Advanced Tools in Macroeconomics

Continuous time models (and methods)

Pontus Rendahl

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Introduction

- In this lecture we will take a look at models in continuous, as opposed to discrete, time.
- There are some advantages and disadvantages
  - Advantages: Can give closed form solutions even when they do not exist for the discrete time counterpart. Can be very fast to solve. Trendier than sourdough bread, fixed gear bicycles, and skinny jeans combined (so if you do continuous time you need neither).
  - Disadvantages: Intuition is a bit tricky. Contraction mapping theorems / convergence results go out the window (but can be somewhat brought back). The latter can create issues for numerical computing. Difficult to deal with certain end-conditions (like finite lives etc.)

Plan for today

- Continuous time methods and models are not as well documented as the discrete time cases.
- Proceed through a series of examples
  1. The Solow growth model (!)
  2. The (stochastic) Ramsey growth model
  3. A monetary economy
  4. Search and matching
- How to solve (turns out to be pretty easy, and we can apply methods we know from earlier parts of the course)

Plan for tomorrow

- Speeding things up and making it more robust
  - Little tips and tricks
- Heterogenous agent models in continuous time
  - Focus on the Aiyagari model
- Useful list of papers to read
  1. "Finite?Difference Methods for Continuous?Time Dynamic Programming" by Candler 2001
  2. `http://www.princeton.edu/ moll/HACT.pdf`
  3. `http://www.princeton.edu/ moll/WIMM.pdf`

The Solow growth model

- The Solow growth model is characterized by the following equations

$$\begin{aligned} Y_t &= K_t^{\alpha}(A_t N_t)^{1-\alpha} \\ K_{t+1} &= I_t + (1-\delta)K_t \\ S_t &= sY_t \\ I_t &= S_t \\ A_{t+1} &= (1+g)A_t \\ N_{t+1} &= (1+\eta)N_t \end{aligned}$$

- To solve this model we rewrite it in intensive form

$$x_t = \frac{X_t}{A_t N_t}, \quad \text{for } X = \{Y, K, S, I\}$$

The Solow growth model

- Using this and substituting in gives

$$\frac{K_{t+1}}{A_t N_t} = sk_t^{\alpha} + (1-\delta)k_t$$

- We can rewrite as

$$\begin{aligned} \frac{K_{t+1}}{A_{t+1}N_{t+1}} \frac{A_{t+1}N_{t+1}}{A_t N_t} &= sk_t^{\alpha} + (1-\delta)k_t \\ k_{t+1}\frac{A_{t+1}N_{t+1}}{A_t N_t} &= sk_t^{\alpha} + (1-\delta)k_t \\ k_{t+1}(1+g)(1+\eta) &= sk_t^{\alpha} + (1-\delta)k_t \end{aligned}$$

The Solow growth model

- Ta-daa!

$$k_{t+1} = \frac{sk_t^{\alpha}}{(1+g)(1+\eta)} + \frac{(1-\delta)k_t}{(1+g)(1+\eta)}$$

- Balanced growth: $k_{t+1} = k_t = k$

$$k = \left(\frac{g+\eta+g\eta+\delta}{s}\right)^{\frac{1}{\alpha-1}}$$

- This is not textbook stuff. Why? Discrete time. More elegant solution in continuous time.

The Solow growth model

- Continuous time is not a state in itself, but is the effect of a limit. A derivative is a limit, an integral is a limit, the sum to infinity is a limit, and so on.
- Continuous time is the name we use for the behavior of an economy as intervals between time periods approaches zero.

The Solow growth model

- The right approach is therefore to derive this behavior as a limit (much like you probably derived derivatives from its limit definition in high school).
- Eventually you may get so well versed in the limit behavior that you can set it up directly (like you can say that the derivative of $\ln x$ is equal to $1/x$, without calculating $\lim_{\varepsilon \to 0}(\ln(x + \varepsilon) - \ln(x))/\varepsilon$)
- I'm not there yet. I have to do this the complicated way. People like Ben Moll at Princeton is. Take a look at his lecture notes on continuous time stuff. They are great.

The Solow growth model

- Back to the model.
- Suppose that before the length of each time period was one month. Now we want to rewrite the model on a biweekly frequency.
- It seems reasonable to assume that in two weeks we produce half as much as we do in one month: $Y_t = 0.5K_t^{\alpha}(A_t N_t)^{1-\alpha}$.
- It also seems reasonable that capital depreciates slower, i.e. 0.5δ.

The Solow growth model

- Notice that we still have N_t worker and K_t units of capital: Stocks are not affected by the length of time intervals (although the accumulation of them will).
- The propensity to save is the same, but with half of the income saving is halved too (and therefore investment)
- What happens to the exogenous processes for A_t and N_t?

The Solow growth model

- Before

$$A_{t+1} = (1+g)A_t, \quad N_{t+1} = (1+\eta)N_t$$

- Now

$$A_{t+0.5} = (1+0.5g)A_t, \quad N_{t+0.5} = (1+0.5\eta)N_t$$

or

$$A_{t+0.5} = e^{0.5g}A_t, \quad N_{t+0.5} = e^{0.5\eta}N_t \quad ?$$

The Solow growth model

- It turns out that this choice does not matter much for our purpose
- Suppose that the time period is not one month but $\Delta \times$ one month. And suppose that

$$A_{t+\Delta} = (1 + \Delta g)A_t$$

- Rearrange

$$\frac{A_{t+\Delta} - A_t}{\Delta} = gA_t.$$

- And take limit $\Delta \to 0$

$$\dot{A}_t = gA_t$$

The Solow growth model

- Suppose that the time period is not one month but $\Delta \times$ one month. And suppose that

$$A_{t+\Delta} = e^{\Delta g} A_t$$

- Rearrange

$$\frac{A_{t+\Delta} - A_t}{\Delta} = \frac{(e^{\Delta g} - 1)}{\Delta} A_t.$$

- Notice

$$\lim_{\Delta \to 0} \frac{(e^{\Delta g} - 1)}{\Delta} = \lim_{\Delta \to 0} \frac{(g e^{\Delta g})}{1} = g$$

The Solow growth model

- So

$$\lim_{\Delta \to 0} \frac{(e^{\Delta g} - 1)}{\Delta} A_t = g A_t$$

  and thus

$$\dot{A}_t = g A_t$$

- Therefore, it doesn't matter if $A_{t+\Delta} = (1 + \Delta g) A_t$ or $A_{t+\Delta} = e^{\Delta g} A_t$. The limits are the same.

