obtain these results. In contrast to Menzio and Shi (2011) and Li and Weng (2016) we can also characterize the optimality conditions in the case of *random search* as in Mortensen (1982) and Pissarides (1985). In this case, the solution is not block recursive as the planner's HJB equation depends on the cross-sectional distribution through the optimal labor market tightness. We obtain an equation for the value of the market tightness, or equivalently of the job finding rate, that makes the model efficient thus providing a useful benchmark to evaluate models with exogenous random search.

Related literature. Few papers have analyzed the constrained-efficient optima in models with heterogeneous agents. Up to our knowledge the first contribution was the discrete-time analysis of Dávila et al. (2012) discussed above. The continuous-time approach presented here differs from this early work in two main aspects. The first is that we characterize the problem in terms of the planner HJB equation instead of the Euler equation. More precisely, we show how the planner's problem can be broken into individual HJB equations in which the value function for each person is her marginal social value under an optimal plan. The HJB equation for this marginal social value can then be compared with the HJB equation in the competitive equilibrium, thereby obtaining an easily interpretable formula that precisely characterizes the externalities causing the planner's allocation to differ from the equilibrium one. The second lies in the approach to compute the evolution of the cross-sectional distribution. Traditional discrete-time methods either simulate a large number of agents by Monte Carlo methods or discretize the state-space. In contrast, in continuous time the distributional dynamics are characterized by a partial differential equation: the KF equation. We take advantage of this fact and develop an efficient and flexible computational algorithm using finite-difference methods that applies to a wide class of planning problems in which a distribution is the relevant state variable.

A couple of recent papers analyze social optima in continuous-time models with heterogeneous agents. Lucas and Moll (2014) solve an optimal planning problem subject to the law of motion of the cross-sectional distribution. Their formulation nevertheless does not consider the possibility of including aggregate constraints, such as market clearing conditions, which are prevalent in most economic problems. Here instead we analyze the general problem. This requires the use of functional analysis, in particular of optimization techniques in infinite-dimensional Hilbert spaces, in order to derive the necessary conditions for a solution. Another continuous-time paper that analyzes the social optimum with heterogeneous agents is Afonso and Lagos (2015). They assume that their state variables can only take a *finite* number of values, in contrast to a continuum, and thus they can avoid the problem of optimization in infinite-dimensional spaces that we analyze here. In many applications it is more natural to work with continuous state variables, for example in models of wealth distribution like the one analyzed in the present paper.

Our paper is related to the literature that employs infinite dimensional analysis in dynamic programming problems, such as Gozzi and Faggian (2004), Fabbri and Gozzi (2008), Boucekkine et al. (2013) or Fabbri et al. (2015). These papers analyze in detail the mathematical properties of particular models in which analytical solutions can be obtained or tightly characterized, such as vintage capital, spatial AK production, forestry management or population dynamics models. In contrast to this literature, here we provide a general methodology and a numerical strategy to solve optimal control problems in models with heterogeneous agents in which an analytical solution may not available.

Finally, the paper is also related to the emerging literature in mathematics and engineering analyzing *mean field games*, introduced by Lasry and Lions (2006a, 2006b) and Huang et al. (2003). The name is borrowed from the mean-field approximation in statistical physics, in which the effect on any given individual of all the other individuals is approximated by a

[6] See also Shi (2009).

single averaged effect. Huang et al. (2012) and Nourian et al. (2013) analyze the decentralization of the social optimum in linear-quadratic models with heterogeneous agents subject to idiosyncratic shocks. Here, instead, we focus on the general case of nonlinear dynamics.[7]

2. General approach

2.1. Competitive equilibrium

2.1.1. Individual problem

State. We consider a continuous-time infinite-horizon economy. Let $(\Sigma, F, \{F_t\}, \mathbb{P})$ be a filtered probability space. There is a continuum of unit measure of ex-ante identical agents indexed by $i \in [0, 1]$. The duration of an agent's life is uncertain. Death is governed by a Poisson random variable with rate η. At the time of death each agent is replaced by a single child so that the size of the population is constant.

Let B_t^i be a n-dimensional F_t-adapted Brownian motion and $X_t^i \in \mathbb{R}^n$ denote the state of an agent i at time $t \in [0, \infty)$. The state evolves according to a multidimensional Itô process of the form

$$dX_t^i = b\left(X_t^i, \mu(t, X_t^i), Z_t\right) dt + \sigma\left(X_t^i\right) dB_t^i, \tag{1}$$

where $\mu \in \mathbb{R}^m$ is a vector of control variables and $Z_t \in \mathbb{R}^p$ is a deterministic vector of aggregate variables. Here the instantaneous drift $b(\cdot)$ and volatility $\sigma(\cdot)$ are measurable functions, $b \in C^1(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p)$ and $\sigma \in C^2(\mathbb{R}^n)$.[8] In the Appendix A we provide some technical assumptions to ensure the existence of a solution of the stochastic differential equation (1).

The control vector $\mu : [0, \infty) \times \mathbb{R}^n \to M \subset \mathbb{R}^m$ is a F_t-adapted Markov control where M is a closed subset of $\mathbb{R}^m$.[9] The control strategy is the same for every agent, but it depends on time and on the specific state of the agent.[10] The control $\mu(t, x)$ is admissible if for any initial point (t, x) such that $X_t^i = x$ the stochastic differential equation (1) has a unique solution. We allow for *state constraints* in which the state X_t^i is restricted to a compact subset $\Omega \subset \mathbb{R}^n$. Given the state x at time t, $X_t^i = x$, we denote $\mathcal{M}(t, x)$ as the space of all admissible controls contained in the set of all Markov controls.[11]

Preferences. Agents maximize their discounted utility

$$U = \mathbb{E}_t\left[\int_t^{\infty} e^{-(\rho+\eta)(s-t)} u(X_s, \mu) ds | X_t = x\right],$$

where utility $u(x, \mu) \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$ is strictly increasing and strictly concave and the subjective discount factor $\rho > 0$ is constant.[12] The optimal *value function* $V(t, x)$ is defined as the maximized value of U subject to the state dynamics (1) where the maximization is over all admissible controls that respect the state constraint.

The transversality condition is

$$\lim_{t\to\infty} e^{-\rho t} V(t, x) = 0. \tag{2}$$

Hamilton–Jacobi–Bellman equation. The solution to this problem is given by a value function $V(t, x)$ and a control strategy $\mu(t, x)$ that satisfy the HJB equation

$$\rho V = \frac{\partial V}{\partial t} + \max_{\mu \in M} \{u(x, \mu) + \mathcal{A}V\}, \tag{3}$$

for all x on the interior of Ω and with state constraint boundary conditions on the boundary of Ω, where $\mathcal{A}$ is a differential operator defined as

$$\mathcal{A}V = \sum_{i=1}^{n} b_i(x, \mu, Z) \frac{\partial V}{\partial x_i} + \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\left(\sigma(x)\sigma(x)^\top\right)_{i,k}}{2} \frac{\partial^2 V}{\partial x_i \partial x_k} - \eta V. \tag{4}$$

Notice that $\frac{\partial}{\partial t} + \mathcal{A}$ is the *infinitesimal generator* of process (1)

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[7] Our approach is related to the mean field control discussed in Bensoussan et al. (2013).

[8] $C^k(\mathbb{R}^n)$ is the set of all k-times continuously differentiable functions on $\mathbb{R}^n$.

[9] For tractability, we restrict our attention to Markovian controls.

[10] For simplicity we restrict ourselves to the case in which the volatility cannot be controlled. Notwithstanding the results in this paper can be extended to the case of controlled volatility.

[11] State constraints are actually restrictions on the admissible controls. The state space is $\mathbb{R}^n$, as stated above. For a detailed analysis on state constraints in dynamic programming problems, please consult Capuzzo-Dolcetta and Lions (1990), Bardi and Capuzzo-Dolcetta (1997, pp. 271–281) or Fleming and Soner (2006, pp. 7, 106–111).

[12] We drop the superindex i from now on as there is no possibility of confusion.

$$\left(\frac{\partial}{\partial t}+\mathcal{A}\right) f(t,x)=\lim_{\tau\to 0}\frac{\mathbb{E}_0\left[f(t+\tau,X_{t+\tau})|X_t=x\right]-f(t,x)}{\tau}, \tag{5}$$

a partial differential operator that encodes the main information about the process.[13]

2.1.2. Aggregate distribution and aggregate variables

Kolmogorov forward equation. We assume than the new cohorts of agents are born with an initial state x_0 drawn from a density $\psi(x)$. Assume that the transition measure of X_t with initial value x_0 has a transition distribution $G(t,x;x_0)$ and a transition density $g(t,x;x_0)\in L^2([0,\infty)\times\mathbb{R}^n)$,[14] i.e., that for any $\varphi\in L^2(\mathbb{R}^n)$:

$$\mathbb{E}_0\left[\varphi(X_t)|X_0=x_0\right]=\int\varphi(x)dG(t,x;x_0)=\int\varphi(x)g(t,x;x_0)dx.$$

The initial density of X_t at time $t=0$ is $g(0,x)=g_0(x)$. The dynamics of the density of agents $g(t,x)=\int g(t,x;x_0)g_0(x_0)dx_0$ are given by the Kolmogorov Forward (KF) equation

$$\frac{\partial g}{\partial t}=\mathcal{A}^*g+\eta\psi, \tag{6}$$

$$\int g(t,x)dx=1, \tag{7}$$

where $\mathcal{A}^*$ is the *adjoint operator* of $\mathcal{A}$[15]:

$$\mathcal{A}^*g=-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[b_i(x,\mu,Z)g(t,x)\right]+\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2}{\partial x_i\partial x_k}\left[\left(\sigma(x)\sigma(x)^\top\right)_{i,k}g(t,x)\right]-\eta g. \tag{8}$$

The first term in the right-hand side of equation (8) describes the changes in the distribution due to the drift (known as "advection" in Physics), the second term describes the changes due to the volatility ("diffusion") and the last term describes changes due to the death of agents.

Market clearing. The vector of aggregate variables is determined by a system of p equations:

$$Z_k(t)=\int f_k(x,\mu)g(t,x)dx,\quad k=1,..,p, \tag{9}$$

where $f_k\in C^1\left(\mathbb{R}^n\times\mathbb{R}^m\right)$. These equations are typically the *market clearing* conditions of the economy.

We may define a competitive equilibrium in this economy.

Definition 1 *(Competitive equilibrium).* A competitive equilibrium is a vector of aggregate variables $Z(t)$, a value function $V(t,x)$, a control $\mu(t,x)$ and a density $g(t,x)$ such that

1. Given $Z(t)$, $V(t,x)$ is the solution of the HJB equation (3) and the optimal control is $\mu(t,x)$.
2. Given $\mu(t,x)$ and $Z(t)$, $g(t,x)$ is the solution of the KF equation (6), (7).
3. Given $\mu(t,x)$ and $g(t,x)$, the aggregate variables $Z(t)$ satisfy the market clearing conditions (9).

2.2. Optimal allocation

Social welfare. We now study the allocation of a planner who chooses a vector of control variables $\mu(t,x)$ to be applied to every agent $i\in[0,1]$ with state dynamics (1). The planner also chooses the vector of aggregate variables Z_t given the constraints (9). The planner chooses the controls and the aggregate variables in order to maximize the discounted *social welfare function* (SWF)

$$U^{oa}=\int_0^\infty e^{-\rho t}\int\omega(t,x)u(x,\mu)g(t,x)dxdt,$$

where $\omega(t,x)$ are state-dependent Pareto weights. If $\omega(t,x)=1$ we have a purely utilitarian social welfare function. Notice that the planner discounts future utility flows at rate ρ, not at rate $(\rho+\eta)$.

[13] See, for example, Øksendal (2010).

[14] $L^2(\Phi)$ is the space of functions on Φ with a square that is Lebesgue-integrable.

[15] The adjoint operator generalizes the concept of matrix transpose for infinite-dimensional operators. See Appendix B.1 for more information.

The planner's optimal value functional is

$$J\left(g\left(0,\cdot\right)\right):=\max_{Z(\cdot),\ \mu(\cdot)\in\mathcal{M}(t,x)} U^{oa}\left(\mu\left(\cdot\right),Z\left(\cdot\right)\right), \tag{10}$$

subject to law of motion of the density (6), (7) and to the market clearing conditions (9).

The optimal value *functional* J is a mapping from the space of initial densities $g\left(0,\cdot\right)$ into the real numbers. The planner's problem with heterogeneous agents is an extension of the classical optimal control problem to an infinite-dimensional setting, in which the state is the whole density of individual states $g(t,x)$. The problem is deterministic, as so is the KF equation.

Planner's HJB. We provide necessary conditions to the problem (10).

Proposition 1 *(Necessary conditions). If a solution to problem (10) exists with $e^{-\rho t}g$, $e^{-\rho t}\mu\in L^2\left([0,\infty)\times\mathbb{R}^n\right)$ and $e^{-\rho t}Z\in L^2[0,\infty)$ it can be represented as an auxiliary competitive equilibrium in which the planner's value function $j\left(t,x\right)$ satisfies the HJB*

$$\rho j=\max_{\mu\in M}\omega(t,x)u(x,\mu)+\sum_{k=1}^{p}\lambda_k(t)\left[f_k(x,\mu)-Z_k\right]+\mathcal{A}j+\frac{\partial j}{\partial t}, \tag{11}$$

for all x on the interior of Ω and with state constraint boundary conditions on the boundary of Ω, where the Lagrange multipliers $\lambda_k(t)$, $k=1,..,p$ are given by

$$\lambda_k(t)=\sum_{i=1}^{n}\int\frac{\partial j}{\partial x_i}\frac{\partial b_i}{\partial Z_k}g(t,x)dx, \tag{12}$$

and the optimal value functional $J\left(g\left(0,\cdot\right)\right)$ can be expressed as

$$J\left(g\left(0,\cdot\right)\right)=\int j(0,x)g(0,x)dx+\eta\int_0^\infty\int e^{-\rho t}j\left(t,x\right)\psi\left(x\right)dxdt. \tag{13}$$

The proof can be found in the Appendix B.1. The idea is to construct a Lagrangian in a subspace of $L^2\left([0,\infty)\times\mathbb{R}^n\right)$ in which $j(t,x)$ is the Lagrange multiplier of the KF equation (6) and $\lambda_k(t)$ are the Lagrange multipliers of the market clearing conditions (9). We then compute the Gateaux derivatives with respect to the policies μ, the density g and the aggregate variables Z_k. We interpret $j(t,x)$ as the marginal social value function, representing the value to the planner of an agent with a state x at time t. The Lagrange multipliers $\lambda_k(t)$ reflect the 'shadow prices' of the market clearing condition. They price, in utility terms, the deviation of an agent from the value of the aggregate variable: $f_k(x,\mu)-Z_k$.

