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Partial differential equation models in macroeconomics

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The purpose of this article is to get mathematicians interested in studying a number of partial differential equations (PDEs) that naturally arise in macroeconomics. These PDEs come from models designed to study some of the most important questions in economics. At the same time, they are highly interesting for mathematicians because their structure is often quite difficult. We present a number of examples of such PDEs, discuss what is known about their properties, and list some open questions for future research.

1. Introduction

Macroeconomics is the study of large economic systems. Most commonly, this system is the economy of a country. But, it may also be a more complex system such as the world as a whole, comprising a large number of interacting smaller geographical regions. Macroeconomics is concerned with some of the most important questions in economics, for example: what causes recessions and what should be done about them? Why are some countries so much poorer than others?

Traditionally, macroeconomic theory has focused on studying systems of difference equations or ordinary differential equations describing the evolution of a relatively small number of macroeconomic aggregates. These systems are typically derived from the optimal control problem of a 'representative agent'. In the past 30 years, however, macroeconomics has seen the development of theories that explicitly model the equilibrium interaction of heterogeneous agents, e.g. heterogeneous consumers, workers and firms (see, in particular, the early contributions of Bewley [1], Aiyagari [2], Huggett [3] and Hopenhayn [4]).

The development of these theories opens up the study of a number of important questions: why are income and wealth so unequally distributed? How is inequality affected by aggregate economic conditions? Is there a trade-off between inequality and economic growth? What are the forces that lead to the concentration of economic activity in a few very large firms? And why do instabilities in the financial sector seem to matter so much for the macroeconomy?

Heterogeneous agent models are usually set in discrete time. While they are workhorses of modern macroeconomics, relatively little is known about their theoretical properties and they often prove difficult to compute. To make progress, some recent papers have therefore studied continuous time versions of such models. Our paper reviews this literature. Macroeconomic models with heterogeneous agents share a common mathematical structure which, in continuous time, can be summarized by a system of coupled nonlinear partial differential equations (PDEs): (i) a Hamilton–Jacobi–Bellman (HJB) equation describing the optimal control problem of a single atomistic individual and (ii) an equation describing the evolution of the distribution of a vector of individual state variables in the population (such as a Fokker–Planck equation, Fisher–KPP equation or Boltzmann equation).[1] While plenty is known about the properties of each type of equation individually, our understanding of the coupled system is much more limited. Lasry & Lions [5–7] and Lions [8] have termed such a system a 'mean field game' and obtained some theoretical characterizations for special cases, but many open questions remain. For useful reference on mean field games, one can see for example Bardi [9], Guéant [10], Guéant *et al.* [11], Gomes *et al.* [12] and Cardaliaguet [13]. The purpose of this article is to present important examples of these systems of PDEs that arise naturally in macroeconomics, to discuss what is known about their properties, and to highlight some directions for future research.

In §2, we present a model describing an economy consisting of a continuum of heterogeneous individuals that face income shocks and can trade a risk-free bond that is in zero net supply. This is the simplest model to illustrate the basic structure of heterogeneous agent frameworks used in macroeconomics, and it is the building block of many models studying the interaction between macroeconomic aggregates and the distribution of income and wealth. In §3, we review PDEs that have been used to describe the distribution of the many economic variables that obey power laws, for example, city and firm size, wealth and executive compensation. One building block of all of these models is the Fokker–Planck equation for a geometric Brownian motion. This equation is then combined with a model of exit and entry, for instance taking the form of a variational inequality of the obstacle type, derived from an optimal stopping time problem. In §4, we present a class of models describing processes of economic growth owing to experimentation and knowledge diffusion, or alternatively the percolation of information in financial markets. These models generate richer, more non-local dynamics, that give rise to Fisher–KPP- or Boltzmann-type equations.

In §5, we introduce a class of models that is substantially more complicated than those in the preceding sections: models with 'aggregate shocks' designed to study business cycle fluctuations. These theories have the property that macroeconomic aggregates, including the distribution of individual states, are stochastic variables rather than just varying deterministically as in the models studied thus far. This creates the difficulty that the distribution—an infinite-dimensional object—has to be included in the state space of the individual optimal control problem. The resulting optimal control problem is no longer a standard HJB equation but instead an 'HJB equation in the space of density functions', a very challenging object mathematically. We present the most canonical version of such a theory: the model in §2 but now with aggregate income shocks. But, in principle, any of the theories in the preceding sections could be enriched

[1]Heterogeneous agent models used in macroeconomics typically make the assumption that individuals have identical preferences (even though they are heterogeneous in other dimensions). It is for this reason that only two equations are sufficient for summarizing such economies. Models with heterogeneous preferences can be considered but they involve more equations.

by introducing such aggregate shocks. Finally, in §6, we note that also models with a finite number of agents, rather than a continuum as in the preceding sections, are of interest in certain macroeconomic applications. We present a model of firm dynamics in an oligopolistic industry which takes the form of a differential game.

Space limitations have forced us to leave out other important areas of macroeconomics and economics more broadly where PDEs, and continuous time methods in general, have played an important role in recent years. A good example is the large literature studying the design of optimal dynamic contracts and policies. See for example the recent work by Sannikov [14], Williams [15] and Farhi & Werning [16]. Another area is given by models of the labour market. See for example the recent contribution by Alvarez & Shimer [17] and the review by Lentz & Mortensen [18]. Finally, throughout this paper, we focus on equilibrium allocations in which individuals take as given the actions of others rather than coordinating with them. As a result, these equilibrium allocations are in general suboptimal from the point of view of society as a whole.[2] Optimal allocations in heterogeneous agent models can be analysed along the lines of Nuño [19] and Lucas & Moll [20].

2. Income and wealth distribution

The discrete time model of Aiyagari [2], Bewley [1] and Huggett [3] is one of the workhorses of modern macroeconomics. This model captures in a relatively parsimonious way the evolution of the income and wealth distribution and its effect on macroeconomic aggregates. It is a natural framework to study the effect of various policies and institutions on inequality. A huge number of problems in macroeconomics have a similar structure and so this is a particularly useful starting point. The simplest formulation of the model is due to Huggett [3] and we here present a continuous time formulation of Huggett's model presented in Achdou *et al.* [21].[3]

There is a continuum of infinitely lived households that are heterogeneous in their wealth a and their income z. Households solve the following optimization problem:

$$\begin{aligned}
\max_{\{c_t\}} \quad & \mathbb{E}_0 \int_0^\infty \mathrm{e}^{-\rho t} u(c_t)\,\mathrm{d}t \quad \text{s.t.} \\
& \mathrm{d}a_t = (z_t + r(t)a_t - c_t)\,\mathrm{d}t \\
& \mathrm{d}z_t = \mu(z_t)\,\mathrm{d}t + \sigma(z_t)\,\mathrm{d}W_t \\
& a_t \geq \underline{a}.
\end{aligned}$$