The Solow growth model

- Solow growth model in Δ units of time

$$
\begin{aligned}
Y_t &= \Delta K_t^{\alpha}(A_t N_t)^{1-\alpha} \\
K_{t+\Delta} &= I_t + (1-\Delta\delta)K_t \\
S_t &= sY_t \\
I_t &= S_t \\
A_{t+\Delta} &= (1+\Delta g)A_t \\
N_{t+\Delta} &= (1+\Delta\eta)N_t
\end{aligned}
$$

The Solow growth model

- Substitute and rearrange as before

$$
\begin{aligned}
\frac{K_{t+\Delta}}{A_{t+\Delta}N_{t+\Delta}}\frac{A_{t+\Delta}N_{t+\Delta}}{A_tN_t} &= s\Delta k_t^{\alpha} + (1-\Delta\delta)k_t \\
k_{t+\Delta}\frac{A_{t+\Delta}N_{t+\Delta}}{A_tN_t} &= s\Delta k_t^{\alpha} + (1-\Delta\delta)k_t \\
k_{t+\Delta}(1+\Delta g)(1+\Delta\eta) &= s\Delta k_t^{\alpha} + (1-\Delta\delta)k_t
\end{aligned}
$$

The Solow growth model

- Simplify and rearrange

$$
\begin{aligned}
& k_{t+\Delta}(1+\Delta g)(1+\Delta\eta) = s\Delta k_t^\alpha + (1-\Delta\delta)k_t \\
\Rightarrow \quad & k_{t+\Delta} - k_t = s\Delta k_t^\alpha - \Delta\delta k_t - \Delta(g+\eta+\Delta g\eta)k_{t+\Delta} \\
\Rightarrow \quad & \frac{k_{t+\Delta} - k_t}{\Delta} = s k_t^\alpha - \delta k_t - (g+\eta+\Delta g\eta)k_{t+\Delta}
\end{aligned}
$$

- Take limits $\Delta \to 0$

$$
\dot{k}_t = s k_t^\alpha - (g+\eta+\delta)k_t
$$

- With steady state

$$
k = \left(\frac{g+\eta+\delta}{s}\right)^{\frac{1}{\alpha-1}}
$$

The Solow growth model: Solution

- How do you solve this model?
- The equation

$$\dot{k}_t = sk_t^{\alpha} - (g + \eta + \delta)k_t$$

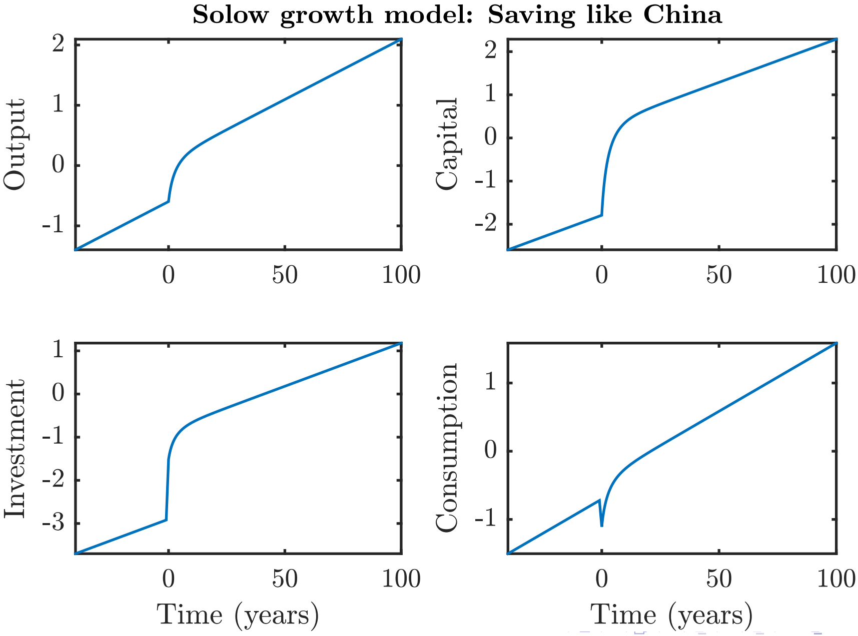
  is an ODE.
- Declare it as a function with respect to time, t, and capital, k, in Matlab as

$$\mathtt{solow} = \mathtt{@}(\mathtt{t},\mathtt{k}) \quad \mathtt{sk}^{\alpha} - (\mathtt{g} + \eta + \delta)\mathtt{k};$$

- The simulate it for, say 100 units of time, with initial condition k_0 as

$$[\mathtt{time},\mathtt{capital}] = \mathtt{ode45}(\mathtt{solow},[\mathtt{0} \ \ \mathtt{100}],\mathtt{k}_0);$$

The Solow growth model: Solution



The Solow growth model: Solution

A few pointers

- Once you got the solution of a deterministic continuous time model, the solution will always be of the form $\dot{x}_t = f(x_t)$, whether or not x_t is a vector.
- The matlab function ode45 (or other versions) can then simulate a transition (such as an impulse response).
- You could also simulate on your own through the approximation

$$\dot{x}_t \approx \frac{x_{t+\Delta} - x_t}{\Delta}$$

  and thus find your solution as $x_{t+\Delta} = x_t + \Delta f(x_t)$.
- For this to be accurate, Δ must be small if there are a lot of nonlinearities.
- The ODE function in matlab uses so-called Runge Kutta methods to vary the step-size Δ in an optimal way.