The proposition is the main result of the paper. It shows how the planner's problem can be represented as an auxiliary competitive equilibrium with an modified HJB (11) including the term $\sum_{k=1}^{p}\lambda_k(t)\left[f_k(x,\mu)-Z_k\right]$ that captures the impact of an individual agent on the aggregate variables. The value of the Lagrange multiplier is given by equation (12). Naturally, in the case of a utilitarian planner ($\omega=1$) if the value of all the Lagrange multipliers is zero the solution of the standard competitive equilibrium coincides with the social optimum.

Corollary 1 *(Optimality of the competitive equilibrium). The competitive equilibrium coincides with the planner's allocation in the utilitarian case ($\omega=1$) if*

$$\tilde{\lambda}_k(\cdot)=0,\ \ k=1,..,p, \tag{14}$$

where $\tilde{\lambda}_k(t)$ are given by

$$\tilde{\lambda}_k(t)=\sum_{i=1}^{n}\int\frac{\partial V}{\partial x_i}\frac{\partial b_i}{\partial Z_k}g(t,x)dx. \tag{15}$$

Notice that we have replaced $j(t,x)$ by $V(t,x)$ in (15), that is, the marginal social value equals the individual value. Therefore, it is enough to solve the competitive equilibrium and to compute (14) to check whether it is socially optimal.

If the value function is twice differentiable, $j\in C^2\left(\Omega\right)$, then the planner's HJB is a "classical solution". If this is not the case, the general concept of a solution to the HJB with state constraints (11) is that of a "*viscosity solution*" (Crandall and Lions, 1983; Crandall et al., 1992). The numerical procedure presented in the next section is able to accommodate viscosity solutions.

It is important to remark that Proposition 1 only provides necessary conditions to the problem (10). Neither existence nor uniqueness of a solution is guaranteed. In particular, it can be the case that multiple solutions satisfy the conditions in Proposition 1. In that case, the optimal allocation will be the one that maximizes the social welfare.

3. Example 1: the constrained-efficient allocation in an incomplete-markets economy with stochastic lifetimes

As a first example, we analyze an extension of the incomplete-markets economy of Aiyagari (1994) with stochastic-life agents as in the "perpetual youth" models of Yaari (1965) and Blanchard (1985). Our aim is to analyze two optimality concepts, the constrained-efficient allocation and the first-best, and to compute them numerically by introducing a new algorithm.

3.1. Model

There is a representative firm with a constant returns to scale production function $Y = F(K, L) = AK^{\alpha}L^{1-\alpha}$, where K is the aggregate capital, L is aggregate labor and A is a positive constant. Capital depreciates at rate δ_K. Since factor markets are competitive, the interest rate and wage are given by

$$r_t = \frac{\partial F\left(K_t, 1\right)}{\partial K} - \delta_K = \alpha \frac{Y_t}{K_t} - \delta_K, \qquad (16)$$
$$w_t = \frac{\partial F\left(K_t, 1\right)}{\partial L} = (1-\alpha)\frac{Y_t}{L_t}.$$

There is a continuum of mass unity of agents that are heterogeneous in their wealth a and labor productivity z. The duration of an agent's life is uncertain. Lifetimes are stochastic and governed by an exponential random variable with mean $1/\eta$. At the time of death each agent is replaced by a single child so that the size of the population is constant. Agents have standard preferences over utility flows from future consumption c_t discounted at rate $\rho > 0$. We assume CRRA preferences, such that $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 1$. The expected discounted utility is

$$U_0 = \mathbb{E}_0\left[\int_0^{\infty} e^{-(\rho+\eta)t}u(c_t)dt\right]. \qquad (17)$$

Individuals have no intergenerational altruism. They purchase an annuity in a perfectly competitive insurance market that pays them a flow ηa_t in exchange of taking control of all the assets when the agent dies.[16] Each agent supplies z_t efficiency units of labor to the labor market and these get valued at wage w_t. An agent's wealth evolves according to

$$da_t = \left[w_t z_t + (r_t + \eta)\, a_t - c_t\right] dt = s\left(a_t, z_t, w_t, r_t, c_t\right) dt, \qquad (18)$$

where $s\left(a_t, z_t, w_t, r_t, c_t\right)$ is the *drift* of the detrended wealth process.

Individual labor productivity evolves stochastically over time on a bounded interval $[\underline{z}, \bar{z}]$ with $\underline{z} \geq 0$, according to a bounded Ornstein–Uhlenbeck process[17]:

$$dz_t = \theta(\hat{z} - z_t)dt + \sigma dB_t, \qquad (19)$$

where B_t is a F_t-adapted idiosyncratic Brownian motion and θ, $\hat{z}$ and σ are positive constants.

Agents also face a borrowing limit,

$$a_t \geq -\phi, \qquad (20)$$

where $\phi \geq 0$ is a constant such that the "natural borrowing limit" is not binding:

$$-\phi > -\underline{z}\int_t^{\infty} e^{-\int_t^s r_\tau d\tau} w_s ds, \quad \forall t \geq 0.$$

3.2. State distribution and market clearing

As described above, agents leave no bequest. New agents begin with zero assets. They are also born with a labor productivity level of $\underline{z}$. The state of the economy is the joint density of wealth and labor, $g(t, a, z)$. The dynamics of the density are given by the Kolmogorov Forward (KF) equation

$$\frac{\partial g}{\partial t} = -\frac{\partial}{\partial a}\left(s\left(a, z, w\left(t\right), r\left(t\right), c\right) g\right) - \frac{\partial}{\partial z}\left(\theta(\hat{z} - z)g\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2 g\right) - \eta g + \eta\delta_0. \qquad (21)$$

[16] The amount of resources collected from expired agents is ηK_t, which equals the flow of payments so that insurance companies make no profits.

[17] This is the continuous-time counterpart of the AR(1).

The term $-\eta g(t,a,z)$ is the outflow of agents due to death and the term $\eta\delta_0 = \eta\delta(a)\,\delta\left(z-\underline{z}\right)$ is the inflow of newborn agents with zero assets and productivity $\underline{z}$.[18] The distribution satisfies the normalization

$$\int\int_{\underline{z}}^{\bar{z}} g(t,a,z)dzda = 1.$$

We assume that the value of $\hat{z}$ in (19) is such that

$$\int\int_{\underline{z}}^{\bar{z}} zg(t,a,z)dzda = 1. \tag{22}$$

There are two market clearing conditions. First, the total amount of capital supplied in the economy equals the total amount of wealth

$$K_t = \int\int_{\underline{z}}^{\bar{z}} ag(t,a,z)dzda. \tag{23}$$

Second, the total amount of effective labor supplied in the economy equals the aggregated productivity, which is one due to assumption (22):

$$L_t = \int\int_{\underline{z}}^{\bar{z}} zg(t,a,z)dzda = 1.$$

Remark 1. The mapping of the variables in this example to the general case described in Section 2 is as follows. The individual state includes wealth and productivity $X_t = [a_t, z_t]'$ and the individual control is consumption $\mu(t,x) = c(t,a,z)$. The only aggregate variable is capital $Z_t = K_t$. The drift and volatility of the state are

$$b(x,\mu,Z) = \begin{bmatrix} (1-\alpha)K^\alpha z + \left[\left(\alpha K^{\alpha-1}-\delta_K\right)+\eta\right]a - c \\ \theta(\hat{z}-z) \end{bmatrix},$$
$$\sigma(x) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma z \end{bmatrix}.$$

The instantaneous utility is $u(x,\mu) = \frac{c^{1-\gamma}}{1-\gamma}$, the exogenous newborn density is given by $\psi(x) = \delta(a)\,\delta\left(z-\underline{z}\right)$ and the market clearing condition is such that $f(x,\mu) = a$. The process z_t is reflected at the boundary $[\underline{z}, \bar{z}]$ and hence it should satisfy the boundary condition

$$-\theta(\hat{z}-z)g + \frac{1}{2}\frac{\partial}{\partial z}\left(\sigma^2 g\right) = 0,\; z = \left\{\bar{z}, \underline{z}\right\}. \tag{24}$$

The state space in this example is the subspace $\mathbb{R}\times[\underline{z},\bar{z}]$ of $\mathbb{R}^2$.[19] We consider a (very large) upper bound $a^{\max}$ introduced in order to make the subdomain $\Omega = [-\phi, a^{\max}]\times[\underline{z},\bar{z}]$ compact. The set of admissible controls is denoted $\mathcal{C}(t,a,z)$.

[18] $\delta(\cdot)$ is the Dirac delta, not to confound with the depreciation rate. The Dirac delta is a *distribution* or generalized function such that

$$\delta(f(\cdot)) = \int_{-\varepsilon}^{\varepsilon} f(x)\delta(x)\,dx = f(0),\;\; \forall\varepsilon>0,\; f\in L^1(-\varepsilon,\varepsilon).$$

A heuristic characterization is that

$$\int_{-\infty}^{\infty}\delta(x)\,dx = 1,\;\; \delta(x) = \begin{cases} \infty, & x=0, \\ 0, & x\neq 0. \end{cases}$$

[19] The fact that the state space is a subspace of $\mathbb{R}^2$ does not modify the results of the previous section. The boundary condition (24) ensures that the operators $\mathcal{A}$ and $\mathcal{A}^*$ are still adjoints in this case. A proof is available upon request.

3.3. Competitive equilibrium

Consider first the competitive equilibrium in which agents decide their individual consumption levels. The optimal value function results in

$$V(t,a,z) = \max_{c(\cdot)\in\mathcal{C}(t,a,z)} U_0(c(\cdot)), \tag{25}$$

subject to evolution of individual wealth (18) and of individual productivity (19).

The Hamilton–Jacobi–Bellman (HJB) equation of the individual problem is

$$(\rho+\eta)V = \max_{c\geq 0} \frac{c^{1-\gamma}}{1-\gamma} + s(a,z,w(t),r(t),c)\frac{\partial V}{\partial a} + \theta(\hat{z}-z)\frac{\partial V}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2} + \frac{\partial V}{\partial t}, \tag{26}$$

for all $a \in (-\phi, \infty)$ and with state constraint boundary conditions on $a = -\phi$.

A competitive equilibrium in this economy is defined as follows.

Definition 2 *(Competitive equilibrium).* A competitive equilibrium is composed by an aggregate capital stock $K(t)$, a value function $V(t,a,z)$, a consumption policy $c(t,a,z)$ and a density $g(t,a,z)$ such that

1. Given K, V is the solution of the individual's problem (25) and the optimal control is c.
2. Given K and c, g is the solution of the KF equation (21).
3. Given g and K, the capital market (23) clears.

The following proposition provides an asymptotic characterization of the solution.

Proposition 2 *(Competitive equilibrium – Aiyagari). Provided that the steady-state interest rate satisfies $r > \rho$, consumption is proportional to wealth for large wealth values*

$$c(a) = \frac{\rho + \gamma\eta - (1-\gamma)r}{\gamma} a, \ \text{as } a \to \infty$$

and the stationary wealth distribution is characterized by an asymptotic power law, $g(a) \sim a^{-(1+\zeta)}$ as $a \to \infty$, with tail exponent

$$\zeta = \frac{\eta\gamma}{r-\rho}. \tag{27}$$

Proof. See Appendix B.2. □

As discussed in Benhabib et al. (2016), an Aiyagari–Bewley–Huggett model with stochastic lifetimes à la Blanchard–Yaari produces a power-law wealth distribution, as it is typically found in the data.

3.4. Constrained-efficient allocation

Consider now the *constrained-efficient* allocation of this economy, in which the planner decides the consumption level of each individual agent, as in Dávila et al. (2012). In this case the planner is constrained to consider allocations with zero net transfers across individuals. The question is whether the planner can improve on the market allocation by simply commanding different levels of consumption, while respecting all individual budget constraints. We consider a utilitarian SWF so that the objective of the social planner is ex-ante expected utility.

The problem of the planner is to choose individual consumption $c(\cdot)$ across agents in order to maximize the discounted aggregate utility

$$J(g(0,\cdot)) = \max_{c(\cdot)\in\mathcal{C}(t,a,z)} \int_0^\infty e^{-\rho t} \int u(c)\, g(t,a,z)\, da dz dt, \tag{28}$$

subject to the law of motion of the aggregate density (21), to the factor prices (16) and to the market clearing condition (23).[20]

We may directly apply the result of Proposition 1.

[20] Notice that the planner gives the same weight to every agent irrespective of its age. This contrasts with the SWF chosen in Benhabib and Bisin (2007) which only considers the welfare of the agents alive at an arbitrary time. Notice also that the planner discounts future aggregate utility flows at the same rate of individual agents ρ, not at rate $(\rho+\eta)$, as it also gives a positive weight to unborn agents.

Proposition 3. *The social value function $j(t, a, z)$ solves the planner's HJB equation*

$$(\rho + \eta)\, j = \max_{c \geq 0} \frac{c^{1-\gamma}}{1-\gamma} + \lambda\,(a - K(t)) + (w(t)\, z + (r(t) + \eta)\, a - c)\, \frac{\partial j}{\partial a} + \theta(\hat{z} - z)\frac{\partial j}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j}{\partial z^2} + \frac{\partial j}{\partial t}, \tag{29}$$

for all $a \in (-\phi, \infty)$ and with state constraint boundary conditions on $a = -\phi$. The Lagrange multiplier $\lambda(t)$ is

$$\begin{aligned} \lambda(t) &= \int \int_{\underline{z}}^{\bar{z}} \frac{\partial j}{\partial a}\left(\frac{\partial r}{\partial K} a + \frac{\partial w}{\partial K} z\right) g(t,a,z) dz da \\ &= -\frac{\alpha\,(1-\alpha)}{K(t)^{2-\alpha}} \int \int_{\underline{z}}^{\bar{z}} \frac{\partial j}{\partial a}\,(a - K(t)\, z)\, g(t,a,z) dz da. \end{aligned} \tag{30}$$

Notice that equation (29) extends the individual HJB (26) to include the term $\lambda(t)\,(a - K(t))$ which reflects the difference between the agent's assets a and the aggregate capital K. If the Lagrange multiplier λ is positive, the social value of agents wealthier than the average is higher than their private value. If λ is negative, the social value of these agents is lower than the private value. This is related to the concept of a *pecuniary externality*: individual agents do not internalize that their saving decisions affect the aggregate amount of capital, which affects the rest of agents through wages and interest rates. The planner does take this effect into account when computing the optimal individual saving decision and thus the optimal wealth allocation.