Households have utility functions $u(c)$ over consumption c that are strictly increasing and strictly concave (e.g. $u(c) = c^{1-\gamma}/(1-\gamma)$, $\gamma > 0$) and they maximize the present discounted value of utility from consumption, discounted at rate ρ. Households can borrow and save at an interest rate $r(t)$ which is determined in equilibrium and they optimally choose how to split their total income $z_t + r(t)a_t$ between consumption and saving. Their income evolves exogenously according to a diffusion process $\mathrm{d}z_t = \mu(z_t)\,\mathrm{d}t + \sigma(z_t)\,\mathrm{d}W_t$ in a closed interval $[\underline{z}, \bar{z}]$ (it is reflected at the boundaries if it ever reaches them). Importantly, there is a *state constraint* $a \geq \underline{a}$ for some scalar $-\infty < \underline{a} \leq 0$. This state constraint has the economic interpretation of a *borrowing constraint*, e.g. if $\underline{a} = 0$ households can only save and cannot borrow at all.

The interest rate $r(t)$ must be such that the following equilibrium condition is satisfied:

$$\int a g(a, z, t)\,\mathrm{d}a\,\mathrm{d}z = 0,$$

where $g(a, z, t)$ denotes the cross-sectional distribution of households with wealth a and income z at time t. The interpretation of this equilibrium condition is as follows: wealth a here takes the form of bonds, and the equilibrium interest rate r is such that bonds are in *zero net supply*. That is, for every dollar borrowed, there is someone else who saves a dollar.

[2]In the sense of the 'Pareto optimality' criterion typically used in economics: there exists an alternative allocation such that all individuals in the economy are weakly better off.

[3]Also see Bayer & Waelde [22,23] who explore a similar model.

The equilibrium can be characterized in terms of an HJB equation for the value function v and a Fokker–Planck equation for the density of households g. In a *stationary* equilibrium, the unknown functions v and g and the unknown scalar r satisfy the following system of coupled PDEs (stationary mean field game) on $(\underline{a}, \infty) \times (\underline{z}, \bar{z})$:

$$\frac{1}{2}\sigma^2(z)\partial_{zz}v + \mu(z)\partial_z v + (z + ra)\partial_a v + H(\partial_a v) - \rho v = 0, \tag{2.1}$$

$$-\frac{1}{2}\partial_{zz}(\sigma^2(z)g) + \partial_z(\mu(z)g) + \partial_a((z + ra)g) + \partial_a(\partial_p H(\partial_a v)g) = 0, \tag{2.2}$$

$$\int g(a, z)\,\mathrm{d}a\,\mathrm{d}z = 1, \quad g \geq 0 \tag{2.3}$$

and

$$\int ag(a, z)\,\mathrm{d}a\,\mathrm{d}z = 0, \tag{2.4}$$

where the Hamiltonian H is given by

$$H(p) = \max_{c \geq 0}(-pc + u(c)). \tag{2.5}$$

The function v satisfies a state constraint boundary condition at $a = \underline{a}$ and Neumann boundary conditions at $z = \underline{z}$ and $z = \bar{z}$.

In general, the boundary value problem including the Bellman equation (2.1) and the boundary condition has to be understood in the sense of viscosity (see Bardi & Capuzzo [24], Crandall *et al.* [25], Barles [26]), whereas the boundary problem with the Fokker–Planck equation (2.3) is set in the sense of distributions. An important issue is to check that (2.1) actually yields an optimal control (verification theorem): this is a direct application of Itô's formula if v is smooth enough; for general viscosity solutions, one may apply the results of Bouchard & Touzi [27] and Touzi [28] (this has not been done yet).

With well chosen initial and terminal conditions, solutions to the HJB equation (2.1) are expected to be smooth and we therefore look for such smooth solutions. If v is indeed smooth, the state constraint boundary condition can be shown to imply

$$(z + r\underline{a})\lambda + H(\lambda) \geq (w + r\underline{a})\partial_a v(\underline{a}, z) + H(\partial_a v(\underline{a}, z)) \quad \forall \lambda \geq \partial_a v(\underline{a}, z)$$

or equivalently

$$z + ra + \partial_p H(\partial_a v) \geq 0, \quad a = \underline{a} \tag{2.6}$$

so that the trajectory of a points towards the interior of the state space. Finally, note that the interest rate r—which is determined by the equilibrium condition (2.4)—is the only variable through which the distribution g enters the HJB equation (2.1).

The time-dependent analogue of (2.1)–(2.4) is also of interest. In the time-dependent equilibrium, the unknown functions v and g satisfy the following system of coupled PDEs (time-dependent mean field game) on $(\underline{a}, \infty) \times (\underline{z}, \bar{z}) \times (0, T)$:

$$\partial_t v + \frac{1}{2}\sigma^2(z)\partial_{zz}v + \mu(z)\partial_z v + (z + r(t)a)\partial_a v + H(\partial_a v) - \rho v = 0, \tag{2.7}$$

$$\partial_t g - \frac{1}{2}\partial_{zz}(\sigma^2(z)g) + \partial_z(\mu(z)g) + \partial_a((z + r(t)a)g) + \partial_a(\partial_p H(\partial_a v)g) = 0, \tag{2.8}$$

$$\int g(a, z, t)\,\mathrm{d}a\,\mathrm{d}z = 1, \quad g \geq 0 \tag{2.9}$$

and

$$\int ag(a, z, t)\,\mathrm{d}a\,\mathrm{d}z = 0, \tag{2.10}$$

where the Hamiltonian H is given by (2.5). The density g satisfies the initial condition $g(a, z, 0) = g_0(a, z)$. For the terminal condition for the value function v, we generally take T large and impose $v(a, z, T) = v_\infty(a, z)$, where v_∞ is the stationary value function, i.e. the solution to the stationary problem (2.1)–(2.4).[4] The function v also still satisfies the state constraint boundary condition (2.6) and Neumann boundary conditions at $z = \underline{z}$ and $z = \bar{z}$.