The Ramsey growth model

- Now consider the Ramsey growth model (without growth)

$$\begin{aligned} v(k_t) = &\max_{c_t, k_{t+1}} \{u(c_t) + (1-\rho)v(k_{t+1})\} \\ \text{s.t.} \quad & c_t + k_{t+1} = k_t^\alpha + (1-\delta)k_t \end{aligned}$$

- In Δ units of time

$$\begin{aligned} v(k_t) = &\max_{c_t, k_{t+\Delta}} \{\Delta u(c_t) + (1-\Delta\rho)v(k_{t+\Delta})\} \\ \text{s.t.} \quad & \Delta c_t + k_{t+\Delta} = \Delta k_t^\alpha + (1-\Delta\delta)k_t \end{aligned}$$

The Ramsey growth model

- Notice that all flows change when the length of the time period on which they are defined changes. Stocks, k, are the same.
- I discount the future with $1 - \Delta\rho$ instead of $(1 - \rho)$ (or with $e^{-\Delta\rho}$ instead of e^{ρ}, but these are, in the limit, equivalent).
- One funny thing: Consumption, c, is still "monthly" consumption, but it now only cost Δ as much, and I only get a Δ fraction of the utility!
- These assumptions are for technical reasons, and it will (hopefully) soon be clear why they are made.

The Ramsey growth model

- Bellman equation

$$v(k_t) = \max_{c_t, k_{t+\Delta}} \{\Delta u(c_t) + (1 - \Delta\rho) v(k_{t+\Delta})\}$$
$$\text{s.t.} \quad \Delta c_t + k_{t+\Delta} = \Delta k_t^\alpha + (1 - \Delta\delta) k_t$$

- Subtract $v(k_t)$ from both sides and insert the budget constraint into $v(k_{t+\Delta})$

$$0 = \max_{c_t} \{\Delta u(c_t) + v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t)) - v(k_t) - \Delta\rho v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t))\}$$

The Ramsey growth model

- From before

$$0 = \max_{c_t}\{\Delta u(c_t) + v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t)) - v(k_t) \\ - \Delta\rho v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t))\}$$

- Divide by Δ

$$0 = \max_{c_t}\{u(c_t) + \frac{v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t)) - v(k_t)}{\Delta} \\ - \rho v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t))\}$$

The Ramsey growth model

- From before

$$0 = \max_{c_t}\{u(c_t) + \frac{v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t)) - v(k_t)}{\Delta} \\ - \rho v(k_t + \Delta(k_t^\alpha - \delta k_t - c_t))\}$$

- Take limit $\Delta \to 0$ and rearrange

$$\rho v(k_t) = \max_{c_t}\{u(c_t) + v'(k_t)(k_t^\alpha - \delta k_t - c_t)\}$$

- This is know as the Hamilton-Jacobi-Bellman (HJB) equation.

The Ramsey growth model: Solving

- Dropping time notation we have

$$\rho v(k) = \max_c \{u(c) + v'(k)(k^\alpha - \delta k - c)\}$$

- This is simple to solve and (can be) blazing fast!

The Ramsey growth model: Solving

- Dropping time notation we have

$$\rho v(k) = \max_{c}\{u(c) + v'(k)(k^{\alpha} - \delta k - c)\}$$

- This is simple to solve and (can be) blazing fast!
- Why fast? Maximization is trivial: First order condition

$$u'(c) = v'(k)$$

- So if we know $v'(k)$ we know optimal c without searching for it!

The Ramsey growth model: Solving

- How do we find $v'(k)$?
- Suppose we have hypothetical values of $v(k)$ on a uniformly spaced grid of k, $\mathcal{K} = \{k_0, k_1, \dots, k_N\}$ with stepsize Δk.
- We can then approximate $v'(k)$ at gridpoint k_i ($i \neq 1, N$) as

$$v'(k_i) = 0.5(v(k_{i+1}) - v(k_i))/\Delta k \\ + 0.5(v(k_i) - v(k_{i-1}))/\Delta k$$

or

$$v'(k_i) = \frac{v(k_{i+1}) - v(k_{i-1})}{2\Delta k}$$

The Ramsey growth model: Solving

- and for k_1 and k_N

$$v'(k_1) = (v(k_2) - v(k_1))/\Delta k$$

  and

$$v'(k_N) = (v(k_N) - v(k_{N-1}))/\Delta k$$

- There are many ways of doing this. If you have a vector of $v(k)$ values – call it `V` – then `dV=gradient(V)/dk`.

The Ramsey growth model: Solving

- I prefer an alternative method.
- Construct the matrix D as

$$D = \begin{pmatrix} -1/dk & 1/dk & 0 & 0 & \dots & 0 \\ -0.5/dk & 0 & 0.5/dk & 0 & \dots & 0 \\ 0 & -0.5/dk & 0 & 0.5/dk & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -1/dk & 1/dk \end{pmatrix}$$

- Then

$$v'(k) \approx D \times v(k)$$

The Ramsey growth model: Solving

Algorithm

1. Construct a grid for k.
2. For each point on the grid, guess for a value of `V_0`.
3. Calculate the derivative as `dV_0=D*V_0`.
4. Find `V_1` from

$$\rho V_1 = u(c_0) + dV_0(k^\alpha - \delta k - c_0),$$
$$\text{with} \quad u'(c_0) = dV_0$$

5. Back to step 3 with V_1 replacing V_0. Repeat until convergence.

The Ramsey growth model: Solving

Algorithm

1. Construct a grid for k.
2. For each point on the grid, guess for a value of $\texttt{V}_0$.
3. Calculate the derivative as $\texttt{dV}_0\texttt{=D*V}_0$.
4. Find $\texttt{V}_1$ from

$$\rho V_1 = u(c_0) + dV_0(k^\alpha - \delta k - c_0),$$
$$\text{with} \quad u'(c_0) = dV_0$$

5. Back to step 3 with V_1 replacing V_0. Repeat until convergence.

Beware: The contraction mapping theorem does not work, so convergence is an issue. Solution: update slowly. That is, $V_1 = \gamma V_1 + (1-\gamma)V_0$, for a low value of γ.