Taking into account the first order condition for the planner's HJB

$$c(t,a,z)^{-\gamma} = \frac{\partial j}{\partial a},$$

the Lagrange multiplier can be expressed as

$$\lambda(t) = -(1-\alpha)\,(r_t + \delta_K) \int \int_{\underline{z}}^{\bar{z}} c(t,a,z)^{-\gamma}\,[a/K(t) - z]\, g(t,a,z) dz da, \tag{31}$$

that is, it is proportional to the average $a/K(t) - z$ weighted by the marginal utility $c^{-\gamma}$.

The asymptotic characterization of the solution is presented in the following proposition. The proof can be found in the Appendix B.3.

Proposition 4 *(Constrained-efficient allocation – Aiyagari). Provided that the steady-state interest rate satisfies $r < \rho$ and the Lagrange multiplier λ is positive, steady-state consumption is constant for large wealth values*

$$\bar{c} = \left(\frac{\lambda}{\rho - r}\right)^{-1/\gamma}, \text{ as } a \to \infty$$

and the stationary wealth distribution is characterized by an asymptotic power law, $g(a) \sim a^{-(1+\zeta)}$ as $a \to \infty$, with tail exponent

$$\zeta = \frac{\eta}{(r+\eta)}. \tag{32}$$

Notice that in the efficient allocation the planner asymptotically assigns a constant level of consumption, irrespective of productivity or level of assets. This contrasts with the competitive equilibrium, in which consumption is directly proportional to wealth. The stationary distribution is also a power law in this case, but the tail exponent differs from that in the competitive equilibrium.

3.5. First-best allocation

An alternative optimality concept is the *first-best* allocation, in which the planner is able to fully redistribute across agents. Consider that the individual wealth in (18) is now given by

$$da_t = (w_t z_t + (r_t + \eta)\, a_t - c_t + \tau_t)\, dt = s\,(a_t, z_t, w_t, r_t, c_t)\, dt, \tag{33}$$

where τ_t are transfers across agents. These transfers can be positive or negative. The aggregate amount of transfers is zero

$$\int \int_{\underline{z}}^{\bar{z}} \tau(t,a,z) g(t,a,z) dz da = 0. \tag{34}$$

The problem is similar to the case of constrained efficiency described above, with the inclusion of a new individual policy $\tau(t,a,z)$ and a new market clearing condition (34). We may again apply the result of Proposition 1 taking into account that no aggregate variable corresponds to the new market clearing condition.

Proposition 5. *The social value function $j(t,a,z)$ solves the planner's HJB equation*

$$(\rho+\eta)\,j = \max_{c\geq 0,\tau} \frac{c^{1-\gamma}}{1-\gamma} + \lambda\,(a - K\,(t)) + \varphi\,(t)\,\tau + (w\,(t)\,z + (r\,(t)+\eta)\,a - c + \tau)\,\frac{\partial j}{\partial a} \\ + \theta(\hat{z}-z)\frac{\partial j}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j}{\partial z^2} + \frac{\partial j}{\partial t}, \tag{35}$$

for all $a \in (-\phi, \infty)$ and with state constraint boundary conditions on $a = -\phi$. The Lagrange multiplier $\lambda\,(t)$ is defined as in equation (30).

The value of the new Lagrange multiplier $\varphi\,(t)$ is pinned-down by the new market clearing condition (34). The complete solution can be obtained analytically.

Proposition 6 *(First-best allocation – Aiyagari). In the first-best allocation the steady-state interest rate is given by $r = \rho$, steady-state consumption is constant for all wealth values*

$$\tilde{c} = \left(\frac{\alpha}{\rho+\delta_K}\right)^{\frac{\alpha}{1-\alpha}} - \delta_K\left(\frac{\alpha}{\rho+\delta_K}\right)^{\frac{1}{1-\alpha}},$$

and the stationary wealth distribution is characterized by an asymptotic power law, $g(a) \sim a^{-(1+\zeta)}$ as $a \to \infty$, with tail exponent

$$\zeta = \frac{\eta}{(\rho+\eta)}.$$

Proof. See Appendix B.4. □

In the first-best allocation the values of the aggregate variables coincide with the case of a representative agent. The asymptotic distribution is again a power-law. Notice that although the exponent is less than 1 the distribution has a well-defined mean as we are assuming that there exists a maximum value $a^{\max} < \infty$ that makes the subdomain compact.

3.6. Numerical analysis

We solve numerically the steady-state of the constrained-efficient allocation using the algorithm described in the Appendix C. In order to solve the HJB and the KF equations, we employ the upwind finite difference method proposed in Achdou et al. (2017). The finite difference method converges to the unique viscosity solution of this problem (Barles and Souganidis, 1991). The idea of the method is to approximate the value function $V(t,a,z)$ and the density $g(t,a,z)$ on a finite grid with steps Δa and Δz and to compute derivatives as differences.[21]

Calibration. We calibrate the model using the Aiyagari (1994) calibration, as in Dávila et al. (2012). Let the unit of time be one year, such that all rates are in annual terms. The capital share parameter, α, is taken to be 0.36 and the depreciation rate of capital, δ_K, is 0.08. The discount rate ρ, is set to 0.04. The intertemporal elasticity of substitution $\frac{1}{\gamma}$ is set to 0.5 so that the risk aversion is 2.[22] The borrowing constraint, ϕ is set to 0. The mean of the productivity process, $\hat{z}$, is set to 1.038 so that $L = 1$. The autocorrelation is 0.6 and the coefficient of variation of 0.2 so that $\theta = 0.4$ and $\sigma = 0.16$.[23] Our model requires an additional parameter, the death rate, η, which we set to 2 percent, equivalent to an average lifetime of 50 years. Finally, in order to solve numerically the model, we employ a grid with 300 points in wealth, ranging from 0 to 100, and 40 points in income, from 0.2 to 1.8. We introduce an upper bound to the wealth distribution of 100, equivalent to around 20 times the average wealth in the competitive equilibrium, in order to capture most of the right tail of the distribution. Results are robust to changes in the size of the domain.

Multiplicity of equilibria. As discussed above, there is the possibility of multiple solutions to the auxiliary competitive equilibrium given by Proposition 1. Any solution should be a fixed point $T\lambda^* = \lambda^*$ of the operator

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[21] See Chapter IX in Fleming and Soner (2006) for an introduction to numerical methods in stochastic control.

[22] This value differs in Dávila et al. (2012) from Aiyagari (1994). We employ the value in Dávila et al. (2012) in order to improve the comparability about the efficient allocation.

[23] In Aiyagari (1994) the logarithm of the productivity follows a zero-mean AR(1) approximated using a Markov chain whereas in our case it is approximated by a unit mean positively bounded Ornstein–Uhlenbeck process in levels.

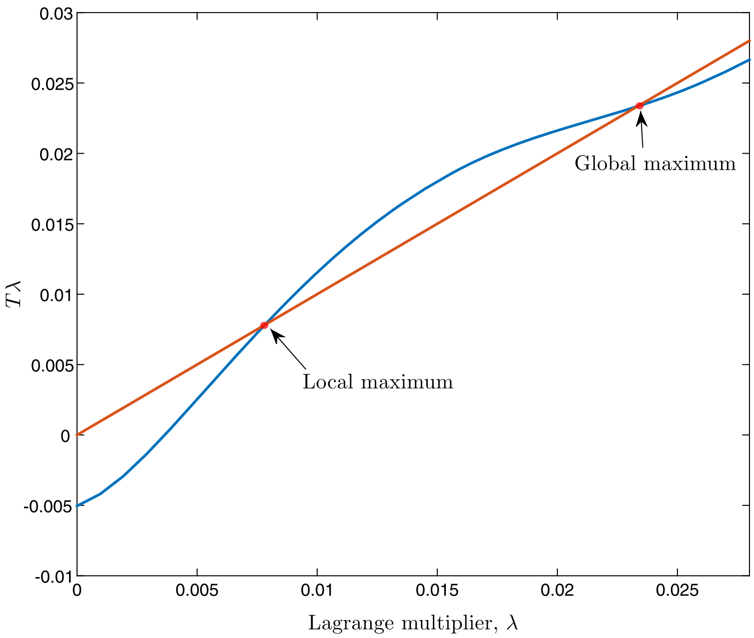

Fig. 1. The operator $T\lambda$ in steady-state.

$$T\lambda = -(1-\alpha)\left[r(t)+\delta_K\right]\int\int_{\underline{z}}^{\bar{z}}\left[c(t,a,z)\right]^{-\gamma}\left[a/K(t)-z\right]g(t,a,z)\,dzda,$$

where r, c, K and g are equilibrium objects. Fig. 1 displays the operator in the steady-state, that is, when it maps constant values λ to values $T\lambda$ after solving the steady-state planning problem. It shows how the problem displays two solutions.[24] We need to compute the social welfare and to compare it to the competitive equilibrium in order to find the global maximum. It turns out that the solution with $\lambda = 0.0078$ is a local maximum whereas the solution with $\lambda = 0.0233$ is the global maximum.

Results. Dávila et al. (2012) found that when the model with infinite lifetimes is calibrated as in Aiyagari (1994) the constrained-efficient allocation involves ever-increasing wealth inequality, as the rich save more and more, a feature that we verify in the case $\eta = 0$. Here, instead, we focus on the case with finite and stochastic lifetimes ($\eta > 0$) and find that a stationary distribution exists even in the case of the original Aiyagari (1994) calibration. The value and policy functions as well as the state density for the constrained-efficient allocation (with $\lambda = 0.0233$) are displayed in Fig. 2 and those for the competitive equilibrium in Fig. 3. The main aggregate variables, including the results for the first best, are reported in Table 1. Notice first that consumption is increasing with wealth in the competitive equilibrium but it is constant for most states in the constrained-efficient allocation. It only declines for low-wealth, low-productivity agents close to the borrowing limit. The constant value of consumption coincides with the one in Proposition 4, $c = \left(\frac{\lambda}{\rho - r}\right)^{-1/\gamma} = 1.506$, and the interest rate is negative $r = -1.29$ percent. In the first best, according to Proposition 6, consumption is constant for *all* states with a value of 1.41. The constrained-efficient allocation thus almost replicates the first-best for unconstrained agents but it is similar to the competitive equilibrium for agents close to the borrowing limit.

In the first-best the level of capital is given by the condition

$$K^{\text{first best}} = \left(\frac{\alpha}{r+\delta_K}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\alpha}{\rho+\delta_K}\right)^{\frac{1}{1-\alpha}}.$$

The interest rates satisfy

$$r^{\text{const. eff.}} < \rho < r^{\text{comp. equil.}},$$

which implies

$$K^{\text{comp. equil.}} < K^{\text{first best}} < K^{\text{const. eff.}}.$$

[24] We have explored a larger region of the λ space without finding any other fixed point.

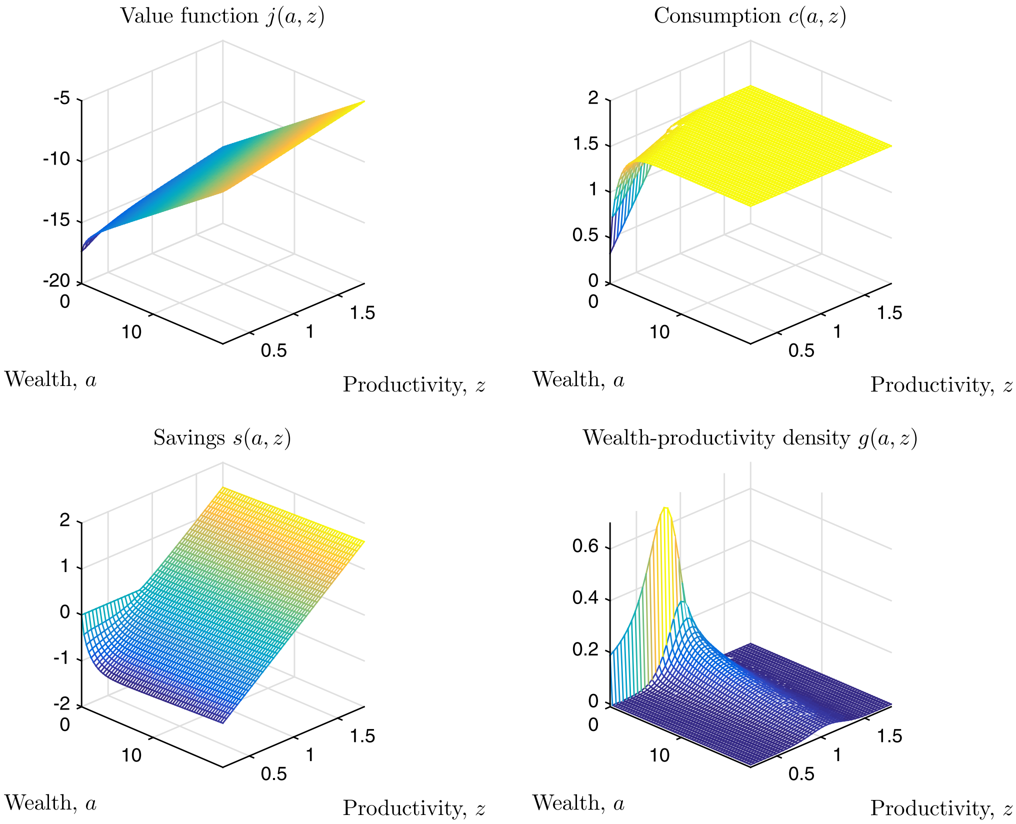

Fig. 2. Constrained-efficient allocation.

The constrained-efficient allocation displays more capital than the first best, which in turn is overcapitalized compared to the competitive equilibrium. The underaccumulation of capital in the competitive equilibrium is in line with the original result of Blanchard (1985) in a stochastic-lifetime setting with complete markets and no idiosyncratic risk. Notice also how wealth inequality, which is proportional to the inverse of the tail exponent, ranks higher in the first-best, then in the constrained-efficient allocation and finally in the competitive equilibrium.