(a) Theoretical results

Achdou *et al.* [21] have analysed some theoretical properties of both the time-varying and stationary problems. We here briefly review the (rather incomplete) theoretical knowledge of these problems, followed by a list of open questions regarding in particular the well-posedness of the problems. Achdou *et al.* [21] first analyse the stationary problem (2.1)–(2.4) under the additional assumption that the state constraint satisfies $\underline{a} > -\underline{z}/r$. They show that under this assumption the state constraint (2.6) binds for low enough income z, that is, the borrowing constraint is 'tight'. Intuitively, individuals with low income z would like to borrow but cannot if their wealth is already at $\underline{a}$. Of particular interest is the stationary *saving policy function*

$$s(a,z) = z + ra + \partial_p H(\partial_a v(a,z)),$$

that is the optimally chosen drift of wealth a, and the behaviour of the implied stationary distribution g. Importantly, one can show that the expansion of the function s around $\underline{a}$ satisfies the following property: there exists z^* with $\underline{z} < z^* < \bar{z}$ such that

$$s(a,z) \sim -\bar{s}_z\sqrt{a - \underline{a}}, \quad \bar{s}_z > 0, \quad \underline{z} \le z \le z^*, \tag{2.11}$$

meaning that in particular the derivative $\partial_a s$ becomes unbounded when we let a go to $\underline{a}$. It then follows from this property that the stationary distribution g is unbounded and has a Dirac mass at $a = \underline{a}$ for $z \le z^*$. The existence of a Dirac mass in the stationary version of (2.7)–(2.10) of course complicates the mathematics substantially. At the same time, it is also one of the economically most interesting predictions of the model. What fraction of individuals in an economy such as that of the USA are borrowing constrained and how we would expect this to change when various features of the environment (say the stochastic process for z) change is an important question with wide-reaching policy implications. That interesting economics and challenging mathematics go hand in hand is one of the main themes of this paper.

Achdou *et al.* [21] prove the existence of a solution to (2.1)–(2.4), i.e. of a stationary equilibrium. The key step in the proof is to analyse solutions v and g to (2.1)–(2.3) *for given* r and to show that the corresponding first moment of g, $m(r) = \int ag(a,z)\,\mathrm{d}a\,\mathrm{d}z$, goes to $\underline{a}$ as $r \to -\infty$ and that it becomes unbounded as we take r to ρ_-. It follows from this that there exists an r such that (2.4) holds. Currently, open theoretical questions are

1. Uniqueness of a solution to (2.1)–(2.4), i.e. of a stationary equilibrium.
2. Existence of a solution to (2.7)–(2.10), i.e. of a time-dependent equilibrium.
3. Uniqueness of a solution to (2.7)–(2.10), i.e. of a time-dependent equilibrium.

The main difficulty in the first question, uniqueness of a stationary solution, lies in showing that (or finding conditions under which) the first moment of g, $m(r)$, is monotone as a function of r. It should also be noted that non-uniqueness is a very real possibility in many equilibrium models arising in economics, and that a better understanding of the conditions under which non-uniqueness can arise is equally interesting to economists as proving uniqueness.

(b) Numerical methods

Achdou *et al.* [21] have also developed numerical methods for solving both the stationary and time-dependent problems, based on Achdou & Capuzzo-Dolcetta [29] and Achdou [30]. Figure 1 plots the optimal stationary saving policy function s and the implied distribution

[4]In principle, the stationary mean-field game (2.1)–(2.4) may not have a unique solution, and hence v_∞ may not be uniquely defined. As we discuss in more detail below, we have not found any examples of such non-uniqueness in our numerical simulations.

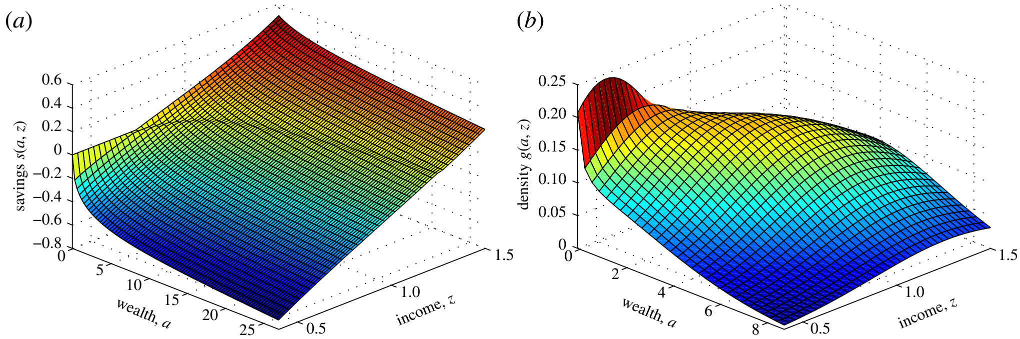

Figure 1. Numerical solution to stationary equilibrium (2.7)–(2.10). (*a*) Saving as function of income and wealth. (*b*) Distribution of income and wealth. (Online version in colour.)

g. These are computed under the assumption that u in (2.5) is given by $u(c)=c^{1-\gamma}/(1-\gamma)$ with $\gamma=2$. In figure 1, one can see that s satisfies (2.11) and g has a Dirac mass for low z (the numerical method computes discretized versions of the equations, so the Dirac mass corresponds to a finite density). Time-dependent solutions can be computed in a similar fashion and the evolution of the distribution over time can be visualized as 'movies' (e.g. http://www.princeton.edu/~moll/aiyagari.mov).

An interesting exercise is to 'calibrate' this model and to compare the resulting distribution of wealth illustrated in figure 1 with that in empirical data for developed countries. In the data, wealth is extremely unequally distributed. For example, in the USA, the top 1% richest individuals own around 35% of aggregate wealth [31,32]. In contrast, it turns out that the degree of wealth inequality generated by this model is substantially smaller than the one observed in the data. This observation was first made by Aiyagari [2]. This has motivated the study of richer models of individual heterogeneity and wealth accumulation. Examples include models in which individuals have access to different returns to their savings [31,33], for instance because they run private enterprises in a world with imperfect capital markets [34–36], and models in which individuals have different preferences for current and future consumption [37]. A close interplay between numerical solutions of calibrated models and data is a central theme in the macroeconomic literature reviewed in this paper (and which we do not discuss in more detail due to space limitations).

3. Models of power laws

One of the most ubiquitous regularities in empirical work in economics and finance is that the empirical distribution of many variables can well be approximated by a power law. Examples are the distributions of income and wealth, of the size of cities and firms, stock market returns, trading volume and executive pay. See Gabaix [38], who reviewed the theoretical and empirical literature on power laws.