The Ramsey growth model: Solving

Alternative algorithm

1. Construct a grid for k.
2. For each point on the grid, guess for a value of $\mathtt{V}_0$.
3. Calculate the derivative as $\mathtt{dV}_0\mathtt{=D*V}_0$.
4. Find $\mathtt{V}_1$ from

$$V_1 = \Gamma(u(c_0) + dV_0(k^\alpha - \delta k - c_0) - \rho V_0) + V_0,$$
$$\text{with} \quad u'(c_0) = dV_0$$

5. Back to step 3 with V_1 replacing V_0. Repeat until convergence.

We will take a look at an alternative way of doing things tomorrow.

The Ramsey growth model: Solving

Alternative vs. standard algorithm

- But to me they sort of look the same
- I.e.

$$\begin{aligned} V_1 &= \gamma\frac{1}{\rho}(u(c_0) + dV_0(k^\alpha - \delta k - c_0)) + (1-\gamma)V_0 \\ &= \frac{\gamma}{\rho}(u(c_0) + dV_0(k^\alpha - \delta k - c_0) - \rho V_0) + V_0 \end{aligned}$$

- So as long as $\frac{\gamma}{\rho} = \Gamma$ they should be identical.

The Ramsey growth model: Euler equation

- Let's go back to the HJB equation.

$$\rho v(k) = u(c) + v'(k)(k^\alpha - \delta k - c)$$
$$\text{with} \quad u'(c) = v'(k)$$

- Thus

$$\rho v'(k) = v''(k)(k^\alpha - \delta k - c) + v'(k)(\alpha k^{\alpha-1} - \delta)$$

- And

$$v''(k) = u''(c)c'(k)$$

The Ramsey growth model: Euler equation

- Using

$$\rho v'(k) = v''(k)(k^\alpha - \delta k - c) + v'(k)(\alpha k^{\alpha-1} - \delta)$$

Together with $v'(k) = u'(c)$ and $v''(k) = u''(c)c'(k)$ gives

$$\rho u'(c) = u''(c)c'(k)(k^\alpha - \delta k - c) + u'(c)(\alpha k^{\alpha-1} - \delta)$$

or

$$-u''(c)c'(k)(k^\alpha - \delta k - c) = u'(c)(\alpha k^{\alpha-1} - \delta - \rho)$$

- Suppose CRRA utility, such that $\frac{u''(c)c}{u'(c)} = -\gamma$

The Ramsey growth model: Euler equation

- Then the last equation

$$-u''(c)c'(k)(k^\alpha - \delta k - c) = u'(c)(\alpha k^{\alpha-1} - \delta - \rho)$$

  is equal to

$$\gamma \frac{c'(k)}{c}(k^\alpha - \delta k - c) = (\alpha k^{\alpha-1} - \delta - \rho)$$

- This is the Euler equation in continuous time.

The Ramsey growth model: Euler equation

Before we attempt to solve the Euler equation, recall that we had

$$k_{t+\Delta} + \Delta c_t = \Delta k_t^\alpha + (1 - \Delta\delta)k_t$$

rearrange

$$k_{t+\Delta} - k_t = \Delta(k_t^\alpha - \delta k_t - c_t)$$

Divide with Δ and take limit $\Delta \to 0$ to get

$$\dot{k}_t = k_t^\alpha - \delta k_t - c_t$$

Or dropping time notation

$$\dot{k} = k^\alpha - \delta k - c$$

The Ramsey growth model: Euler equation

- Our Euler equation is

$$\frac{c'(k)}{c}(k^\alpha - \delta k - c) = \frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)$$

or now

$$\gamma \frac{c'(k)}{c}\dot{k} = (\alpha k^{\alpha-1} - \delta - \rho)$$

- What is $c'(k)\dot{k}$? Recall chain rule

$$\dot{c} = \frac{\partial c_t}{\partial t} = \frac{\partial c_t}{\partial k}\frac{\partial k}{\partial t} = c'(k)\dot{k}$$

- Thus

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)$$

The Ramsey growth model: Exploring

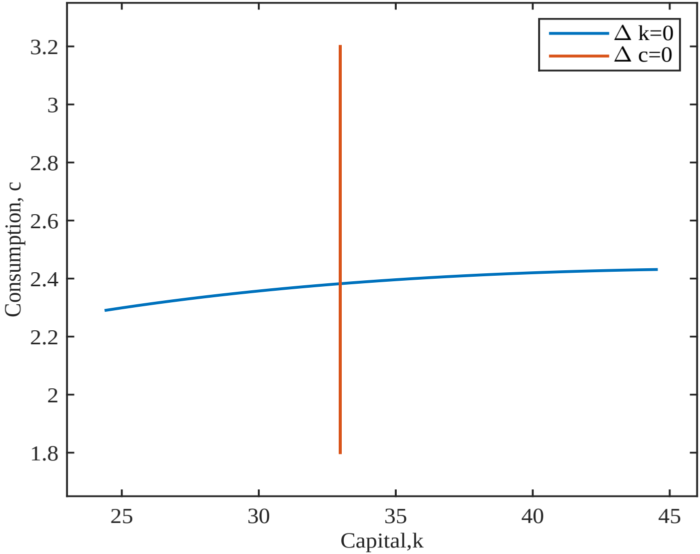
- Two equations

$$\dot{c} = \frac{c}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)$$
$$\dot{k} = k^{\alpha} - \delta k - c$$