These results are in stark contrast with the conclusions in Aiyagari (1994) and Dávila et al. (2012) about the model with infinite lifetimes. First, in that model the interest rate in the competitive equilibrium $r^{comp.\ equil.}$ is *less* than the subjective discount factor ρ and the competitive equilibrium displays capital overaccumulation due to precautionary savings by individual agents. Second, in the constrained-efficient allocation interest rates $r^{const.\ eff.}$ can be higher or lower than those in the competitive equilibrium and, at least in the numerical exercises, they are below the ones in the first best. The constrained-efficient allocation in that model can thus display capital over- or underaccumulation depending on the numerical calibration. In the case of the original Aiyagari calibration considered here, Dávila et al. (2012) find that the constrained-efficient allocation displays ever-increasing wealth inequality and thus no stationary allocation can be computed.

Welfare. Aggregate welfare in the stationary case is

$$U = \frac{1}{(\rho+\eta)} \int \int_{\underline{z}}^{\bar{z}} u(c)\, g(a,z)\, dz da. \tag{36}$$

In order to compare the three allocations, we express the ratio of welfare in consumption equivalent terms, that is, we express it as the proportion Θ of increase in the stationary consumption $c(a,z)$ of all agents in the competitive equilibrium to attain the same welfare as in the optimal allocation (constrained-efficient or first-best):

$$U^{oa} = \int \int_{\underline{z}}^{\bar{z}} u^{oa}(c)\, g^{oa}(a,z)\, dz da = \int \int_{\underline{z}}^{\bar{z}} u^{ce}(c(1+\Theta))\, g^{ce}(a,z)\, dz da = (1+\Theta)^{1-\gamma}\, U^{ce},$$

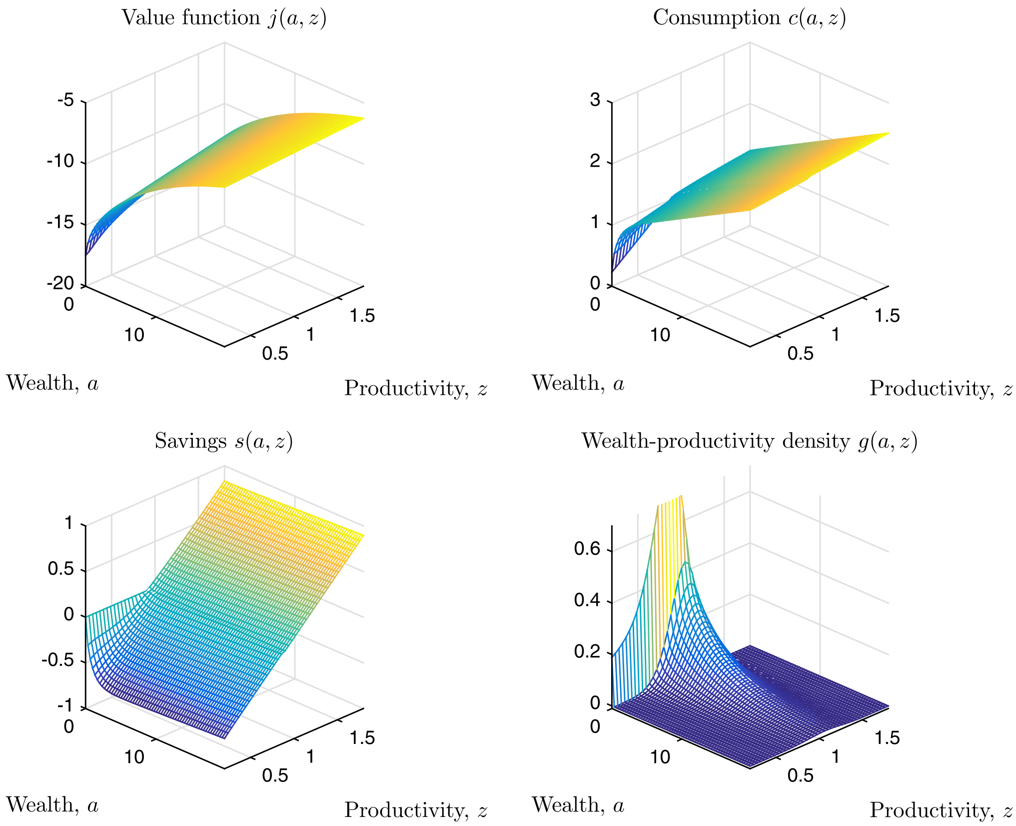

Fig. 3. Competitive equilibrium.

Table 1
Model results.

	Constrained-efficient	Competitive equilibrium	First-best
Aggregate capital, K	13.82	5.04	5.57
Output, Y	2.57	1.79	1.86
Capital-output ratio, K/Y	5.37	2.82	1.57
Aggregate consumption, C	1.45	1.39	1.41
Wages, w	1.65	1.15	1.19
Interest rate (%), r	−1.29	4.79	4.00
Tail exponent, ζ	2.83	5.08	0.33
Welfare (% cons. c.e.), Θ	15.13	–	15.41

and hence

$$\Theta = \left(\frac{U^{oa}}{U^{ce}}\right)^{\frac{1}{1-\gamma}} - 1. \tag{37}$$

In the case of the constrained-efficient allocation there is a gain of 15.13 percent in consumption-equivalent terms, inferior but very close to the gain in the first-best (15.41 percent).[25]

4. Example 2: efficient learning and matching in the labor market

As a second example, we analyze the efficient allocation in a continuous-time version of Menzio and Shi (2011) with Gaussian learning, similar to Li and Weng (2016). This problem differs from the one presented in Section 2 in the introduction of a Markov chain in addition to the Brownian motion. As we show below, this does not pose any significant difference in terms of the methodology.

[25] In order to prevent numerical mistakes when computing welfare associated with the upper bound on wealth, for asset values larger than 50 we set the value equal to the theoretical constant.

4.1. Model

The economy is populated by a unit measure of workers and a sufficiently large measure of long-lived firms to ensure free entry. Workers and firms are risk-neutral and *ex-ante* homogeneous. The discount factor is r. A consumption good is produced by pairwise firm-worker matches. Matches are destroyed according to a Poisson distribution with rate ρ. The cumulative output of a match of duration t, X_t, follows a Brownian motion with drift μ and known variance σ^2

$$dX_t = \mu dt + \sigma dB_t,$$

where μ is ex ante uncertain, idiosyncratic, and randomly assigned by Nature upon matching. μ can take two values $\mu \in \{\mu_H, \mu_L\}$, where μ_H denotes a high-quality match and μ_L denotes a low-quality match such that $\mu_H > \mu_L$.

The realized performance is public information so both parties update their beliefs about the quality of the match μ using Bayes' rule. Let α_t denote the posterior belief $\alpha_t := \Pr\left(\mu = \mu_H | X^t\right)$ conditional on all previous information $X^t = \{X_s\}_{s=0}^t$. Given the prior beliefs

$$\Pr(\mu = \mu_H) = \alpha_0 \text{ and } \Pr(\mu = \mu_L) = 1 - \alpha_0, \quad \alpha_0 \in (0,1),$$

the posterior belief α_t is given by

$$d\alpha_t = \alpha_t (1-\alpha_t) \frac{s}{\sigma} \left[dX_t - \alpha_t \mu_H dt - (1-\alpha_t)\mu_L dt\right] = \alpha_t (1-\alpha_t) s dB_t,$$

where $s = (\mu_H - \mu_L)/\sigma$.

The individual state is given by $\alpha_t \in [0,1]$ when a worker is employed and by $\alpha_t = -1$ when the worker is unemployed. The expected output of a worker is

$$y(\alpha) = \begin{cases} \alpha\mu_H - (1-\alpha)\mu_L, & \text{if } \alpha \in [0,1], \\ b, & \text{if } \alpha = -1, \end{cases}$$

where b is a positive constant.

At any instant, the planner sends workers and firms searching for new matches at different locations and chooses which locations each (employed and unemployed) worker should search. The planner finds it optimal to send workers in different individual states to search in different locations but has no incentive to send workers in the same individual state to different locations. At each location, the workers and the vacancies meet (and match) according to a constant returns-to-scale matching technology that can be described in terms of the tightness of the location $\theta(t,\alpha)$: a worker (employed or unemployed) meets a vacancy at a rate $p(\theta)$ where $p \in C^2(\mathbb{R}_+)$ is a strictly increasing and concave function such that $p(0) = 0$, and $\lim_{\theta\to\infty} p(\theta) = \bar{p} < \infty$ The per-worker social cost of maintaining a tightness θ is given by $k\theta$ where k is a positive constant.

4.2. State distribution

The state of the economy is given by the probability density $g(t,\alpha) : [0,\infty) \times \mathbb{R} \to \mathbb{R}_+$, which can be expressed as

$$g(t,\alpha) = \begin{cases} g^e(t,\alpha), & \text{if } \alpha \in [0,1], \\ u(t)\,\delta_{-1}(\alpha), & \text{if } \alpha = -1, \\ 0, & \text{if } \alpha \notin [0,1] \times \{-1\}, \end{cases}$$

where $g^e(t,\alpha)$ is the density of employed workers, $u(t)$ is the measure of unemployed workers and δ_{-1} is the Dirac delta centered at -1. It satisfies

$$\int g(t,\alpha)\,d\alpha = \int_0^1 g^e(t,\alpha)\,d\alpha + u(t) = 1.$$

The dynamics of the density $g(t,\alpha)$ are given by the KF equation

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial t}\begin{bmatrix} g^e \\ u\delta_{-1}(\alpha) \end{bmatrix} = \mathcal{A}^* g, \tag{38}$$

where the operator $\mathcal{A}^*$ is given by

$$\mathcal{A}^* g = \begin{bmatrix} -(\rho + p(\theta))\,g^e + \frac{\partial^2}{\partial\alpha^2}\left(\Sigma(\alpha)\,g^e\right) + \left(p(\theta)\,u + \int_0^1 p(\theta)\,g^e d\alpha\right)\delta_{\alpha_0}(\alpha) \\ -p(\theta)\,u\delta_{-1}(\alpha) + \left(\rho \int_0^1 g^e d\alpha\right)\delta_{-1}(\alpha) \end{bmatrix}.$$

Here δ_{α_0} is the Dirac delta centered at α_0 and

$$\Sigma(\alpha) = \frac{1}{2}s^2\alpha^2(1-\alpha)^2.$$

The term $-(\rho + p(\theta))g^e$ accounts for the random destruction of matches: $\rho\int_0^1 g^e d\alpha$ go to unemployment whereas $\int_0^1 p(\theta)g^e d\alpha$ are reinserted back into employment with initial state α_0. The term $\frac{\partial^2}{\partial\alpha^2}\left(\Sigma(\alpha)g^e\right)$ captures the density changes due to arriving information. Finally, the term $p(\theta)u$ is the measure of unemployed workers finding a job.

Lemma 1. *The adjoint of $\mathcal{A}^*$ is the infinitesimal generator $\mathcal{A}$ of the underlying process, given by*

$$\mathcal{A}V(\alpha) = \left[\begin{array}{c} \rho\left[V(-1) - V(\alpha)\right] + p(\theta)\left[V(\alpha_0) - V(\alpha)\right] + \Sigma(\alpha)\frac{\partial^2 V}{\partial\alpha^2}, \ \text{if } \alpha \in [0,1], \\ p(\theta)\left[V(\alpha_0) - V(-1)\right], \ \text{if } \alpha = -1. \end{array}\right]$$

Proof. See Appendix B.5. □

This is the infinitesimal generator of the Poisson-diffusion process of the individual state. It is a diffusion with volatility $\sqrt{2\Sigma(\alpha)} = s\alpha(1-\alpha)$ in the interval $[0,1]$ which may "jump" to state -1 at a rate ρ and to state $\alpha_0 \in [0,1]$ at a rate $p(\theta(\alpha))$. In state -1 the process remains constant until it jumps to state α_0 at a rate $p(\theta(-1))$.

Remark 2. The mapping of the variables in this example to the general case described in Section 2 is as follows. The individual state is the posterior belief $X_t = \alpha_t$. The regime (employed or unemployed) is modeled as a 2-state Markov chain with transition rates towards a new employment $p(\theta(-1))$ and to unemployment ρ. The individual control is the tightness of the location $\mu(x) = \theta(\alpha)$. There are no aggregate variables. The drift and volatility of the state are

$$b(x,\mu) = 0,$$
$$\sigma(x) = \Sigma(\alpha).$$

The instantaneous utility is $u(x,\mu) = y(\alpha) - k\theta$. There are no exogenous newborns density or market clearing conditions. The domain is $\mathbb{R}$ but the process is only nonzero in the compact set $[0,1] \cup \{-1\}$. The set of admissible controls is $L^2(\mathbb{R};\mathbb{R}_+)$ as there are no state constraints. Notice that the variables are stationary, that is, they do not depend on time. This is a consequence of the absence of any aggregate variable linking the aggregate state with the individual ones.

4.3. Planner's problem

At time t, a benevolent social planner observes the state of the economy g and chooses the tightness at a location $\theta(\alpha) \in \mathbb{R}_+$. The planners maximizes net aggregate output, that is, aggregate output minus the aggregate social cost of vacancy creation. The planner's value functional is

$$J(g(0,\cdot)) = \max_{\theta(\cdot)\in\mathcal{M}} \int_0^\infty e^{-rt}\left[\int (y(\alpha) - k\theta(\alpha))\,g(t,\alpha)\,d\alpha\right]dt, \tag{39}$$

subject to the KF equation (38). This problem is just a particular instance of the one described in Section 2 in which there are no aggregate market clearing conditions. The solution is just a particular case of Proposition 1 in a 2-dimensional setting.