Gabaix [39] has proposed a simple explanation of power law phenomena that naturally leads to PDEs: many variables follow geometric Brownian motions, combined with a 'small friction' such as a minimum size in the form of a reflecting barrier or small 'death shocks'. The following material is based on Gabaix [38]. Consider a stochastic process

$$\frac{dz_t}{z_t}=\bar{\mu}\,dt+\bar{\sigma}\,dW_t, \tag{3.1}$$

where $\bar{\mu}<0$ and $\bar{\sigma}>0$ are scalars. For sake of concreteness, consider the case where z represents the size or productivity of a firm, and we are interested in the firm size distribution. But of course z could be city size or any other variable as well. Further assume that there is a minimum firm size $z_{\min}$ in the form of a reflecting barrier (other mechanisms are possible as well and we explore some below). The stationary firm size distribution f satisfies the Fokker–Planck equation

$$\tfrac{1}{2}\partial_{zz}(\bar{\sigma}^2 z^2 f) - \partial_z(\bar{\mu} z f) = 0 \tag{3.2}$$

and

$$\int f(z)\,\mathrm{d}z = 1, \quad f \geq 0, \tag{3.3}$$

on $(z_{\min}, \infty)$, with boundary condition

$$\tfrac{1}{2}\partial_z(\bar{\sigma}^2 z^2 f) - \bar{\mu} z f = 0, \quad z = z_{\min}.$$

It is easy to see that the solution to (3.2) is

$$f(z) = \zeta z_{\min}^{\zeta} z^{-\zeta-1}, \quad \zeta = 1 - \frac{2\bar{\mu}}{\bar{\sigma}^2}, \tag{3.4}$$

that is, a power law with exponent $\zeta > 1$. This basic idea can be generalized in a number of ways and applied in a number of different contexts and we here review some of these other applications.

(a) Entry, exit and firm size distribution

An important paper by Luttmer [40] has applied the same logic to the question why the size distribution of firms follows a power law. We here review a simplified version of Luttmer's model. The problem also corresponds to a continuous time formulation of that originally studied by Hopenhayn [4]. Each firm has a profit function $\pi(z, m[f])$ which is strictly increasing in its own productivity z, and strictly decreasing in a geometric average of all other firms' productivities[5]

$$m[f] = \left(\int z^{\theta} f(z)\,\mathrm{d}z\right)^{1/\theta}, \quad \theta > 0.$$

The value of a firm is the present discounted value of profits, discounted at rate ρ. Firms' productivity and hence profits evolve according to the stochastic process $\mathrm{d}z_t = \mu(z_t)\,\mathrm{d}t + \sigma(z_t)\,\mathrm{d}W_t$ which we later specialize to (3.1), following Luttmer [40]. Firms have only one choice: whether to remain active or whether to exit the industry. If a firm exits the industry, it obtains a scrap value ψ, but it can never re-enter the industry. When firms exit, they mechanically get replaced by a group of entrants of equal mass whose initial productivity is given by some finite and positive z_0.

Firms therefore solve a *stopping time* problem

$$v(z_0) = \max_{\tau} \mathbb{E}_0 \left[\int_0^{\tau} \mathrm{e}^{-\rho t} \pi(z_t, m[f])\right] \mathrm{d}t + \mathrm{e}^{-\rho\tau}\psi,$$
$$\mathrm{d}z_t = \mu(z_t)\,\mathrm{d}t + \sigma(z_t)\,\mathrm{d}W_t.$$

[5]This dependence is motivated as follows. Firms face demand functions $(p(z)/P)^{-\gamma}$, $\gamma > 1$ where $p(z)$ is the price of firm z and P is a 'price index' $P = (\int p(z)^{1-\gamma} f(z)\,\mathrm{d}z)^{1/(1-\gamma)}$. This specification of the demand function is standard in economics (so-called isoelastic demand obtained from Spence–Dixit–Stiglitz preferences). Each firm's profit function is given by

$$\pi = \max_p\, p\left(\frac{p}{P}\right)^{-\gamma} - \frac{1}{z}\left(\frac{p}{P}\right)^{-\gamma} = (\bar{p}-1) z^{\gamma-1} \bar{p}^{-\gamma} P^{\gamma}, \quad \bar{p} = \frac{\gamma}{\gamma-1}$$

and the optimal price is $p(z) = \bar{p}/z$, so that $P = \bar{p}(\int z^{\gamma-1} f(z)\,\mathrm{d}z)^{1/(1-\gamma)}$ and hence $\pi(z, m[f]) = (\bar{p}-1) z^{\gamma-1} (m[f])^{-\gamma}$ with $\theta = \gamma - 1$.

The value function $v(z)$ can be characterized by a *variational inequality of the obstacle type* [24,41]. As before the density of firms f satisfies a Fokker–Planck equation. In the stationary version of this problem, the unknown functions v and f satisfy

$$\min\left\{\rho v - \frac{1}{2}\sigma^2(z)\partial_{zz}v - \mu(z)\partial_z v - \pi(z, m[f]), v - \psi\right\} = 0, \tag{3.5}$$

$$\frac{1}{2}\partial_{zz}(\sigma^2(z)f) - \partial_z(\mu(z)f) = 0, \tag{3.6}$$

$$\int f(z)\,\mathrm{d}z = 1, \quad f \geq 0 \tag{3.7}$$

and

$$m[f] = \left(\int z^\theta f(z)\,\mathrm{d}z\right)^{1/\theta} \tag{3.8}$$

on $\mathbb{R}^+$.[6] The variational inequality (3.5) now determines an endogenous threshold $z_{\min}$ at which firms exit. Because firms exit immediately when their productivity reaches $z_{\min}$, f satisfies the boundary condition $f(z_{\min}) = 0$. Luttmer [40] shows that under the assumption that $\mu(z) = \bar{\mu}z$, $\sigma(z) = \sigma z$ with $\bar{\mu} < 0$ and $\bar{\sigma} > 0$ (i.e. z_t follows (3.1)) and some other appropriately chosen assumption (e.g. that π is a power function with appropriately chosen exponents), the system can be solved explicitly. He further shows that, for $z > z_0$, the stationary distribution satisfies $f(z) = cz^{-\zeta-1}$ for some constant $c > 0$ and with ζ given by the same formula as in (3.4). The model of firm dynamics considered here therefore generates the empirical regularity that the right tail of the firm size distribution follows a power law.

While the case in which z_t follows a geometric Brownian motion (3.1) is very well understood, a natural question is what the exit decision and the firm size distribution look like for more general stochastic processes and perhaps also more general interdependencies between firms $m[f]$. For this more general set-up, open questions are

1. Existence and uniqueness of a stationary equilibrium, i.e. solutions to (3.5)–(3.8).
2. Existence and uniqueness of the time-dependent counterpart.
3. Development of numerical methods for solving both stationary and time-dependent equilibria.

Stokey [42] discusses other examples of stopping time problems in economics, many of them describing richer versions of the model of firm dynamics introduced in this section. This includes the problem of firms that set their price subjected to an adjustment cost. These models are important in macroeconomics, because the existence of frictions to the adjustment of prices is the main motivation for the use of monetary policy to stabilize business cycle fluctuations. Recent examples are given by Golosov & Lucas [43] and Alvarez & Lippi [44].

(b) Other applications of theories of power laws

The ideas presented in the preceding two sections have been used to understand the emergence of power laws in a number of different contexts. For example, Benhabib *et al.* [31] and in particular Benhabib *et al.* [33] develop models of the wealth distribution whose mathematical structure is quite similar to the one presented here. Similarly, Jones [45] applies the same insights into the question why the top of the income distribution (the infamous 'one percent') can be well described by a power law.