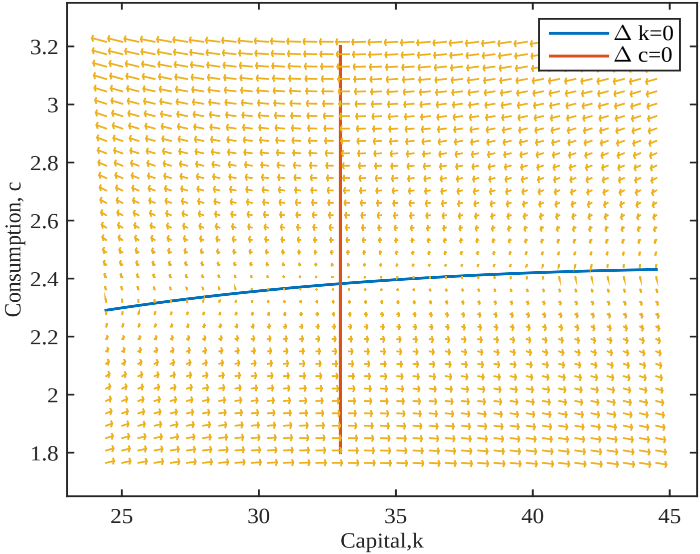
- Nullclines

$$0 = \alpha k^{\alpha-1} - \delta - \rho$$
$$0 = k^{\alpha} - \delta k - c$$

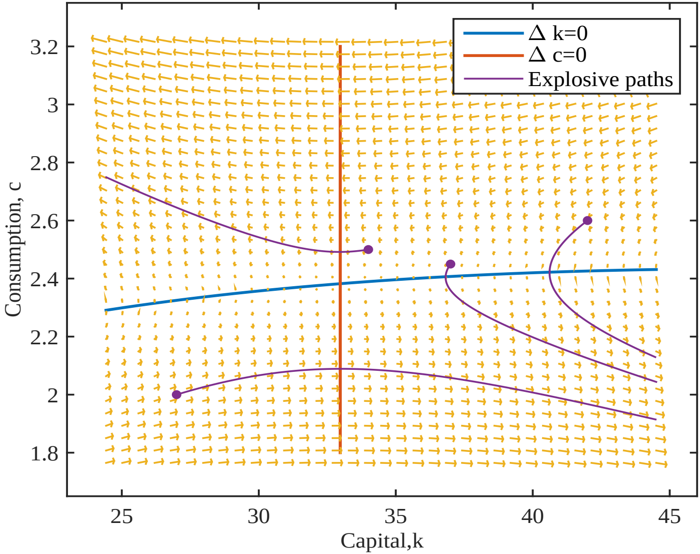
The Ramsey growth model: Exploring



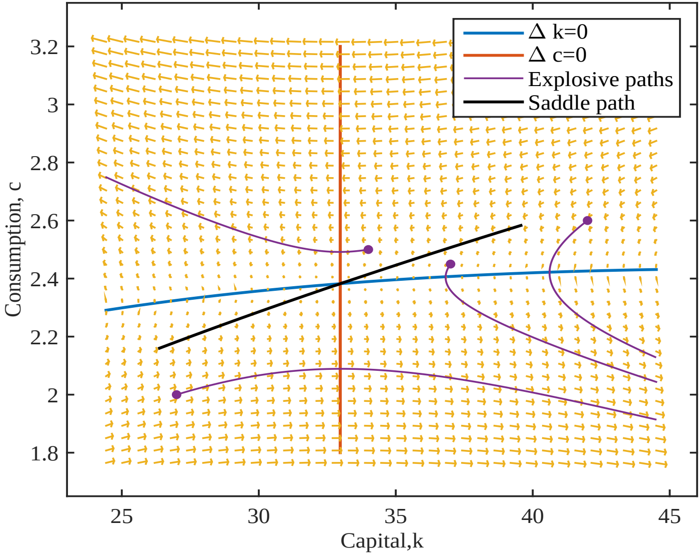
The Ramsey growth model: Exploring



The Ramsey growth model: Exploring



The Ramsey growth model: Exploring



The Ramsey growth model: Exploring

- How did I do that?
- I created a grid for k and c, and found $\dot{c}$ and $\dot{k}$ through

$$\dot{c} = \frac{c}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)$$
$$\dot{k} = k^{\alpha} - \delta k - c$$

- Then I used Matlab's command `quiver(k,c,`$\dot{k}$`,`$\dot{c}$`)`
  - This creates the swarm of arrows
- I then used Matlab's command `streamline(k,c,`$\dot{k}$`,`$\dot{c}$`)` at various starting values to get the explosive paths.
- Lastly I solved for the saddle path and plotted it.

The Ramsey growth model: Euler equation solution

- Back to the "recursive" Euler

$$\frac{c'(k)}{c}(k^{\alpha} - \delta k - c) = \frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho)$$

- Solve for c

$$c = \frac{c'(k)(k^{\alpha} - \delta k)}{\frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho) + c'(k)}$$

The Ramsey growth model: Euler equation solution

Algorithm

1. Construct a grid for k.
2. For each point on the grid, guess for a value of c_0.
3. Calculate the derivative as `dc0=D*c0`.
4. Find c_1 from

$$c_1 = \frac{dc_0(k^\alpha - \delta k)}{\frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho) + dc_0}$$