Proposition 7 *(Directed search). If a solution to problem (39) exists with $e^{-rt}g(t,\alpha) \in L^2([0,\infty)\times\mathbb{R})$, $\theta(\alpha) \in L^2(\mathbb{R})$ then the optimal value functional $J(g(0,\cdot))$ is separable*

$$J(g(0,\cdot)) = \int_0^1 j(\alpha)g^e(0,\alpha)d\alpha + j(-1)u(t), \tag{40}$$

where $j(\alpha)$ is the marginal social value function. The social value function satisfies the HJB

$$rj(\alpha) = \max_{\theta\geq 0} y(\alpha) - k\theta + \mathcal{A}j,$$

which can be expressed as

$$rj(\alpha) = \max_{\theta\geq 0} y(\alpha) - k\theta + \rho(j(-1) - j(\alpha)) + p(\theta)(j(\alpha_0) - j(\alpha)) + \Sigma(\alpha)j''(\alpha), \ \text{if } \alpha \in [0,1]$$
$$rj(-1) = \max_{\theta\geq 0} y(-1) - k\theta + p(\theta)(j(\alpha_0) - j(-1)), \ \text{if } \alpha = -1.$$

Proposition 3 is the same as Theorem 1 in Li and Weng (2016) with no endogenous separation. Notwithstanding, the proof is based on a completely different strategy to that in Li and Weng (2016). Here it is just a particular case of the general theory exposed in Section 2 whereas Li and Weng (2016) develop a specific proof for this particular model based on the separability of the planner's value functional, the verification of the optimality of the controls and the convexity of $j(\alpha)$.

Following Corollary 1, as the Lagrange multipliers are zero the efficient allocation can be replicated by a competitive equilibrium in which each worker decides her individual tightness, a result first discussed in Menzio and Shi (2011).

4.4. Random search

Menzio and Shi (2011) and Li and Weng (2016) underline the importance of the assumption of directed search or state-contingent tightness $\theta(\alpha)$ in their solution strategy, as it yields to the separability of the planner's problem. Here, however, we may modify this assumption and consider instead random search as in Mortensen (1982) and Pissarides (1985). In the case of random search the planner assigns the same tightness $\theta(t)$ to every worker. The result is given by the following Proposition. The proof can be found in Appendix B.6.

Proposition 8 *(Random search). If a solution to problem (39) with random search $\theta(t)$ exists with $e^{-rt}g(t,\alpha) \in L^2([0,\infty)\times\mathbb{R})$, $e^{-rt}\theta(t) \in L^2([0,\infty))$ then the optimal value functional $J(g(0,\cdot))$ can be expressed as in (40) with a social value function*

$$rj(t,\alpha) = y(\alpha) - k\theta(t) + \rho\left(j(t,-1) - j(t,\alpha)\right) + p(\theta)\left(j(t,\alpha_0) - j(t,\alpha)\right) + \Sigma(\alpha)\frac{\partial^2 j}{\partial\alpha^2} + \frac{\partial j}{\partial t}, \quad \text{if } \alpha \in [0,1]$$

$$rj(t,-1) = y(-1) - k\theta(t) + p(\theta)\left(j(t,\alpha_0) - j(t,-1)\right) + \frac{\partial j}{\partial t}, \quad \text{if } \alpha = -1.$$

If the optimal tightness solution is interior, it is given by

$$p'(\theta) = -\frac{k}{\left[\int_0^1 j(t,\alpha)\,g^e d\alpha + u(t)\,j(t,-1) - j(t,\alpha_0)\right]}. \tag{41}$$

In the steady-state the value of the optimal tightness θ is constant and given by equation (41). The optimal job finding rate $p(\theta)$ is also constant. This provides a criterion to evaluate the optimality of models with exogenous job finding rates as Jovanovic (1984), Moscarini (2005) or Papageorgiou (2014). If the exogenous rate equals $p(\theta)$ where θ is as in (41), then the decentralized allocation with exogenous rates is efficient.[26]

Proposition 8 shows that the efficient allocation in this case is not block recursive. Block recursiveness is lost as the optimal tightness is a function of the cross-sectional distribution.

5. Conclusions

This paper analyzes the problem of a social planner who maximizes aggregate welfare in a model with heterogeneous agents subject to idiosyncratic shocks. If the problem is formulated in continuous time, the KF equation provides a deterministic law of motion of the entire distribution of state variables across agents. The problem can thus be analyzed as one of deterministic optimal control in a suitable function space. If a solution exists, we show how it can be decentralized as an auxiliary competitive equilibrium in which the individual dynamic programming equation includes an extra term capturing the marginal impact of each agent on the aggregate variables. We also provide a new numerical algorithm to solve the problem in the stationary case.

We illustrate this methodology with two examples. First we analyze the welfare properties of an Aiyagari economy with stochastic lifetimes. We consider two alternative concepts of optimality: the constrained-efficient allocation, in which the planner maximizes aggregate welfare subject to the same equilibrium budget constraints as the individual agents, and the first best in which the planner may redistribute freely across agents. We provide theoretical results characterizing the optimal policies and the distribution in both cases and in the competitive equilibrium and we compute numerically the steady-state. We find that, in sharp contrast to the case with infinite lifetimes, in the competitive equilibrium the aggregate capital stock is depressed relative to the first best allocation. Furthermore, capital in the constrained-efficient allocation is higher than capital in the other two cases. We also find that in the constrained-efficient allocation the planner provides the same consumption level to agents with asset levels far enough from the borrowing limit. This contrasts both with the first best, in which the planner provides the same consumption level to *all* agents, and with the competitive equilibrium, in which consumption grows with individual assets.

[26] The criterion is adapted to the particular model presented here and small changes would need to be included in order to address issues such as different job finding rates between employed and unemployed as in Jovanovic (1984) or Moscarini (2005) or the possibility of not accepting an offer as in Papageorgiou (2014).

Second, we analyze the efficient allocation in a model of on-the-job search with Gaussian learning. The case with directed search, in which the job finding rate is state-dependent and the solution is block recursive, has already been discussed in previous studies such as Menzio and Shi (2011) or Li and Weng (2016). Here we show how this case is just a particular instance of our general methodology. In addition, we consider the case with random (i.e. non state-contingent) search. We provide an equation that determines the job finding rate that makes the allocation optimal.

The methodology presented in this paper can, in principle, also be used to analyze the transitional dynamics of the social optimum given an initial distribution, which is left for future research. Nonetheless, in many economic problems such as the ones considered in Examples 1 and 2 much of the economic insights come from comparing the differences between the stationary solution in the competitive equilibrium and in the social optima.[27]

Appendix A. Technical assumptions

The assumptions are similar to those in Bensoussan et al. (2016):

1. *Lipschitz continuity.* There exists $K > 0$ such that

$$\left|b\left(x,\mu,Z\right)-b\left(x',\mu',Z'\right)\right| \le K\left(\left|x-x'\right|+\left|\mu-\mu'\right|+\left|Z-Z'\right|\right),$$
$$\left|\sigma\left(x\right)-\sigma\left(x'\right)\right| \le K\left(\left|x-x'\right|\right).$$

2. *Linear growth.* There exists $K > 0$ such that

$$\left|b\left(x,\mu,Z\right)\right| \le K\left(1+|x|+|\mu|+|Z|\right),$$
$$\left|\sigma\left(x\right)\right| \le K\left(1+|x|\right).$$

3. *Quadratic condition on the objective.* There exists $K > 0$ such that

$$\left|e^{-\rho t}\omega(t,x)u\left(x,\mu\right)g\left(t,x\right)-e^{-\rho t'}\omega(t,x')u\left(x',\mu'\right)g\left(t',x'\right)\right| \le K\begin{pmatrix}1+|t|+\left|t'\right|+|x|+\left|x'\right|\\ +|\mu|+\left|\mu'\right|\end{pmatrix} \cdot \begin{pmatrix}\left|t-t'\right|+\left|x-x'\right|\\ +\left|\mu-\mu'\right|\end{pmatrix}.$$

Appendix B. Proofs

B.1. Proof of Proposition 1

The idea of the proof is to solve problem (10) using differentiation techniques in infinite-dimensional Hilbert spaces.

Mathematical preliminaries. First we need to introduce some mathematical concepts.[28] Let $L^2(\Phi)$ be the space of functions with a square that is Lebesgue-integrable over $\Phi \subset \mathbb{R}^n$. It is a Banach space with the norm

$$\|g\|_{L^2(\Phi)} = \sqrt{\int_\Phi |g(x)|^2 \, dx},$$

that is, it is a complete normed vector space. In contrast to n-dimensional Banach spaces such as $\mathbb{R}^n$, $L^2(\Phi)$ is infinite-dimensional.

The space of Lebesgue-integrable functions $L^2(\Phi)$ with the inner product

$$\langle u, g\rangle_\Phi = \int_\Phi ugdx, \quad \forall u, g \in L^2(\Phi),$$

is a Hilbert space.

An *operator* T is a mapping from one vector space to another. For example, given the process X_t described in (1), its *infinitesimal generator* is an operator in $L^2(\Phi)$ defined by (4). The *adjoint operator* T^* of a linear operator T in a Hilbert space is defined by the equation

$$\langle u, Tg\rangle_\Phi = \left\langle T^* u, g\right\rangle_\Phi .$$

[27] Note, however, that this does *not* imply that one could just compute the social optimum, imposing stationarity from the from the very beginning. This is exactly the same distinction as that between the steady-state and the "Golden Rule" consumption level in the neoclassical growth model.

[28] All the contents here are adapted from Luenberger (1969), Gelfand and Fomin (1991), Sagan (1992) and Brezis (2011).

Example 1. We may verify that $\mathcal{A}$ and $\mathcal{A}^*$, defined in (4) and (8) respectively, are adjoint operators in $\mathbb{R}^n$. Given $\forall u, g \in L^2(\mathbb{R}^n)$

$$\begin{aligned}\langle g, \mathcal{A}u\rangle &= \int g\mathcal{A}u dx = \int g(x)\left(\sum_{i=1}^{n} b_i(x,\mu,Z)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\left(\sigma(x)\sigma(x)^\top\right)_{i,k}}{2}\frac{\partial^2 u}{\partial x_i \partial x_k} - \eta u(t,x)\right)dx \\ &= \int u\left(-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(gb_i)dx + \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2}{\partial x_i \partial x_k}\left(g\frac{\left(\sigma\sigma^\top\right)_{i,k}}{2}\right)dx - \eta g\right)dx \\ &= \int u\mathcal{A}^* g dx = \langle \mathcal{A}^* g, u\rangle,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbb{R}^n)$ and we have integrated by parts.[29]

Let $J(g): L^2(\Phi) \to \mathbb{R}$ be a *functional* of g. There are two concepts of differentials in Hilbert spaces.

Definition 3 *(Gateaux differential).* Let $J(g)$ be a functional and let h be arbitrary in $L^2(\Phi)$. If the limit

$$\delta J(g;h) = \lim_{\alpha\to 0}\frac{J(g+\alpha h) - J(g)}{\alpha} \tag{42}$$

exists, it is called the *Gateaux derivative* of J at g in the direction h. If the limit (42) exists for each $h \in L^2(\Phi)$, the functional J is said to be *Gateaux differentiable* at g.

If the limit exists, it can be expressed as $\delta J(g;h) = \frac{d}{d\alpha}J(g+\alpha h)\,|_{\alpha=0}$.
A more restricted concept is that of the Fréchet differential.

Definition 4 *(Fréchet differential).* Let h be arbitrary in $L^2(\Phi)$. If for fixed $g \in L^2(\Phi)$ there exists $\delta J(g;h)$ which is linear and continuous with respect to h such that

$$\lim_{\|h\|_{L^2(\Phi)}\to 0}\frac{|J(g+h) - J(g) - \delta J(g;h)|}{\|h\|_{L^2(\Phi)}} = 0,$$

then J is said to be *Fréchet differentiable* at g and $\delta J(g;h)$ is the *Fréchet differential* of J at g with increment h.

The following proposition links both concepts.

Proposition 9. *If the Fréchet differential of J exists at g, then the Gateaux differential exists at g and they are equal.*

Proof. See Luenberger (1969, p. 173). □

The familiar technique of maximizing a function of a single variable by ordinary calculus can be extended in infinite dimensional spaces to a similar technique based on more general differentials. We use the term *extremum* to refer to a maximum or a minimum over any set. A function $g \in L^2(\Phi)$ is a maximum of $J(g)$ if for all functions h, $\|h-g\|_{L^2(\Phi)} < \varepsilon$ then $J(g) \geq J(h)$. The following theorem is the Fundamental Theorem of Calculus.

Theorem 1. *Let J have a Gateaux differential, a necessary condition for J to have an extremum at g is that $\delta J(g;h) = 0$ for all $h \in L^2(\Phi)$.*

Proof. Luenberger (1969, p. 173), Gelfand and Fomin (1991, pp. 13–14) or Sagan (1992, p. 34). □

Finally, we can extend this result to the case of constrained optimization.

Theorem 2 *(Lagrange multipliers). Let H be a mapping from $L^2(\Phi)$ into $\mathbb{R}^n$. If J has a continuous Fréchet differential, a necessary condition for J to have an extremum at g under the constraint $H(g) = 0$ at the function g is that there exists a function $\lambda \in L^2(\Phi)$ such that the Lagrangian functional*

$$\mathcal{L}(g) = J(g) + \langle\lambda, H(g)\rangle_\Phi \tag{43}$$

is stationary in g, i.e., $\delta\mathcal{L}(g;h) = 0$.

[29] Notice that the boundary conditions are zero due to the fact that the domain of integration is $\mathbb{R}^n$ and $g \in L^2(\mathbb{R}^n)$.

Proof. Luenberger (1969, p. 243). □

We introduce a new space that will be quite useful in the proofs below.

Definition 5. We define $L^2(\Phi)_{(\cdot,\cdot)_\Phi}$ as the space of functions f such that

$$\int_\Phi e^{-\rho t}|f|^2 dx < \infty.$$

This is the space of functions f such that $e^{-\rho t} f \in L^2(\Phi)$.