[6] Equation (3.5) can also be written somewhat more intuitively as

$$0 = \begin{cases} \rho v - \frac{1}{2}\sigma^2(z)\partial_{zz}v - \mu(z)\partial_z v - \pi(z, m[f]), & v - \psi \geq 0 \\ v - \psi, & \rho v - \frac{1}{2}\sigma^2(z)\partial_{zz}v - \mu(z)\partial_z v - \pi(z, m[f]) \geq 0. \end{cases}$$

4. Knowledge diffusion and growth

We now present some models of knowledge diffusion that have recently been used in macroeconomics, international trade and finance. These differ from the classic mean field games presented in §§2 and 3 in that the law of motion of the distribution no longer takes the form of a Fokker–Planck equation describing a local diffusion process. Instead, this law of motion takes the form of equations describing richer more 'non-local' dynamics, for example Fisher–KPP- or Bolztmann-type equations. They also differ in that the long-run behaviour of the systems they describe is not stationary. Instead, these models are designed to feature sustained growth. As such they can be used to try to answer some of the most important questions in economics, for example: what generates long-run growth? What is the relation between growth and inequality? In §4*a*, we first present some problems that are purely 'mechanical' in the sense that they do not feature an optimal control problem. We then add such a control problem in §4*b*.

(a) Diffusion and experimentation as an engine of growth

The following is based on Alvarez *et al.* [46], Lucas [47] and in particular Luttmer [48]. Consider an economy populated by a continuum of individuals indexed by their productivity or knowledge $z \in \mathbb{R}^+$. The economy is described by its distribution of knowledge with cdf $G(z,t)$. The evolution of G is modelled as a process of individuals meeting others from the same economy, comparing ideas, improving their own productivity. Meetings happen at Poisson intensity α, and from the point of view of an individual, a meeting is simply a random draw from the distribution G. When a meeting occurs, a person z compares his or her productivity with the person he or she meets and leaves the meeting with the best of the two productivities $\max\{z, z'\}$. Individual productivities also fluctuate in the absence of a meeting. In particular individuals 'experiment' and their productivity may either increase or decrease according to the process $\mathrm{d}\log z_t = \sigma\,\mathrm{d}W_t,\ \ \sigma > 0$. Given this structure, it is convenient to work with $x = \log z$ and the corresponding distribution F, and one can show that this distribution satisfies

$$\partial_t F - \frac{\sigma^2}{2}\partial_{xx} F = -\alpha F(1-F), \tag{4.1}$$

on $\mathbb{R} \times \mathbb{R}^+$, and with boundary conditions

$$\lim_{x\to-\infty} F(x,t) = 0, \quad \lim_{x\to\infty} F(x,t) = 1, \quad F(x,0) = F_0(x), \tag{4.2}$$

where $F_0(x)$ is the initial productivity distribution. As Luttmer [48] points out, this is a Fisher–KPP-type equation [49,50] whose theoretical properties are well understood [51]. In particular, one can show that (4.1) admits 'travelling wave' solutions, i.e. solutions of the form

$$F(x,t) = \Phi(x - \gamma t). \tag{4.3}$$

One can further show that if the initial distribution is a Dirac point mass, the limiting distribution is a travelling wave with $\gamma = \sigma\sqrt{2\alpha}$. If the distribution F is a travelling wave (4.3), productivity $z = \mathrm{e}^x$ is on average growing at the constant rate γ and hence one can say that the economy is on a 'balanced growth path' with growth rate γ. The interpretation of the formula for the growth rate $\gamma = \sigma\sqrt{2\alpha}$ is also very natural: it says that it is the combination of 'experimentation' parametrized by σ and 'diffusion' parametrized by α that is the engine of growth in this economy. Either force in isolation would lead to stagnation, but the two together create sustained growth. Similarly, applying some results from the literature studying (4.1), Luttmer [48] shows that the distribution Φ on this balanced growth path satisfies $(1-\Phi(x))/\mathrm{e}^{-\zeta x} \to c$ as $x \to \infty$ with $\zeta = \sqrt{2\alpha}/\sigma$ meaning that the distribution of $z = \mathrm{e}^x$ follows an asymptotic power law with parameter ζ with a low ζ

indicating a fat tail, i.e. a high degree of inequality. Summarizing, on a balanced growth path, the model generates the following predictions for the growth rate and inequality in the tail of the income distribution:

$$\text{growth} = \gamma = \sigma\sqrt{2\alpha}, \quad \text{inequality} = \frac{1}{\zeta} = \frac{\sigma}{\sqrt{2\alpha}}. \tag{4.4}$$

The theory presented in this section therefore has some non-trivial implications for one of the big questions raised in the introduction of this paper: is there a trade-off between inequality and growth? In particular, varying the parameter σ which parametrizes the degree of uncertainty at the individual level, one can see that there is indeed such a trade-off: a rise in σ leads not only to higher growth, but also higher inequality. Interestingly, however, the same is *not* true for variations in the parameter α measuring how much knowledge diffusion takes place: a rise in α leads to both higher growth and *lower* inequality. Now, to make things more interesting, the reader should imagine an extension of the model presented here where α and σ are outcomes of choices and/or can be affected by economic policy. We pursue one such extension in §4*b*. In such an environment, policies that increase the amount of knowledge diffusion α have the twin benefits of stimulating growth while at the same time reducing inequality.

Economists have studied various versions of the Fisher–KPP equation (4.1). Lucas [47] and Alvarez *et al.* [46] study the version of (4.1) with $\sigma = 0$:

$$\partial_t F = -\alpha F(1-F), \tag{4.5}$$

on $\mathbb{R} \times \mathbb{R}^+$. To generate sustained growth, they assume that the initial distribution satisfies $(1 - F_0(x))/\mathrm{e}^{-\zeta x} \to c$ as $x \to \infty$ for some constants $c, \zeta > 0$, meaning that the initial distribution for $z = \mathrm{e}^x$ is asymptotically a power law as in (3.4).[7] Luttmer [52] studies the equation

$$\partial_t F - \frac{\sigma^2}{2}\partial_{xx} F = -\alpha \min\{F, 1-F\}$$

on $\mathbb{R} \times \mathbb{R}^+$ which can be solved explicitly.

(b) Knowledge diffusion and search

While the models in the previous section are interesting in that they describe environments in which there is sustained growth, they are somewhat less interesting than those in §§2 and 3 in that individuals in the economy did not make any choices, i.e. solve optimal control problems. Lucas & Moll [20] extend the set-up in the previous section to feature such an optimal choice. In this extension, one can then ask questions such as: is the equilibrium growth rate of the economy optimal or should policy makers intervene to boost (or perhaps depress) economic growth?