5. Back to step 3 with c_1 replacing c_0. Repeat until convergence.

The Ramsey growth model: Euler equation solution

Algorithm

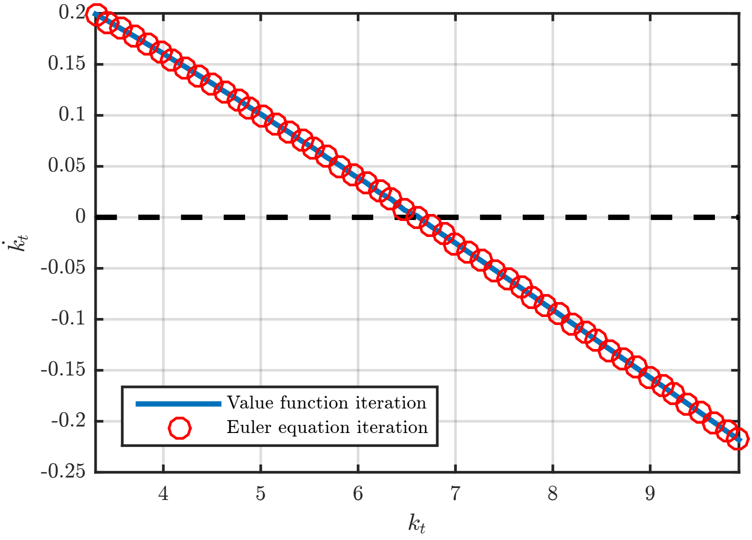
1. Construct a grid for k.
2. For each point on the grid, guess for a value of c_0.
3. Calculate the derivative as `dc0=D*c0`.
4. Find c_1 from

$$c_1 = \frac{dc_0(k^\alpha - \delta k)}{\frac{1}{\gamma}(\alpha k^{\alpha-1} - \delta - \rho) + dc_0}$$

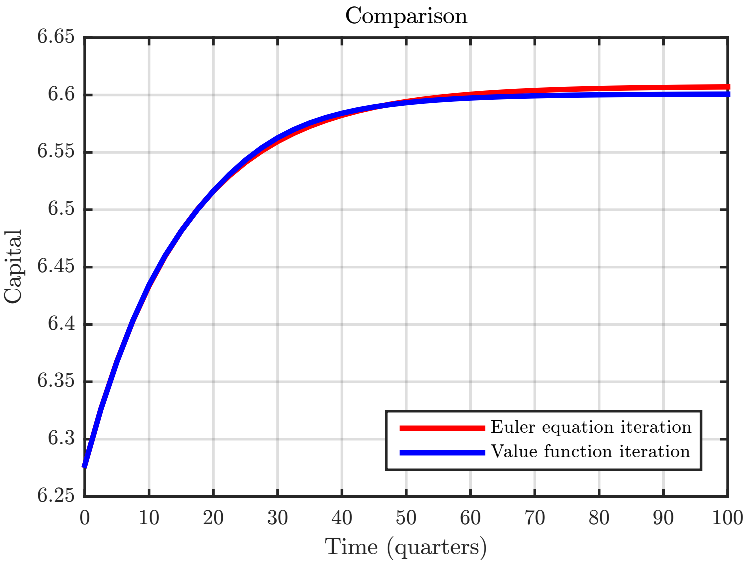
5. Back to step 3 with c_1 replacing c_0. Repeat until convergence.

Beware: No guaranteed convergence. Update slowly. Fewer gridpoints appears to provide some stability.

The Ramsey growth model: Solution



The Ramsey growth model: Solution



More on continuous time Euler equations

- We derived the Euler equation in a slightly roundabout way
  1. Discrete time Bellman equation
  2. To continuous time HJB equation
  3. To continuous time Euler equation using the envelope condition.
- This can be done more directly from the discrete time Euler equation.

More on continuous time Euler equations

- The discrete time Euler equation is given by

$$u'(c_t) = (1-\rho)(1+\alpha k_{t+1}^{\alpha-1}-\delta)u'(c_{t+1})$$

- In Δ units of time

$$u'(c_t) = (1-\Delta\rho)(1+\Delta(\alpha k_{t+\Delta}^{\alpha-1}-\delta))u'(c_{t+\Delta})$$

- Use the approximation $x_{t+\Delta} \approx x_t + \dot{x}_t\Delta$ to get

$$u'(c_t) = (1-\Delta\rho)(1+\Delta(\alpha k_{t+\Delta}^{\alpha-1}-\delta))u'(c_t+\dot{c}_t\Delta)$$

More on continuous time Euler equations

$$u'(c_t) = (1 - \Delta\rho)(1 + \Delta(\alpha k_{t+\Delta}^{\alpha-1} - \delta))u'(c_t + \dot{c}_t\Delta)$$

- Move the $u'(c_t + \dot{c}_t\Delta)$ term to the left-hand side and expand

$$\begin{aligned} & u'(c_t) - u'(c_t + \dot{c}_t\Delta) \\ & \quad = \Delta[\alpha k_{t+\Delta}^{\alpha-1} - \delta - \rho - \rho\Delta(\alpha k_{t+\Delta}^{\alpha-1} - \delta)]u'(c_t + \dot{c}_t\Delta) \end{aligned}$$

- Divide by Δ and take limits $\Delta \to 0$

$$-u''(c_t)\dot{c}_t = [\alpha k_t^{\alpha-1} - \delta - \rho]u'(c_t)$$

More on continuous time Euler equations

$$-u''(c_t)\dot{c}_t = [\alpha k_t^{\alpha-1} - \delta - \rho]u'(c_t)$$

- Lastly, use the CRRA property to get

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}[\alpha k_t^{\alpha-1} - \delta - \rho]$$

More on continuous time Euler equations

- Now consider a stochastic model with a "good", g, and a "bad", b, state

$$
\begin{aligned}
u'(c_t^g) = (1-\rho)[(1-p)(1+z_{t+1}^g\alpha(k_{t+1}^g)^{\alpha-1}-\delta)u'(c_{t+1}^g) \\
+ p(1+z_{t+1}^b\alpha(k_{t+1}^b)^{\alpha-1}-\delta)u'(c_{t+1}^b)]
\end{aligned}
$$

and

$$
\begin{aligned}
u'(c_t^b) = (1-\rho)[(1-q)(1+z_{t+1}^g\alpha(k_{t+1}^g)^{\alpha-1}-\delta)u'(c_{t+1}^g) \\
+ q(1+z_{t+1}^b\alpha(k_{t+1}^b)^{\alpha-1}-\delta)u'(c_{t+1}^b)]
\end{aligned}
$$