Lemma 2. *The space $L^2(\Phi)_{(\cdot,\cdot)_\Phi}$ with the inner product $(f,g)_\Phi = \int_\Phi e^{-\rho t} fg dx$ is a Hilbert space.*

Proof. We need to show that $L^2(\Phi)_{(\cdot,\cdot)_\Phi}$ is complete, that is, that given a Cauchy sequence $\{f_n\}$ with limit $f : \lim_{n\to\infty} f_n = f$ then $f \in L^2(\Phi)_{(\cdot,\cdot)_\Phi}$. If $\{f_n\}$ is a Cauchy sequence then

$$\|f_n - f_m\|_{(\cdot,\cdot)_\Phi} \to 0, \text{ as } n, m \to \infty,$$

or

$$\|f_n - f_m\|^2_{(\cdot,\cdot)_\Phi} = \int_\Phi e^{-\rho t}|f_n - f_m|^2 dx = \left\langle e^{-\frac{\rho}{2}t}(f_n - f_m), e^{-\frac{\rho}{2}t}(f_n - f_m)\right\rangle_\Phi = \left\|e^{-\frac{\rho}{2}t}(f_n - f_m)\right\|^2_\Phi \to 0,$$

as $n, m \to \infty$. This implies that $\left\{e^{-\frac{\rho}{2}t} f_n\right\}$ is a Cauchy sequence in $L^2(\Phi)$. As $L^2(\Phi)$ is a complete space, then there is a function $\hat{f} \in L^2(\Phi)$ such that

$$\lim_{n\to\infty} e^{-\frac{\rho}{2}t} f_n = \hat{f} \tag{44}$$

under the norm $\|\cdot\|^2_\Phi$. If we define $f = e^{\frac{\rho}{2}t}\hat{f}$ then

$$\lim_{n\to\infty} f_n = f$$

under the norm $\|\cdot\|_{(\cdot,\cdot)_\Phi}$ that is, for any $\varepsilon > 0$ there is an integer N such that

$$\|f_n - f\|^2_{(\cdot,\cdot)_\Phi} = \left\|e^{-\frac{\rho}{2}t}(f_n - f)\right\|^2_\Phi = \left\|e^{-\frac{\rho}{2}t} f_n - \hat{f}\right\|^2_\Phi < \varepsilon,$$

where the last inequality is due to (44). It only remains to prove that $f \in L^2(\Phi)_{(\cdot,\cdot)_\Phi}$:

$$\|f\|^2_{(\cdot,\cdot)_\Phi} = \int_\Phi e^{-\rho t}|f|^2 dx = \int_\Phi \left|\hat{f}\right|^2 dx < \infty. \qquad \square$$

Construction of the Lagrangian. We define the extended domain $\Phi := [0,\infty) \times \mathbb{R}^n$. The problem of the planner is to maximize

$$\int_\Phi e^{-\rho t}\omega(t,x)u(x,\mu)g(t,x)dxdt = \left\langle e^{-\rho t}\omega u, g\right\rangle_\Phi = (\omega u, g)_\Phi,$$

subject to the KF equation (6)

$$-\frac{\partial g}{\partial t} + \mathcal{A}^* g + \eta\psi = 0, \ \forall (t,x) \in \Phi$$

and the market clearing condition (9)

$$\int (f_k(x,\mu) - Z_k(t))\, g(t,x)dx, \ \ k = 1, .., p, \ \forall t \in [0,\infty). \tag{45}$$

If a solution to the planner's problem (10) exists with $e^{-\rho t}g$, $e^{-\rho t}\mu \in L^2(\Phi)$ and $e^{-\rho t}Z \in L^2[0,\infty)$, we can express the Lagrangian functional (43) for this problem as

$$\mathcal{L}\left(g,\mu_1,..,\mu_m,Z_1,..,Z_p\right)=(\omega u,g)_\Phi+\left(j,-\frac{\partial g}{\partial t}+\mathcal{A}^*g+\eta\psi\right)_\Phi+\sum_{k=1}^{p}(\lambda_k,(f_k-Z_k)\,g)_\Phi \tag{46}$$
$$=\langle e^{-\rho t}\omega u,g\rangle_\Phi+\left\langle j,e^{-\rho t}\left(-\frac{\partial g}{\partial t}+\mathcal{A}^*g+\eta\psi\right)\right\rangle_\Phi+\sum_{k=1}^{p}\langle\lambda_k,e^{-\rho t}(f_k-Z_k)\,g\rangle_\Phi,$$

where $e^{-\rho t}j(t,x)\in L^2(\Phi)$ and $e^{-\rho t}\lambda_k(t)\in L^2[0,\infty)$, $k=1,..,p$ are the Lagrange multipliers associated to equations (6) and (9), respectively. As stated in Theorem 2, a necessary condition for $\left(g,\mu_1,..,\mu_m,Z_1,..,Z_p\right)$ to be a maximum of (46) is that the Gateaux derivative with respect to each of these functions equals zero.

It will prove useful to modify the second term in the Lagrangian

$$\left\langle j,e^{-\rho t}\left(-\frac{\partial g}{\partial t}+\mathcal{A}^*g+\eta\psi\right)\right\rangle_\Phi=\int_0^\infty\int e^{-\rho t}j(t,x)\left(-\frac{\partial g}{\partial t}+\mathcal{A}^*g+\eta\psi\right)dxdt \tag{47}$$
$$=-\int e^{-\rho t}j(t,x)\,g(t,x)\big|_0^\infty dx+\int_0^\infty\int e^{-\rho t}\left(\frac{\partial j}{\partial t}-\rho j(t,x)\right)gdxdt$$
$$+\eta\int_0^\infty\int e^{-\rho t}j(t,x)\,\psi(t,x)\,dxdt+\langle e^{-\rho t}\mathcal{A}j,g\rangle_\Phi$$
$$=-\lim_{T\to\infty}\int e^{-\rho T}j(T,x)\,g(T,x)\,dx+\int j(0,x)\,g(0,x)\,dx$$
$$+\eta\int_0^\infty\int e^{-\rho t}j(t,x)\,\psi(t,x)\,dxdt+\left\langle e^{-\rho t}\left(\frac{\partial j}{\partial t}-\rho j+\mathcal{A}j\right),g\right\rangle_\Phi,$$

where we have integrated by parts with respect to time in the term $\frac{\partial g}{\partial t}$ and applied the fact that $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$ and thus

$$\langle j,\mathcal{A}^*g\rangle=\langle\mathcal{A}j,g\rangle=\langle g,\mathcal{A}j\rangle.$$

Necessary conditions. The Gateaux derivative with respect to g in the direction $h(t,x)\in L^2(\Phi)_{(\cdot,\cdot)_\Phi}$ is

$$\frac{\partial}{\partial\alpha}\langle e^{-\rho t}\omega u,g+\alpha h\rangle_\Phi\Big|_{\alpha=0}+\frac{\partial}{\partial\alpha}\left\langle e^{-\rho t}\left(\frac{\partial j}{\partial t}-\rho j+\mathcal{A}j\right),g+\alpha h\right\rangle_\Phi\Big|_{\alpha=0}$$
$$+\frac{\partial}{\partial\alpha}\sum_{k=1}^{p}\langle e^{-\rho t}\lambda_k,(f_k-Z_k)(g+\alpha h)\rangle_\Phi\Big|_{\alpha=0}-\lim_{T\to\infty}\frac{\partial}{\partial\alpha}\int e^{-\rho T}j(T,x)\,(g(T,x)+\alpha h(T,x))\,dx\Big|_{\alpha=0}$$
$$+\frac{\partial}{\partial\alpha}\int j(0,x)\,(g(0,x)+\alpha h(0,x))\,dx\Big|_{\alpha=0}$$
$$=\langle e^{-\rho t}\omega u,h\rangle_\Phi+\left\langle e^{-\rho t}\left(\frac{\partial j}{\partial t}-\rho j+\mathcal{A}j\right),h\right\rangle_\Phi+\sum_{k=1}^{p}\langle e^{-\rho t}\lambda_k,(f_k-Z_k)\,h\rangle_\Phi-\lim_{T\to\infty}\int e^{-\rho T}j(T,x)\,h(T,x)\,dx,$$

where we have applied the fact that $h(0,\cdot)=0$ as the initial density $g(0,\cdot)=0$ is given. The Gateaux derivative should be zero for any $h(t,x)$. Therefore

$$\rho j(t,x)=\omega u+\sum_{k=1}^{p}\lambda_k(f_k-Z_k)+\mathcal{A}j+\frac{\partial j}{\partial t},\ \forall(t,x)\in\Phi, \tag{48}$$
$$\lim_{T\to\infty}e^{-\rho T}j(T,\cdot)=0, \tag{49}$$

which is the HJB equation of the planner (11).

The Gateaux derivative with respect to the individual control μ_j is

$$\frac{\partial}{\partial\alpha}\langle e^{-\rho t}\omega u\left(x,\mu_1,..,\mu_j+\alpha h..,\mu_m\right),g\rangle_\Phi\Big|_{\alpha=0}+\frac{\partial}{\partial\alpha}\left\langle e^{-\rho t}\left(\frac{\partial j}{\partial t}-\rho j+\mathcal{A}_{(\mu_j+\alpha h)}j\right),g\right\rangle_\Phi\Big|_{\alpha=0} \tag{50}$$
$$+\frac{\partial}{\partial\alpha}\sum_{k=1}^{p}\langle e^{-\rho t}\lambda_k,\left(f_k\left(x,\mu_1,..,\mu_j+\alpha h..,\mu_m\right)-Z_k\right)g\rangle_\Phi\Big|_{\alpha=0},$$

where $\mathcal{A}_{(\mu_j+\alpha h)}j$ is defined as

$$\mathcal{A}_{(\mu_j+\alpha h)}j := \sum_{i=1}^{n} b_i\left(x,\mu_1,..,\mu_j+\alpha h,..,\mu_m,Z\right)\frac{\partial j}{\partial x_i} + \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\left(\sigma(x)\sigma(x)^\top\right)_{i,k}}{2}\frac{\partial^2 j}{\partial x_i \partial x_k} - \eta j.$$

The space of admissible controls $\mathcal{M}(t,x)$ is defined as

$$\mathcal{M}(t,x) := \left\{\mu(\cdot) \in L^2\left([t,\infty)\times\mathbb{R}^n; M\right) : X_s \in \Omega,\ \text{for } t \le s,\ \text{with } X_t = x\right\}.$$

This definition implies that $\mu(t,x) \in M$ if $x \in \text{int}(\Omega)$ and it is such that X_t does not abandon Ω if $x \in \partial\Omega$. Provided that $x \in \text{int}(\Omega)$ then the condition that the Gateaux differential is zero yields

$$\mu = \arg\max_{\tilde{\mu}\in M}\left\{\omega u\left(x,\tilde{\mu}\right) + \sum_{k=1}^{p}\lambda_k f_k\left(x,\tilde{\mu}\right) + \mathcal{A}_{\tilde{\mu}} j\right\}. \tag{51}$$

The Gateaux derivative with respect to the aggregate variable Z_k is

$$\frac{\partial}{\partial\alpha}\left\langle e^{-\rho t}j, \mathcal{A}^*_{(Z_k+\alpha h)}g\right\rangle_\Phi\Big|_{\alpha=0} + \frac{\partial}{\partial\alpha}\left\langle e^{-\rho t}\lambda_k, \left(f_k - (Z_k+\alpha h)\right)g\right\rangle_\Phi\Big|_{\alpha=0}, \tag{52}$$

for any $h(t) \in L^2[0,\infty)_{(\cdot,\cdot)_{[0,\infty)}}$. $\mathcal{A}^*_{(Z_k+\alpha h)}$ is defined as

$$\mathcal{A}^*_{(Z_k+\alpha h)}g := -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[b_i\left(x,\mu,Z_1,..,Z_k+\alpha h,..,Z_p\right)g\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2}{\partial x_i\partial x_k}\left[\left(\sigma(x)\sigma(x)^\top\right)_{i,k}g\right] - \eta g.$$

Equation (52) can be expressed as

$$\frac{\partial}{\partial\alpha}\int_0^\infty\int e^{-\rho t}\left\{j(t,x)\left(-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[b_i\left(x,\mu,Z_1,..,Z_k+\alpha h,..,Z_p\right)g(t,x)\right]\right) - \lambda_k(t)\left(Z_k+\alpha h\right)g(t,x)\right\}dxdt\Bigg|_{\alpha=0}.$$

Taking the derivatives with respect to α and computing the limit this results in:

$$-\int_0^\infty e^{-\rho t}h(t)\left\{\int j(t,x)\left(\sum_{i=1}^{n}\left[\frac{\partial^2 b_i}{\partial Z_k\partial x_i}g(t,x) + \sum_{j=1}^{m}\frac{\partial^2 b_i}{\partial Z_k\partial\mu_j}\frac{\partial\mu_j}{\partial x_i}g(t,x) + \frac{\partial b_i}{\partial Z_k}\frac{\partial g}{\partial x_i}\right]\right)dx + \lambda_k(t)\right\}dt.$$

As the Gateaux derivative should be zero for any $h(t) \in L^2[0,\infty)_{(\cdot,\cdot)_{[0,\infty)}}$, we obtain an expression for $\lambda_k(t)$:

$$\lambda_k(t) = -\int j(t,x)\left\{\sum_{i=1}^{n}\left[\frac{\partial^2 b_i}{\partial Z_k\partial x_i}g(t,x) + \sum_{j=1}^{m}\frac{\partial^2 b_i}{\partial Z_k\partial\mu_j}\frac{\partial\mu_j}{\partial x_i}g(t,x) + \frac{\partial b_i}{\partial Z_k}\frac{\partial g}{\partial x_i}\right]\right\}dx.$$

Finally, integrating by parts[30]

$$\lambda_k(t) = \sum_{i=1}^{n}\int\frac{\partial j}{\partial x_i}\frac{\partial b_i}{\partial Z_k}g(t,x)dx. \tag{53}$$

Aggregate welfare. Finally, in order to show that the Lagrange multiplier j is indeed the social value function we multiply by $e^{-\rho t}g(t,x)$ at both sides of the planner's HJB equation (48) and integrate over Φ:

$$\int_0^\infty\int \rho e^{-\rho t}jgdxdt = \int_0^\infty\int e^{-\rho t}\left(\omega u + \sum_{k=1}^{p}\lambda_k\left(f_k - Z_k\right) + \mathcal{A}j + \frac{\partial j}{\partial t}\right)gdxdt. \tag{54}$$

Taking into account that, due to the market clearing condition (9),

$$\int\left(f_k - Z_k\right)fdx = 0,$$

and, as $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$,

[30] The boundary conditions are zero as the domain of integration is $\mathbb{R}^n$ and $g(t,\cdot) \in L^2\left(\mathbb{R}^n\right)$.

$$\int g\mathcal{A}jdx = \int j\mathcal{A}^* gdx,$$

then

$$\int_0^\infty e^{-\rho t} g\frac{\partial j}{\partial t}dt = e^{-\rho t} g(t,x)j(t,x)\big|_0^\infty - \int_0^\infty j\frac{\partial}{\partial t}\left(e^{-\rho t} g\right)dt$$
$$= -g(0,x)j(0,x) + \int_0^\infty e^{-\rho t}\rho jgdt - \int_0^\infty e^{-\rho t} j\frac{\partial g}{\partial t}dt,$$

and (54) results in

$$\int_0^\infty \int e^{-\rho t}\omega ugdxdt + \int_0^\infty \int e^{-\rho t} j\left(-\frac{\partial g}{\partial t} + \mathcal{A}^* g\right)dxdt = \int g(0,x)j(0,x)dx.$$