In Lucas & Moll [20], individuals have one unit of time and they can split it between producing with the knowledge they already have, or they can search for productivity enhancing ideas. Search increases the likelihood of meeting other individuals. In particular, the Poisson meeting rate of an individual who searches a fraction s of their time is $\alpha(s)$ which is strictly increasing and concave. Conditional on a meeting the knowledge diffusion process is exactly as described in the previous section. The cost of search is that it interferes with production. In particular, the output of an individual with productivity $z = \mathrm{e}^x$ who searches a fraction s of their time is $(1-s)\mathrm{e}^x$. Individuals maximize the present discounted value of future output

$$v(x,0) = \max_{s_t \in [0,1]} \mathbb{E}_0 \int_0^\infty \mathrm{e}^{-\rho t}(1-s_t)\mathrm{e}^{x_t}\,\mathrm{d}t$$
$$\mathrm{d}x_t = \sigma\,\mathrm{d}W_t + \mathrm{d}J_t,$$

[7] As shown by Luttmer [48], the travelling wave solution obtained in the case (4.1) with $\sigma > 0$ satisfies this property and hence this is a relatively innocuous assumption.

where J_t is a Poisson process with intensity $\alpha(s_t)$ that jumps when individuals learn something useful. The equilibrium of this economy can be described in terms of a system of two integro-PDEs for the value function v and the density of the productivity distribution f:

$$\partial_t v + \frac{\sigma^2}{2}\partial_{xx} v + \max_{s\in[0,1]}\left\{(1-s)\mathrm{e}^x + \alpha(s)\int_x^\infty (v(y,t) - v(x,t))f(y,t)\,\mathrm{d}y\right\} - \rho v = 0, \tag{4.6}$$

$$\partial_t f - \frac{\sigma^2}{2}\partial_{xx} f + \alpha(s^*(x,t))f(x,t)\int_x^\infty f(y,t)\,\mathrm{d}y - f(x,t)\int_{-\infty}^x \alpha(s^*(y,t))f(y,t)\,\mathrm{d}y = 0 \tag{4.7}$$

and
$$\int f(z,t)\,\mathrm{d}z = 1, \quad f \geq 0, \tag{4.8}$$

on $\mathbb{R}\times\mathbb{R}^+$ and where s^* is the maximand of (4.6). There is also an initial condition $f(x,0) = f_0(x)$. It can be seen that (4.1) is the special case of (4.7) in which the optimal control s^* and hence also α are constant across x-types, and written in terms of the cdf $F(x,t) = \int_0^x f(x,t)\,\mathrm{d}x$. However, in general, it will not be true that s^* is constant for all x. Instead, s^* is usually decreasing in x. Lucas & Moll [20] study the special case of (4.6)–(4.8) with $\sigma = 0$. They show that the system admits solutions of the travelling wave type, that is

$$v(x,t) = w(x-\gamma t), \quad f(x,t) = \phi(x-\gamma t),$$

and they develop numerical methods for computing such solutions numerically, and in particular to find the growth rate γ of the system. However, there remain many open questions, among which are

1. Existence and uniqueness of a solution to (4.6)–(4.8).
2. Asymptotic behaviour of f for different initial conditions f_0, in particular the one where f_0 is a Dirac point mass. Does the solution converge to a travelling wave $f(x,t) = \phi(x-\gamma t)$? If so, what does this limiting distribution look like? And what is the growth rate γ and the degree of inequality?
3. Development of numerical methods for solving the time-dependent problem (4.6)–(4.8).

Regarding the second question, a natural conjecture would be that the limiting distribution is a travelling wave with growth rate and tail inequality

$$\gamma = \sigma\sqrt{2\int_{-\infty}^\infty \alpha(s^*(x))\phi(x)\,\mathrm{d}x}, \quad \frac{1}{\zeta} = \sigma\left(\sqrt{2\int_{-\infty}^\infty \alpha(s^*(x))\phi(x)\,\mathrm{d}x}\right)^{-1}.$$

These are the natural generalizations of the formulae (4.4) in §4*a* to the case where s^* varies across productivity types. If this conjecture turns out to be correct, one prediction of the model would be that policies that increase s^* for part of the population have the benefit of simultaneously stimulating growth and reducing inequality.

Ideas similar to those presented in this section in the context of search and knowledge diffusion have been applied to different contexts. For example, Duffie *et al.* [53] and Lagos & Rocheteau [54] and others use search theory to model the trading frictions that are characteristic of over-the-counter markets, and to examine the effects of these frictions on asset prices and trading volumes. A mathematical analysis of a similar model is provided by Gomes & Ribeiro [55].

(c) Diffusion and international trade

An alternative route to enrich the model of knowledge diffusion is to consider explicit mechanisms mediating the interactions among individuals. One possible avenue is explored by Alvarez *et al.* [56], who consider a multi-country model in which knowledge is transmitted through the interaction with the sellers of goods to a country. In their theory, barriers to trade affect the composition of sellers to a country, and therefore they impact the diffusion of knowledge. The higher the trade costs are, the more likely it is that sellers in a country are given by relatively inefficient local producers.

The central object in their theory is the distribution of productivities across potential producers of different goods $G(z,t)$, denoting the fraction of goods that can be produced with productivity less than z. Similar to the previous models, an individual producer meets other producers at the constant Poisson rate α. The main difference is that now draws come from the distribution of sellers, which depends on the distribution of productivities of producers from all countries in the world, and trade costs $1/\kappa$, $\kappa \in [0,1]$. As before, it is convenient to work with $x = \log(z)$ and the corresponding distribution F, and define $\delta = \log(\kappa)$. For the case of a world with n symmetric countries, the evolution of the distribution $F(x,t)$ solves the following non-local Fisher–KPP-type equation:[8]

$$\partial_t F = -\alpha(1-M)F \tag{4.9}$$

on $\mathbb{R} \times \mathbb{R}^+$ where

$$M(x,t) = \int_{-\infty}^{x} (F(y-\delta,t)^{n-1} + (n-1)F(y+\delta,t)F(y,t)^{n-2})\partial_x F(y,t)\,\mathrm{d}y \tag{4.10}$$

is the distribution of productivity of sellers to a country. The boundary conditions are given by (4.2). For $\kappa = 1$ $(\delta = 0)$ and $n = 1$, this equation simplifies to the one analysed in §4*a*, but more generally only the behaviour of the solution for large x is fully understood. One can show that this equation admits solutions of the travelling wave type

$$F(x,t) = \Phi(x - \gamma t), \tag{4.11}$$

provided that $(1 - F_0(x))/\mathrm{e}^{-\zeta x} \to c$ as $x \to \infty$ for some constants $c, \zeta > 0$, that is the initial distribution of productivity $z = \mathrm{e}^x$ follows an asymptotic power law. It can also be shown that the growth rate is $\gamma = n\alpha/\zeta$. That is, in the present model, there are growth benefits from openness to international trade: the higher is the number of trading partners of a country n, the higher is its growth rate. This is in contrast to most standard trade models in which trade only confers static benefits, that is trade typically leads to a higher *level* of a country's gross domestic product (GDP) but not a higher growth rate.