- We will focus on the good state (the treatment of the bad state is symmetric)

More on continuous time Euler equations

- Good state Euler equation in Δ units of time

$$u'(c_t^g) = (1-\Delta\rho)[(1-\Delta p)(1+\Delta(z_{t+\Delta}^g \alpha(k_{t+\Delta}^g)^{\alpha-1}-\delta)) \\ \times u'(c_{t+\Delta}^g) + \Delta p(1+\Delta(z_{t+\Delta}^b \alpha(k_{t+\Delta}^b)^{\alpha-1}-\delta))u'(c_{t+\Delta}^b)]$$

- Use $u'(c_{t+\Delta}^g) \approx u'(c_t^g + \dot{c}_t^g \Delta)$ again, move to the left-hand side, divide by Δ and take limits

$$-u''(c_t^g)\dot{c}_t^g = (z_t^g \alpha(k_t^g)^{\alpha-1} - \delta - \rho))u'(c_t^g) \\ + p(u'(c_t^b) - u'(c_t^g))$$

- Or

$$\frac{\dot{c}_t^g}{c_t^g} = \frac{1}{\gamma}(z_t^g \alpha(k_t^g)^{\alpha-1} - \delta - \rho)) + p\frac{(u'(c_t^b) - u'(c_t^g))}{u'(c_t^g)}$$

More on continuous time Euler equations

- For the bad state

$$\frac{\dot{c}_t^b}{c_t^b} = \frac{1}{\gamma}(z_t^b \alpha (k_t^b)^{\alpha-1} - \delta - \rho)) + q\frac{(u'(c_t^b) - u'(c_t^g))}{u'(c_t^b)}$$

- These can be solved using the previous methods. The only difference is that we now iterate on two equations instead of one. But the procedure is the same.

More on continuous time Euler equations

- As a last step, I just want to give you a hint on how these ideas can be applied in different settings.
- For instance, the Euler equation for a standard deterministic monetary model is given by

$$u'(c_t) = (1-\rho)(1+i_{t+1})\frac{p_t}{p_{t+1}}u'(c_{t+1})$$

- In Δ units of time

$$u'(c_t) = (1-\Delta\rho)(1+\Delta i_{t+1})\frac{p_t}{p_{t+\Delta}}u'(c_{t+\Delta})$$

More on continuous time Euler equations

- Use the approximations $u'(c_{t+\Delta}) \approx u'(c_t + \dot{c}_t\Delta)$, and $p_t \approx p_{t+\Delta} - \dot{p}_t\Delta$ and rewrite

$$u'(c_t) = (1 - \Delta\rho)(1 + \Delta i_{t+\Delta})\frac{p_{t+\Delta} - \dot{p}_t\Delta}{p_{t+\Delta}}u'(c_t + \dot{c}_t\Delta)$$

- Expand

$$u'(c_t) = (1 - \Delta\rho + \Delta i_{t+\Delta} - \Delta^2 i_{t+\Delta}\rho)(1 - \frac{\dot{p}_t\Delta}{p_{t+\Delta}})u'(c_t + \dot{c}_t\Delta)$$

- Thus

$$\begin{aligned} u'(c_t) - u'(c_t + \dot{c}_t\Delta) &= (-\Delta\rho + \Delta i_{t+\Delta} - \Delta^2 i_{t+\Delta}\rho) \\ &\times (1 - \frac{\dot{p}_t\Delta}{p_{t+\Delta}})u'(c_t + \dot{c}_t\Delta) - \frac{\dot{p}_t\Delta}{p_{t+\Delta}}u'(c_t + \dot{c}_t\Delta) \end{aligned}$$

More on continuous time Euler equations

- Previous equation

$$u'(c_t) - u'(c_t + \dot{c}_t\Delta) = (-\Delta\rho + \Delta i_{t+\Delta} - \Delta^2 i_{t+\Delta}\rho)$$
$$\times (1 - \frac{\dot{p}_t\Delta}{p_{t+\Delta}})u'(c_t + \dot{c}_t\Delta) - \frac{\dot{p}_t\Delta}{p_{t+\Delta}}u'(c_t + \dot{c}_t\Delta)$$

- Divide by Δ

$$\frac{u'(c_t) - u'(c_t + \dot{c}_t\Delta)}{\Delta} = (-\rho + i_{t+\Delta} - \Delta i_{t+\Delta}\rho)$$
$$\times (1 - \frac{\dot{p}_t\Delta}{p_{t+\Delta}})u'(c_t + \dot{c}_t\Delta) - \frac{\dot{p}_t}{p_{t+\Delta}}u'(c_t + \dot{c}_t\Delta)$$

- And take limit $\Delta \to 0$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}(i_t - \frac{\dot{p}_t}{p_t} - \rho)$$

The Mortensen-Pissarides model

- Continuous time is frequently used in the theoretical labor literature.
- In the remainder of this lecture I will go through the workhorse model developed by Christopher Pissarides and Dale Mortensen.
- In today's exercise you will be asked to solve this model.

The Mortensen-Pissarides model

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The Mortensen-Pissarides model: Workers

- In the simplest case workers are risk-neutral and value a job according to

$$V_t = w_t + (1-\rho)[(1-\delta)V_{t+1} + \delta U_{t+1}]$$

- The value of being unemployed is given by

$$U_t = b + (1-\rho)[(1-f_{t+1})U_{t+1} + f_{t+1}V_{t+1}]$$

- The variables w_t and f_{t+1} will be endogenously determined, but we will, for the moment, treat them as exogenous.