Given the KF equation $\frac{\partial g}{\partial t} = \mathcal{A}^* g + \eta\psi$;

$$\int_0^\infty \int e^{-\rho t} j\left(-\frac{\partial g}{\partial t} + \mathcal{A}^* g\right)dxdt = -\eta\int_0^\infty \int e^{-\rho t} j(t,x)\,\psi(x)\,dxdt.$$

The social welfare functional can then be expressed as

$$J\left(g(0,\cdot)\right) = \int_0^\infty \int e^{-\rho t}\omega ugdxdt = \int g(0,x)j(0,x)dx + \eta\int_0^\infty \int e^{-\rho t} j(t,x)\,\psi(x)\,dxdt.$$

B.2. Proof of Proposition 2

The proof is similar to that of Proposition 6 in Achdou et al. (2017). First we show that individual consumption is asymptotically linear in a as $a \to \infty$: $c \propto a$. We consider the auxiliary problem without labor income, wz, and without borrowing constraint ($\phi = \infty$), characterized by the HJB equation

$$(\rho+\eta)\,V(a) = \max_c \frac{c^{1-\gamma}}{1-\gamma} + ((r+\eta)\,a - c)\,V'(a)\,. \tag{55}$$

We guess and verify a solution of the form $V(a) = \kappa^{-\gamma}\frac{a^{1-\gamma}}{1-\gamma}$, so that the first-order condition is

$$c^{-\gamma} = V'(a) = \kappa^{-\gamma}a^{-\gamma}.$$

The HJB results in

$$(\rho+\eta)\,\kappa^{-\gamma}\frac{a^{1-\gamma}}{1-\gamma} = \kappa^{1-\gamma}\frac{a^{1-\gamma}}{1-\gamma} + ((r+\eta)-\kappa)\,\kappa^{-\gamma}a^{1-\gamma},$$

and then

$$\kappa = \frac{\rho+\gamma\eta-(1-\gamma)\,r}{\gamma}.$$

The optimal consumption is

$$c = \frac{\rho+\gamma\eta-(1-\gamma)\,r}{\gamma}a, \tag{56}$$

which has a positive slope as long as $\rho+\gamma\eta-(1-\gamma)\,r > 0$.

Second, given the HJB equation (11), for any $\xi > 0$,

$$V(a,z) = \xi^{1-\gamma}V_\xi(a/\xi, z),$$

where $V_\xi(a,z)$ solves

$$(\rho+\eta)V_{\xi}(a,z)=\max_{c_{\xi}}\frac{c_{\xi}^{1-\gamma}}{1-\gamma}+\left(wz/\xi+(r+\eta)a-c_{\xi}\right)\frac{\partial V_{\xi}(a,z)}{\partial a}$$
$$+\theta(\hat{z}-z)\frac{\partial V_{\xi}(a,z)}{\partial z}+\frac{\sigma^2}{2}\frac{\partial^2 V_{\xi}(a,z)}{\partial z^2},\;\; a\geq-\phi\xi.$$

This can be easily verified as

$$V(a,z)=\xi^{1-\gamma}V_{\xi}(a/\xi,z),\quad \frac{\partial V(a,z)}{\partial a}=\xi^{-\gamma}\frac{\partial V_{\xi}(a/\xi,z)}{\partial(a/\xi)},$$
$$\frac{\partial V(a,z)}{\partial z}=\xi^{1-\gamma}\frac{\partial V_{\xi}(a/\xi,z)}{\partial z},\quad \frac{\partial^2 V(a,z)}{\partial z^2}=\xi^{1-\gamma}\frac{\partial^2 V_{\xi}(a/\xi,z)}{\partial z^2}.$$

Third, notice how in the asymptotic limit $\xi\to\infty$:

$$\lim_{\xi\to\infty}V_{\xi}(a,z)=V(a)$$

and

$$\lim_{\xi\to\infty}c_{\xi}(a,z)=c(a),$$

the latter given by equation (56). This is equivalent to state that, for a large enough, we have

$$c(a,z)=\frac{\rho+\gamma\eta-(1-\gamma)r}{\gamma}a.$$

The stationary KF equation for large a then results in

$$0=-\frac{d}{da}\left[\left(r+\eta-\frac{\rho+\gamma\eta-(1-\gamma)r}{\gamma}\right)ag(a)\right]-\eta g(a).$$

We may guess that $g(a)\sim a^{-(1+\zeta)}$ and then verify

$$\zeta=\frac{\gamma\eta}{r-\rho}.$$

B.3. Proof of Proposition 4

We proceed as in Proposition 2. We consider the auxiliary problem without labor income, wz, and without borrowing constraint ($\phi=\infty$)

$$(\rho+\eta)j(a)=\max_{c}\frac{c^{1-\gamma}}{1-\gamma}+\lambda(a-K)+((r+\eta)a-c)j'(a),\tag{57}$$

and we guess and verify a solution of the form $j(a)=\kappa a+\varkappa$, so that the first-order condition is

$$c^{-\gamma}=j'(a)=\kappa,$$

that is, optimal consumption is constant. The HJB results in

$$(\rho+\eta)(\kappa a+\varkappa)=\frac{\kappa^{\frac{\gamma-1}{\gamma}}}{1-\gamma}+\lambda(a-K)+\left((r+\eta)a-\kappa^{-\frac{1}{\gamma}}\right)\kappa,$$

and then

$$\kappa=\frac{\lambda}{(\rho-r)},$$
$$\varkappa=\frac{1}{(\rho+\eta)}\left(\frac{\kappa^{\frac{\gamma-1}{\gamma}}}{1-\gamma}-\lambda K-\kappa^{\frac{\gamma-1}{\gamma}}\right).$$

Defining $j(a,z)=\xi^{1-\gamma}j_{\xi}(a/\xi,z)$ it is trivial to verify that in the asymptotic limit $\xi\to\infty$:

$$\lim_{\xi\to\infty}j_{\xi}(a,z)=j(a)$$

and

$$\lim_{\xi\to\infty} c_\xi (a, z) = c (a) .$$

Finally, the stationary KF equation for large a is

$$0 = -\frac{d}{da} [(r + \eta) ag (a)] - \eta g (a) .$$

We may guess that $g (a) \sim a^{-(1+\zeta)}$ and then verify

$$\zeta = \frac{\eta}{(r + \eta)}.$$

B.4. Proof of Proposition 6

First we compute the first-order condition with respect to τ and c in the stationary version of the HJB equation (35):

$$\varphi + \frac{\partial j}{\partial a} = 0, \tag{58}$$

$$c^{-\gamma} - \frac{\partial j}{\partial a} = 0,$$

so that

$$c = (-\varphi)^{-1/\gamma} ,$$

is a constant. If we take the derivative with respect to a in the HJB equation and apply the envelope theorem:

$$(\rho + \eta) \frac{\partial j}{\partial a} = \lambda + (r + \eta) \frac{\partial j}{\partial a} + (wz + (r + \eta) a - c + \tau) \frac{\partial^2 j}{\partial a^2} + \theta(\hat{z} - z) \frac{\partial^2 j}{\partial z \partial a} + \frac{\sigma^2}{2} \frac{\partial^3 j}{\partial z^2 \partial a},$$

and taking into account that, given (58), $j (a, z) = -a\varphi + \tilde{j} (z)$ where $\tilde{j} (z)$ is a given function, it simplifies to

$$(\rho - r) \varphi = -\lambda.$$

Second, we compute the value of the Lagrange multiplier

$$\begin{aligned}\lambda &= \frac{\alpha (1 - \alpha) \varphi}{K^{2-\alpha}} \int \int_{\underline{z}}^{\bar{z}} (a - Kz)\, g(a, z) dz da \\ &= \frac{\alpha (1 - \alpha) \varphi}{K^{1-\alpha}} \left(1 - \int \int_{\underline{z}}^{\bar{z}} z g(a, z) dz \right) = 0.\end{aligned}$$

Third, we compute the law of motion of aggregate capital

$$\begin{aligned}\frac{dK}{dt} &= \int \int_{\underline{z}}^{\bar{z}} a \frac{\partial g}{\partial t} dz da = -\int \int_{\underline{z}}^{\bar{z}} a \frac{\partial}{\partial a} (sg)\, dz da \\ &\quad - \int \int_{\underline{z}}^{\bar{z}} a \frac{\partial}{\partial z} \left(\theta(\hat{z} - z) g\right) da dz + \frac{1}{2} \int \int_{\underline{z}}^{\bar{z}} a \frac{\partial^2}{\partial z^2} \left(\sigma^2 g\right) da dz - \eta \int \int_{\underline{z}}^{\bar{z}} a g da dz,\end{aligned}$$

and integrating by parts[31]

$$\frac{dK}{dt} = \int \int_{\underline{z}}^{\bar{z}} (w (t) z + (r (t) + \eta) a - c (t) + \tau (t))\, g dz da - \eta K (t) = w (t) + r (t) K (t) - c.$$

In steady-state $\frac{dK}{dt} = 0$ and

$$c = w + rK = K^\alpha - \delta_K K.$$

[31] Notice that as z_t is a reflected process at the boundary, g satisfies the boundary condition (24): $-\theta(\hat{z} - z) g + \frac{1}{2} \frac{\partial}{\partial z} \left(\sigma^2 g\right) = 0$ for $z = \{\bar{z}, \underline{z}\}$.

Combining these results, we have

$$\rho = r = \alpha K^{\alpha-1} - \delta_K \Longrightarrow K = \left(\frac{\alpha}{\rho+\delta_K}\right)^{\frac{1}{1-\alpha}},$$

$$c = \left(\frac{\alpha}{\rho+\delta_K}\right)^{\frac{\alpha}{1-\alpha}} - \delta_K \left(\frac{\alpha}{\rho+\delta_K}\right)^{\frac{1}{1-\alpha}}.$$

Finally, the stationary KF equation for large a values is

$$0 = -\frac{d}{da}\left[(\rho+\eta)\, a g(a)\right] - \eta g(a).$$

We may guess that $g(a) \sim a^{-(1+\zeta)}$ and then verify

$$\zeta = \frac{\eta}{(\rho+\eta)}.$$

B.5. Proof of Lemma 1

We compute the product

$$\begin{aligned}
\langle V, \mathcal{A}^* g\rangle_{[0,1]\cup\{-1\}} &= \int\limits_{[0,1]} V(\alpha)\left[-(\rho+p(\theta))\, g^e + \frac{\partial^2}{\partial\alpha^2}\left(\Sigma(\alpha)\, g^e\right)\right] d\alpha \\
&\quad + V(\alpha_0)\left(p(\theta)\, u + \int_0^1 p(\theta)\, g^e d\alpha\right) - V(-1)\left(p(\theta)\, u - \rho\int_0^1 g^e d\alpha\right) \\
&= \int\limits_{[0,1]} g^e \left[-(\rho+p(\theta))\, V(t,\alpha) + \Sigma(\alpha)\frac{\partial^2 V}{\partial\alpha^2}\right] d\alpha \\
&\quad + V(\alpha_0)\left(p(\theta)\, u + \int_0^1 p(\theta)\, g^e d\alpha\right) - V(-1)\left(p(\theta)\, u - \rho\int_0^1 g^e d\alpha\right) \\
&= \int\limits_{[0,1]} g^e \left[\rho\left(V(-1) - V(\alpha)\right) + p(\theta)\left(V(\alpha_0) - V(\alpha)\right) + \Sigma(\alpha)\frac{\partial^2 V}{\partial\alpha^2}\right] d\alpha \\
&\quad + p(\theta)\left[V(\alpha_0) - V(-1)\right] u \\
&= \langle \mathcal{A} V, g\rangle_{[0,1]\cup\{-1\}},
\end{aligned}$$

where in the second equality we have integrated by parts and taken into account the fact that $\Sigma(0) = \Sigma(1) = 0$. The operator $\mathcal{A}$ is given by

$$\mathcal{A}V(\alpha) = \left[\begin{array}{c} \rho\left[V(-1) - V(\alpha)\right] + p(\theta)\left[V(\alpha_0) - V(\alpha)\right] + \Sigma(\alpha)\frac{\partial^2 V}{\partial\alpha^2}, \ \text{if}\ \alpha \in [0,1], \\ p(\theta)\left[V(\alpha_0) - V(-1)\right], \ \text{if}\ \alpha = -1. \end{array}\right]$$

B.6. Proof of Proposition 8

We proceed as in the proof of Proposition 1 in the case of no individual controls and no market clearing conditions. Here $\theta(t)$ plays the role as an aggregate variable. Therefore equation (52) in Appendix B.1 now results in

$$\frac{\partial}{\partial\theta}\left(-k\theta + \int j(t,\alpha)\, \mathcal{A}^* g d\alpha\right) = 0,$$

$$\frac{\partial}{\partial\theta}\left\{-k\theta - \int_0^1 j(t,\alpha)\, p(\theta) g^e d\alpha - p(\theta)\, u(t)\, j(t,-1) + \left(p(\theta)\, u(t) + \int_0^1 p(\theta)\, g^e d\alpha\right) j(t,\alpha_0)\right\} = 0,$$

so that an interior solution satisfies

$$p'(\theta) = -\frac{k}{\left[\int_0^1 j(t,\alpha)\, g^e d\alpha + u(t)\, j(t,-1) - j(t,\alpha_0)\right]}.$$

Appendix C. Description of the numerical algorithm

Here we present an algorithm to compute the stationary solution of the optimal planning problem.