Natural open questions are

1. Existence and uniqueness of a solution to (4.9) and (4.10).
2. Development of numerical methods for computing both stationary and time-dependent solutions.

Another interesting extension could be the addition of noise in the form of a geometric Brownian motion to (4.9) along the lines of equation (4.1).

(d) Information percolation in finance

A related class of models arises when studying the distribution of information across individuals in an economy, e.g. beliefs about the value of a particular financial asset. These models are useful to understand the dynamics of asset prices and how these are affected when market participants do not share common beliefs about the 'intrinsic' value of a financial asset. A simple example is provided by Duffie & Manso [58], who consider the beliefs about the realization of a binary random variable. Individuals are initially endowed with a prior about this realization. Over time, individuals randomly meet at a constant Poisson rate α. Upon a meeting, individuals exchange their information and update their beliefs. In their example, they show that beliefs are characterized by a distribution over a sufficient statistic x, and the updating of beliefs after

[8] Related equations have been studied by Berestycki *et al.* [57].

a meeting with an individual of belief x' is simply given by the sum of x and x'. The evolution of the distribution of the sufficient statistic $F(x,t)$ is given by the PDE

$$\partial_t f(x,t) = -\alpha f(x,t) + \alpha \int_{-\infty}^{+\infty} f(y,t) f(x-y,t)\, \mathrm{d}y.$$

This equation can be solved explicitly using Fourier transforms. A natural extension is to endogenize the search effort α, similar to §4*b*. This is pursued in Duffie *et al.* [59]. Other recent contributions in this area include Amador & Weill [60] and Golosov *et al.* [61].

5. Business cycles: models with aggregate shocks

Some of the most important questions in macroeconomics are concerned with business cycle fluctuations, that is the fluctuations of macroeconomic aggregates like GDP, aggregate investment and asset prices such as the interest rate. The models presented so far are not well suited to address these questions, because all of them featured macroeconomic aggregates that are deterministic. Instead, we want theories in which these aggregates are stochastic. Denhaan [62,63] and Krusell & Smith [37] have extended theories with heterogeneity at the individual level to feature aggregate risk. We here present a continuous time formulation from Achdou *et al.* [64].

To introduce these ideas in the simplest possible way, consider the model of §2 but assume that income is $z_t A_t$, where z_t is an idiosyncratic income process as before but now income also has an aggregate component A_t. That is, if A_t falls by 10%, it means that the income of *everyone* in the economy falls by 10%. For the sake of simplicity, assume that $A_t \in \{A_1, A_2\}$ is a two-state Poisson process. The process jumps from state 1 to state 2 with intensity ϕ_1 and vice versa with intensity ϕ_2. The introduction of aggregate shocks creates a major difficulty: in contrast to the case without aggregate uncertainty studied in §2, it becomes necessary to include the entire distribution of income and wealth g as a state variable in the optimal control problem of individuals. This distribution is now itself a random variable and hence calendar time t is no longer a sufficient statistic to describe the behaviour of the system.

The aggregate state is (A_i, g), $i=1,2$ and the individual state is (a,z), so that the value function of an individual is $v_i(a,z,g)$. This value function satisfies the equation

$$\begin{aligned} 0 = {} & \frac{1}{2}\sigma^2(z)\partial_{zz} v_i + \mu(z)\partial_z v_i + (A_i z + r_i(g) a)\partial_a v_i \\ & + \phi_i (v_j - v_i) + \int T[g, \partial_a v_i](a,z) \frac{\delta v_i}{\delta g(a,z)}\, \mathrm{d}a\, \mathrm{d}z \\ & + H(\partial_a v_i) - \rho v_i \end{aligned} \tag{5.1}$$

and

$$T[g, \partial_a v_i] = \tfrac{1}{2}\partial_{zz}(\sigma^2(z) g) - \partial_z(\mu(z) g) - \partial_a((zA_i + ra)g) - \partial_a(\partial_p H(\partial_a v_i) g) \tag{5.2}$$

for $i = 1,2, j \neq i$. The domain of this equation is $(\underline{a}, \infty) \times (\underline{z}, \bar{z}) \times \mathcal{S}$, where $\mathcal{S}$ is the space of density functions. The Hamiltonian H is defined in (2.5), and there is again a state constraint at $a = \underline{a}$. $\delta v_i / \delta g(a,z)$ denotes the functional derivative of V_i with respect to g at point (a,z) and T defined in (5.2) is the 'Fokker–Planck' operator that maps functions g and $\partial_a v_i$ to the time derivative of g. Note that (5.1) is not an ordinary HJB equation because of the presence of g in the state space. The difficulty, of course, is that g is an infinite-dimensional object.

If the functions $v_i(a,z,g)$ were known, the value function corresponding to a given path (A_t) would then be found by solving a Fokker–Planck equation for g_t and plugging g_t into the functions v_i. However, (5.1) is a PDE with a variable lying in an infinite dimensional space. Therefore, its numerical approximation is very difficult.

For this reason, instead of using the infinite-dimensional Bellman equation (5.1) coupled with a stochastic Fokker–Planck equation, Achdou *et al.* [64] consider a situation in which aggregate shocks occur only finitely many times and at finite time intervals of length Δ, that is at times $\tau_n = n\Delta$, $n = 1, \ldots, N$, $N = 1/\Delta$. There is therefore only a finite (but possibly large) number of paths (v_t, g_t). For each path, (v_t, g_t) solve a system of forward–backward PDEs between the aggregate shocks and satisfy suitable transmission conditions at the shocks. It is crucial that the variables of the PDEs are (t, a, z), therefore lie in a finite dimensional space. Hence, a situation with, for example, 10 shocks leads to $2^{10} = 1024$ paths and can be simulated numerically. The hope is that the model with a finite number of shocks approximates the model when A_t is a two-state Poisson process, as $\Delta \to 0$.