The Mortensen-Pissarides model: Workers

- Let's rewrite these equations in Δ units of time

$$V_t = \Delta w_t + (1 - \Delta\rho)[(1 - \Delta\delta)V_{t+\Delta} + \Delta\delta U_{t+\Delta}]$$
$$U_t = \Delta b + (1 - \Delta\rho)[(1 - \Delta f_{t+\Delta})U_{t+1} + \Delta f_{t+\Delta} V_{t+1}]$$

- Or

$$\begin{aligned} V_t - V_{t+\Delta} &= \Delta w_t - \Delta[\delta + \rho - \Delta\delta\rho]V_{t+\Delta} \\ &\qquad + (1 - \Delta\rho)\Delta\delta U_{t+\Delta} \\ U_t - U_{t+\Delta} &= \Delta b - \Delta[f_{t+\Delta} + \rho - \Delta f_{t+\Delta}\rho]U_{t+\Delta} \\ &\qquad + (1 - \Delta\rho)\Delta f_{t+\Delta} V_{t+\Delta} \end{aligned}$$

The Mortensen-Pissarides model: Workers

- Dividing through by Δ gives

$$\begin{aligned}\frac{V_t - V_{t+\Delta}}{\Delta} &= w_t - [\delta + \rho - \Delta\delta\rho]V_{t+\Delta} \\ &\qquad + (1-\Delta\rho)\delta U_{t+\Delta} \\ \frac{U_t - U_{t+\Delta}}{\Delta} &= b - [f_{t+\Delta} + \rho - \Delta f_{t+\Delta}\rho]U_{t+\Delta} \\ &\qquad + (1-\Delta\rho)f_{t+\Delta}V_{t+\Delta}\end{aligned}$$

- And taking limits $\Delta \to 0$ yields

$$\begin{aligned}-\dot{V}_t &= w_t - (\delta + \rho)V_t + \delta U_t \\ -\dot{U}_t &= b - (f_t + \rho)U_t + f_t V_t\end{aligned}$$

The Mortensen-Pissarides model: Workers

- These equations are commonly written as

$$\rho V_t = w_t + \dot{V}_t + \delta(U_t - V_t)$$
$$\rho U_t = b + \dot{U}_t + f_t(V_t - U_t)$$

- Define the surplus of having a job as $S_t = V_t - U_t$, that is

$$(\rho + \delta + f_t)S_t = w_t - b + \dot{S}_t$$

The Mortensen-Pissarides model: Firms

- The value to a firm of having an employed worker is

$$J_t = z_t - w_t + (1-\rho)[(1-\delta)J_{t+1} + \delta W_{t+1}]$$

- We will assume free entry, such that $W_t = 0$ for all t.
- Following the same procedure as before we find that in continuous time

$$(\rho + \delta)J_t = z_t - w_t + \dot{J}_t$$

The Mortensen-Pissarides model: Wages

- Collecting equations

$$\rho S_t = w_t - b + \dot{S}_t + (\delta + f_t) S_t$$
$$\rho J_t = z_t - w_t + \dot{J}_t - \delta J_t$$

- Wages are set according to Nash bargaining, which are renegotiated period-by-period

$$w_t = \text{argmax}\{J_t^{\eta} S_t^{1-\eta}\}$$

- First order condition

$$\eta S_t = (1 - \eta) J_t$$

The Mortensen-Pissarides model: Wages

- Expanding

$$\eta(w_t - b + \dot{S}_t + (\delta + f_t)S_t) = (1-\eta)(z_t - w_t + \dot{J}_t - \delta J_t)$$

and using the fact that

$$S_t = \frac{1-\eta}{\eta}J_t, \quad \text{and} \quad \dot{S}_t = \frac{1-\eta}{\eta}\dot{J}_t$$

gives

$$w_t = \eta b + (1-\eta)z_t + f_t(1-\eta)J_t$$

- Inserting into the firm's value function gives

$$\rho J_t = \eta(z_t - b) + \dot{J}_t - (f_t(1-\eta) + \delta)J_t$$

The Mortensen-Pissarides model: Matching

- Suppose that there are v_t vacancies posted and u_t unemployed individuals. Then the measure of matches in a given period is given as

$$M_t = \Delta\psi v_t^{\omega} u_t^{1-\omega}$$

- The probability that an unemployed individual finds a job, Δf_t, is then given as

$$\Delta f_t = \frac{M_t}{u_t} = \Delta\psi\theta_t^{\omega}, \quad \text{with } \theta = \frac{v_t}{u_t}$$

The Mortensen-Pissarides model: Matching

- The probability that a vacant position is filled, Δh_t, is then given as

$$\Delta h_t = \frac{M_t}{v_t} = \Delta\psi\theta_t^{\omega-1}$$

- Suppose the cost of posting a vacancy is given by $\Delta\kappa$. Free entry then ensures that

$$\kappa = h_t J_t$$

- To see this more clearly, a firm that is considering posting a vacancy faces the optimization problem

$$\max_{v_{t,i}}\{-\kappa\Delta v_{t,i} + v_{t,i}\Delta h_t J_t\}$$

The Mortensen-Pissarides model: Matching

- Lastly, employment, $n_t = 1 - u_t$, satisfies the law of motion

$$n_{t+\Delta} = (1 - n_t)\Delta f_t + (1 - \Delta\delta)n_t$$

which can be rearranged to

$$\frac{n_{t+\Delta} - n_t}{\Delta} = (1 - n_t)f_t - \delta n_t$$

taking limits

$$\dot{n}_t = (1 - n_t)f_t - \delta n_t$$

The Mortensen-Pissarides model:

The standard Mortensen-Pissarides model is therefore characterized by the three equations

$$
\begin{aligned}
\rho J_t &= \eta(z_t - b) + \dot{J}_t - (f_t(1-\eta) + \delta)J_t \\
\kappa &= h_t J_t \\
\dot{n}_t &= (1 - n_t)f_t - \delta n_t
\end{aligned}
$$

in the three unknowns $J_t, \theta_t, \dot{n}_t$.