Step 1: solution to the Hamilton–Jacobi–Bellman equation. The HJB equation is solved by a finite difference scheme following Achdou et al. (2017). It approximates the value function $V(a,z)$ on a finite grid with steps Δa and Δz: $a \in \{a_1, ..., a_I\}$, $z \in \{z_1, ..., z_J\}$.[32] We use the notation $V_{i,j} := V(a_i, z_j)$, $i = 1, ..., I$; $j = 1, ..., J$. The derivative of V with respect to a can be approximated with either a forward or a backward approximation:

$$\frac{\partial V(a_i, z_j)}{\partial a} \approx \partial_{a,F} V_{i,j} := \frac{V_{i+1,j} - V_{i,j}}{\Delta a}, \tag{59}$$

$$\frac{\partial V(a_i, z_j)}{\partial a} \approx \partial_{a,B} V_{i,j} := \frac{V_{i,j} - V_{i-1,j}}{\Delta a}, \tag{60}$$

where the decision between one approximation or the other depends on the sign of the savings function $s_{i,j} = wz_j + (r+\eta)a_i - c_{i,j}$ through an "upwind scheme" described below. The derivative of V with respect to z is approximated using a forward approximation

$$\frac{\partial V(a_i, z_j)}{\partial z} \approx \partial_z V_{i,j} := \frac{V_{i,j+1} - V_{i,j}}{\Delta z}, \tag{61}$$

$$\frac{\partial^2 V(a_i, z_j)}{\partial z^2} \approx \partial_{zz} V_{i,j} := \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{(\Delta z)^2}. \tag{62}$$

The HJB equation (11) is

$$(\rho+\eta)V = u(c) + (wz + (r+\eta)a - c)\frac{\partial V}{\partial a} + \theta(\hat{z} - z)\frac{\partial V}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2},$$

where

$$c = (u')^{-1}\left(\frac{\partial V}{\partial a}\right),$$

and $u(c) = \frac{c^{1-\gamma}}{1-\gamma} + \lambda(a-K)$. The HJB equation is approximated by an upwind scheme

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \rho V_{i,j}^{n+1} = u(c_{i,j}^n) + \partial_{a,F} V_{i,j}^{n+1} s_{i,j,F}^n \mathbf{1}_{s_{i,j,F}^n > 0} + \partial_{a,B} V_{i,j}^{n+1} s_{i,j,B}^n \mathbf{1}_{s_{i,j,B}^n < 0}$$
$$+ \theta(\hat{z} - z_j)\partial_z V_{i,j}^{n+1} + \frac{\sigma_z^2 z_j}{2}\partial_{zz} V_{i,j}^{n+1} - \eta V_{i,j}^{n+1},$$

where

$$s_{i,j,F}^n = wz_j + (r+\eta)a_i - (u')^{-1}(\partial_{a,F} V_{i,j}^n),$$
$$s_{i,j,B}^n = wz_j + (r+\eta)a_i - (u')^{-1}(\partial_{a,B} V_{i,j}^n).$$

This can be expressed as:

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \rho V_{i,j}^{n+1} = u(c_{i,j}^n) + V_{i-1,j}^{n+1}\varrho_{i,j} + V_{i,j}^{n+1}\beta_{i,j} + V_{i+1,j}^{n+1}\chi_{i,j} + V_{i,j-1}^{n+1}\xi_j + V_{i,j+1}^{n+1}\varsigma_j, \tag{63}$$

where

$$c_{i,j}^n = (u')^{-1}\left(\partial_{a,F} V_{i,j}^n \mathbf{1}_{s_{i,j,F}^n > 0} + \partial_{a,B} V_{i,j}^n \mathbf{1}_{s_{i,j,B}^n < 0} + u'(wz_j + ra_i)\mathbf{1}_{s_{i,j,F}^n < 0, s_{i,j,B}^n > 0}\right), \tag{64}$$

$$\varrho_{i,j} = -\frac{s_{i,j,B}^n \mathbf{1}_{s_{i,j,B}^n < 0}}{\Delta a},$$

$$\beta_{i,j} = -\frac{s_{i,j,F}^n \mathbf{1}_{s_{i,j,F}^n > 0}}{\Delta a} + \frac{s_{i,j,B}^n \mathbf{1}_{s_{i,j,B}^n < 0}}{\Delta a} - \frac{\theta(\hat{z} - z_j)}{\Delta z} - \frac{\sigma^2}{(\Delta z)^2} - \eta,$$

[32] Notice that subindexes i and j have a different meaning here than in the main text. We use $V(a,z)$ to denote the value function instead of $j(a,z)$ to avoid confusions.

$$\chi_{i,j} = \frac{s^n_{i,j,F} \mathbf{1}_{s^n_{i,j,F}>0}}{\Delta a},$$
$$\xi = \frac{\sigma^2}{2(\Delta z)^2},$$
$$\varsigma_j = \frac{\sigma^2}{2(\Delta z)^2} + \frac{\theta(\hat{z} - z_j)}{\Delta z}.$$

The state constraint (20) $a \geq -\phi$ is enforced by setting $s^n_{i,j,B} = 0$. Similarly, $a \leq a^{\max}$ requires $s^n_{I,j,F} = 0$. Therefore, the values $V^{n+1}_{0,j}$ and $V^{n+1}_{I+1,j}$ are never used. The boundary conditions with respect to z are

$$\frac{\partial V(a, \underline{z})}{\partial z} = \frac{\partial V(a, \bar{z})}{\partial z} = 0,$$

as the process is reflected. At the boundaries in the j dimension, equation (63) becomes

$$\frac{V^{n+1}_{i,j} - V^n_{i,j}}{\Delta} + \rho V^{n+1}_{i,j} = u(c^n_{i,1}) + V^{n+1}_{i-1,j} \varrho_{i,1} + V^{n+1}_{i,1} \left(\beta_{i,1} + \xi\right) + V^{n+1}_{i+1,1} \chi_{i,1} + V^{n+1}_{i,2} \varsigma_1,$$
$$\frac{V^{n+1}_{i,j} - V^n_{i,j}}{\Delta} + \rho V^{n+1}_{i,j} = u(c^n_{i,J}) + V^{n+1}_{i-1,J} \varrho_{i,J} + V^{n+1}_{i,J} \left(\beta_{i,J} + \varsigma_J\right) + V^{n+1}_{i+1,J} \chi_{i,J} + V^{n+1}_{i,J-1} \xi_J.$$

Equation (63) is a system of $I \times J$ linear equations which can be written in matrix notation as:

$$\frac{\mathbf{V}^{n+1} - \mathbf{V}^n}{\Delta} + \rho \mathbf{V}^{n+1} = \mathbf{u}^n + \mathbf{A}^n \mathbf{V}^{n+1},$$

where the matrix $\mathbf{A}^n$ and the vectors $\mathbf{V}^{n+1}$ and $\mathbf{u}^n$ are defined by:

$$\mathbf{A}^n = \begin{bmatrix}
\beta_{1,1}+\xi & \chi_{1,1} & 0 & \cdots & 0 & \varsigma_1 & 0 & 0 & \cdots & 0 \\
\varrho_{2,1} & \beta_{2,1}+\xi & \chi_{2,1} & 0 & \cdots & 0 & \varsigma_1 & 0 & \cdots & 0 \\
0 & \varrho_{3,1} & \beta_{3,1}+\xi & \chi_{3,1} & 0 & \cdots & 0 & \varsigma_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \varrho_{I,1} & \beta_{I,1}+\xi & \chi_{I,1} & 0 & 0 & \cdots & 0 \\
\xi & 0 & \cdots & 0 & \varrho_{1,2} & \beta_{1,2} & \chi_{1,2} & 0 & \cdots & 0 \\
0 & \xi & \cdots & 0 & 0 & \varrho_{2,2} & \beta_{2,2} & \chi_{2,2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \varrho_{I-1,J} & \beta_{I-1,J}+\varsigma_J & \chi_{I-1,J} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \varrho_{I,J} & \beta_{I,I}+\varsigma_J
\end{bmatrix},$$

$$\mathbf{V}^{n+1} = \begin{bmatrix} V^{n+1}_{1,1} \\ V^{n+1}_{2,1} \\ \vdots \\ V^{n+1}_{1,2} \\ V^{n+1}_{2,2} \\ \vdots \\ V^{n+1}_{I-1,J} \\ V^{n+1}_{I,J} \end{bmatrix}, \quad \mathbf{u}^n = \begin{bmatrix} u(c^n_{1,1}) \\ u(c^n_{2,1}) \\ \vdots \\ u(c^n_{1,2}) \\ u(c^n_{2,2}) \\ \vdots \\ u(c^n_{I-1,J}) \\ u(c^n_{I,J}) \end{bmatrix}.$$

The system can in turn be written as

$$\mathbf{B}^n \mathbf{V}^{n+1} = \mathbf{d}^n, \tag{65}$$

where $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right) \mathbf{I} - \mathbf{A}^n$ and $\mathbf{d}^n = \mathbf{u}^n + \frac{\mathbf{V}^n}{\Delta}$. $\mathbf{I}$ is the identity matrix. Matrix $\mathbf{B}^n$ is a sparse matrix, and the system (65) can be efficiently solved in Matlab.

The algorithm to solve the HJB equation runs as follows. Begin with an initial guess $V^0_{i,j} = u(ra_i + wz_j)/\rho$, set $n = 0$. Then:

1. Compute $\partial_{a,F} V^n_{i,j}$, $\partial_{a,B} V^n_{i,j}$, $\partial_z V^n_{i,j}$ and $\partial_{zz} V^n_{i,j}$ using (59)–(62).
2. Compute $c^n_{i,j}$ using (64).
3. Find $V^{n+1}_{i,j}$ solving the linear system of equations (65).
4. If $V^{n+1}_{i,j}$ is close enough to $V^n_{i,j}$, stop. If not set $n := n+1$ and go to step 1.

Step 2: solution to the Kolmogorov forward equation. The KF equation is also solved using an upwind finite difference scheme. The equation (21) in this case is

$$0 = -\frac{\partial}{\partial a}\left[(wz + (r+\eta)a - c)\, g\right] - \frac{\partial}{\partial z}\left[\theta(\hat{z} - z) g\right] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\sigma^2 g - \eta g + \eta\delta_0, \tag{66}$$

$$\int g(a,z) da dz = 1. \tag{67}$$

This case is simpler than the previous one, as the problem is linear in g, so no iterative procedure is needed. We use the notation $g_{i,j} := g(a_i, z_j)$. The system can be expressed as

$$\begin{aligned} 0 = &-\frac{g_{i,j} s^n_{i,j,F} \mathbf{1}_{s^n_{i,j,F}>0} - g_{i-1,j} s^n_{i-1,j,F} \mathbf{1}_{s^n_{i-1,j,F}>0}}{\Delta a} - \frac{g_{i+1,j} s^n_{i+1,j,B} \mathbf{1}_{s^n_{i+1,j,B}<0} - g_{i,j} s^n_{i,j,B} \mathbf{1}_{s^n_{i,j,B}<0}}{\Delta a} \\ &- \frac{g_{i,j}\theta(\hat{z} - z_j) - g_{i,j-1}\theta(\hat{z} - z_{j-1})}{\Delta z} + \frac{g_{i,j+1}\sigma^2 + g_{i,j-1}\sigma^2 - 2g_{i,j}\sigma^2}{2(\Delta z)^2} - \eta g_{i,j} + \eta\delta_0, \end{aligned}$$

or equivalently

$$g_{i-1,j}\chi_{i-1,j} + g_{i+1,j}\varrho_{i+1,j} + g_{i,j}\beta_{i,j} + g_{i,j+1}\xi + g_{i,j-1}\varsigma_j = -\eta\delta_0, \tag{68}$$

then (68) is also a system of $I \times J$ linear equations which can be written in matrix notation as:

$$\mathbf{A}^{\mathbf{T}}\mathbf{g} = \mathbf{h}, \tag{69}$$

where $\mathbf{A}^{\mathbf{T}}$ is the transpose of $\mathbf{A} = \lim_{n\to\infty} \mathbf{A}^n$ and $\mathbf{h}$ is a vector of zeros with a -1 at the first position. We solve the system (69) and obtain a solution g. Then we renormalize as

$$g_{i,j} = \frac{g_{i,j}}{\sum_{i=1}^{I}\sum_{j=1}^{J} g_{i,j}\Delta a \Delta z}.$$

Step 3: finding the equilibrium aggregate capital. In order to find the aggregate capital K, we employ a relaxation method. Given $\nu \in (0,1)$, begin with an initial guess of the aggregate capital K^0, set $n = 0$. Then:

1. Compute r^n and w^n as a function of K^n.
2. Given r^n and w^n, solve the planner's HJB equation as in Step 1 to obtain an estimate of the value function V^n and of the consumption c^n.
3. Given c^n, solve the KF equation as in Step 2 and compute the aggregate density g^n.
4. Compute the aggregate capital stock $S = \sum_{i=1}^{I}\sum_{j=1}^{J} a_i g_{i,j}\Delta a \Delta z$.
5. Compute $K^{n+1} = \nu S^n + (1-\nu) K^n$. If K^{n+1} is close enough to K^n, stop. If not set $n := n+1$ and go to step 1.

Step 4: finding the Lagrange multiplier. In order to find the value of the optimal Lagrange multiplier in the planning problem (30) we also employ a relaxation method. Given constant $\vartheta \in (0,1)$, begin with an initial guess of the Lagrange multiplier $\lambda^0 = 0$, set $m = 0$:

1. Compute the value function V^m, consumption c^m, density g^m and aggregate capital K^m given λ^m as in Step 3 above.
2. Compute $\tilde{\lambda}^{m+1}$ as[33]

$$\tilde{\lambda}^{m+1} = \frac{\alpha(1-\alpha)}{(K^m)^{2-\alpha}} \sum_{i=1}^{I}\sum_{j=1}^{J}\left[g^m_{i,j} + a_i \frac{g^m_{i+1,j} - g^m_{i,j}}{\Delta a} - K^m z_j \frac{g^m_{i+1,j} - g^m_{i,j}}{\Delta a}\right] V^m_{i,j}\Delta a \Delta z. \tag{70}$$

[33] We discretize the expression

$$\lambda = \frac{\alpha(1-\alpha)}{K^{2-\alpha}} \int \int_{\underline{z}}^{\bar{z}} \left[g(a,z) + (a - Kz)\frac{\partial g}{\partial a}\right] V(a,z)\, dz da,$$

which can be obtained integrating by parts equation (30).

3. If $\tilde{\lambda}^{m+1} \neq \lambda^n$, set $\lambda^{m+1} = \vartheta \tilde{\lambda}^{m+1} + (1-\vartheta)\lambda^m$, update $m := m+1$ and return to step 1.

If the algorithm converges, it should produce the steady-state value function V, the optimal policy c, the aggregate capital K, the Lagrange multiplier λ and the density g.[34]

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[34] We do not provide any proof of convergence of the numerical algorithm.

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