To see why economists find it useful to have a model like the present one that generates predictions about the consumption and saving behaviour of individuals over the business cycle and at different points in the wealth distribution, let us come back to one of the questions raised in the introduction: what should be done if the economy is hit by a recession? A policy that is often advocated is fiscal stimulus, that is a one-time transfer from government to households with the aim of increasing their disposable income (in practice, this is achieved, for example, by sending households tax rebate cheques). The crucial question is usually whether such fiscal stimulus will be effective and in particular whether households will actually increase their spending. The critical object one would like to know is the *marginal propensity to consume* (MPC) out of income which, in the model, is $\mathrm{MPC}_i(a, z, g) = \partial_z c_i(a, z, g)$, where $c_i = -\partial_p H(\partial_a v_i)$ is optimal consumption. This object answers the question: if a household receives an unexpected increase in income z, what fraction of it will it consume and what fraction will it save? Again, the presence of the state constraint is important here. For the same reasons discussed in §2, individuals with wealth equal to $\underline{a}$ and low enough income z will have $c_i(\underline{a}, z, g) = zA_i + r\underline{a}$, i.e. they consume their entire income rather than saving it, and hence have a high MPC. However, similar to the model in §2, it turns out that calibrated versions of the model do not generate high enough average MPCs when compared with the data, mainly because not enough individuals are borrowing constrained for reasonable parameter values. This has motivated the development of alternative models, for example models with more than one asset (e.g. Kaplan & Violante [65], who argue for the importance of distinguishing between liquid and illiquid assets).

Models with aggregate shocks such as (5.1) are by far the most challenging in terms of the mathematics, and many open questions remain. Among these are

1. Existence and uniqueness of solutions to (5.1).
2. A theoretical understanding of the behaviour of g. For example, given a stationary process for A_t (such as the two-state Poisson process), does there exist a 'stationary equilibrium' for g? Similarly, are there certain regions of the space of density functions $\mathcal{S}$ in which g lives 'most of the time'?
3. Development of efficient and robust approximation schemes to (5.1) and results regarding their convergence.

Regarding the first question, it should again be noted that non-uniqueness is quite possible and understanding non-uniqueness is equally interesting to economists as proving uniqueness. One approach to obtaining more tractable formulations of models with aggregate shocks has been to simplify the heterogeneity at the individual level. For example, Brunnermeier & Sannikov [66], He & Krishnamurthy [67,68], Adrian & Boyarchenko [69] and Di Tella [70] all study business cycles in models of financial intermediation with frictions and argue that these frictions give rise to interesting nonlinear behaviour of macroeconomic aggregates. For example, GDP may have a bimodal stationary distribution even if the driving stochastic process is unimodal. These papers all make the assumption that there are only two (or three) types of agents, so that the wealth distribution can be summarized by the share of wealth of one of the two types. The big advantage of these two approaches is that this is a one-dimensional rather than an infinite-dimensional object. Related, business cycle fluctuations can also be generated from theories without aggregate shocks. An important early paper by Scheinkman & Weiss [71] demonstrates that in a model with only a finite number of agents (two in their framework) idiosyncratic shocks (in combination with missing insurance markets) can lead to aggregate fluctuations. See Conze *et al.* [72] and Lippi *et al.* [73] for other applications of their framework. These authors again make assumptions that avoid dealing with an infinite-dimensional problem. However, for many interesting economic questions, it may be necessary to consider richer forms of heterogeneity. Our hope is therefore that some progress can be made on infinite dimensional problems such as (5.1).

6. Models with finite number of agents

In this paper, we have mostly focused on models with a continuum of individuals (mean field games). While these frameworks are useful to study a very large class of macroeconomic phenomena, their applicability to other important macro questions is limited. In some industries, production is concentrated in a very small number of producers, who act strategically when making their production, innovation and pricing decisions. The strategic nature of their decision could have important aggregate implications. For example, Atkeson & Burstein [74] consider a model with a continuum of sectors and a finite number of firms in each sector to explain why there are large and systematic deviations of the law of one price across countries. Aghion *et al.* [75] study a model of innovation in duopolist industries to analyse the relationship between competition and innovation.

In this section, we introduce a continuous time version of the canonical model of firm dynamics in an oligopolistic industry introduced by Ericson & Pakes [76], and recently studied by Weintraub *et al.* [77] and Doraszelski & Judd [78], among many others. We show that this model takes the form of a differential game.

There are two firms $i=1,2$ that compete with each other. Firm i has profits $\pi(z_i, q_i, q_j)$, where $j \neq i$. Profits π are increasing in productivity z_i and own quantity q_j, but decreasing in the quantity of the other firm q_j. The quantity choice also affects the evolution of the firm's productivity which evolves as $\mathrm{d}z_{it} = \mu(z_{it}, q_{it})\,\mathrm{d}t + \sigma(z_{it})\,\mathrm{d}W_{it}$. We assume that there is 'learning-by-doing', so that μ is increasing in q_{it} (of course, other assumptions also are possible). We assume that the two firms play a *Nash equilibrium*, that is their choices of q_{it} are best responses to each other. Given the symmetry of the problem, we look for a symmetric Nash equilibrium. To this end denote by z a firm's own productivity and by x the productivity of the other firm. In a symmetric Nash equilibrium, the value function $v(z,x)$ of a firm satisfies

$$\frac{\sigma^2(z)}{2}\partial_{zz}v + \frac{\sigma^2(x)}{2}\partial_{xx}v + \mu(x, q^*(x,z,\partial_x v, \partial_z v))\partial_x v + H(z,x,\partial_z v, \partial_x v) - \rho v = 0 \tag{6.1}$$

on $\mathbb{R}^+ \times \mathbb{R}^+$, and where the Hamiltonians H and optimal choice q^* jointly satisfy

$$H(z,x,p_z,p_x) = \max_q(\pi(z,q,q^*(x,z,p_x,p_z)) + \mu(z,q)p_z)$$

$$q^*(z,x,p_z,p_x) = \arg\max_q(\pi(z,q,q^*(x,z,p_x,p_z)) + \mu(z,q)p_z).$$

There are many possible extensions of this simple framework. Naturally, the model can be generalized to $n>2$. One can also consider versions of the model with entry and exit of firms, along the lines of the analysis in §3*a*. One way to model this process is to consider a maximum number of potential firms $\bar{n}$. In this case, the relevant 'aggregate' state is given by the vector of characteristics of all the active and potential firms, for example, their respective z. An alternative route, which is the one that is typically followed in the literature, is to assume that the state describing an individual firm takes only a finite set of values. In this case, one can describe the aggregate state with the distribution of firms over these (finite) characteristics. The first route leads naturally to PDE methods. We are not aware of a general characterization of these problems. As in the previous examples, the open questions are

1. Existence and uniqueness of a solution to (6.1).
2. Development of numerical methods for computing both stationary and time-dependent solutions when the state variable is continuous.

7. Conclusion

We have surveyed a large literature in macroeconomics that studies theories that explicitly model the equilibrium interaction of heterogeneous agents. These theories share a common mathematical structure which can be summarize by a system of coupled nonlinear PDEs or mean field game. Some of our examples are well-understood problems in the theory of PDEs, whereas others present new and challenging mathematical problems. The development of numerical methods for actually solving these in practice is equally important. We view this to be a very promising area for future research, or, as economists like to say, we see large 'gains from trade' between macroeconomists and mathematicians working on PDEs.

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