# Lecture 8

## The Workhorse Model of Income and Wealth Distribution in Macroeconomics

ECO 521: Advanced Macroeconomics I

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Princeton University, Fall 2016

# Plan for rest of my part of course

- Only 5 lectures left
    - Chris will take over on Wednesday 10/26 (before break)
- Lecture 8 (10/10): Textbook heterogeneous agent model without aggregate shocks
- Lecture 9 (10/12): Het agent models with aggregate shocks
    - “No more excuses!” (Reiter perturbation method)
- Lecture 10 (10/17): Dirk Krueger guest lecture
    - “Macroeconomics and Household Heterogeneity”
- Lecture 11 (10/19): Het agent models with nominal rigidities
    - HANK & friends
- Lecture 12 (10/24): Estimation of heterogeneous agent models
    - Parra-Alvarez, Posch and Wang (2015)

# Outline

1. Textbook heterogeneous agent model (no aggregate shocks)
   - the Aiyagari-Bewley-Huggett model

2. Some theoretical results

3. Computations

# What this lecture is about

- Many interesting questions require thinking about distributions
  - Why are income and wealth so unequally distributed?
  - Is there a trade-off between inequality and economic growth?
  - What are the forces that lead to the concentration of economic activity in a few very large firms?
- Modeling distributions is hard
  - closed-form solutions are rare
  - computations are challenging
- Main idea: solving heterogeneous agent model = solving PDEs
  - main difference to existing continuos-time literature: handle models for which closed-form solutions do not exist

# Solving het. agent model = solving PDEs

- More precisely: a system of two PDEs
  1. Hamilton-Jacobi-Bellman equation for individual choices
  2. Kolmogorov Forward equation for evolution of distribution
- Many well-developed methods for analyzing and solving these
  - codes: `http://www.princeton.edu/~moll/HACTproject.htm`
- Apparatus is very general: applies to any heterogeneous agent model with continuum of atomistic agents
  1. heterogeneous households (Aiyagari, Bewley, Huggett,...)
  2. heterogeneous producers (Hopenhayn,...)
- can be extended to handle aggregate shocks (Krusell-Smith,...)

# Computational Advantages relative to Discrete Time

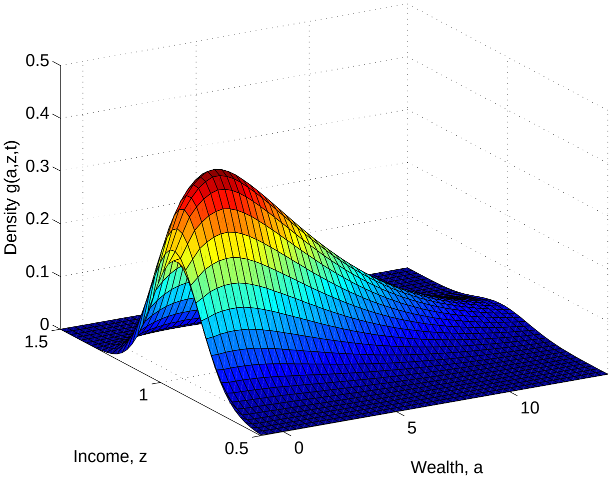
1. Borrowing constraints only show up in boundary conditions
   - FOCs always hold with "$=$"
2. "Tomorrow is today"
   - FOCs are "static", compute by hand: $c^{-\gamma} = v_a(a, y)$
3. Sparsity
   - solving Bellman, distribution = inverting matrix
   - but matrices very sparse ("tridiagonal")
   - reason: continuous time $\Rightarrow$ one step left or one step right
4. Two birds with one stone
   - tight link between solving (HJB) and (KF) for distribution
   - matrix in discrete (KF) is transpose of matrix in discrete (HJB)
   - reason: diff. operator in (KF) is adjoint of operator in (HJB)

# Real Payoff: extends to more general setups

- non-convexities
- stopping time problems
- multiple assets
- aggregate shocks

# What you'll be able to do at end of this lecture

- Joint distribution of income and wealth in Aiyagari model



# What you'll be able to do at end of this lecture

- Experiment: effect of one-time redistribution of wealth

# What you'll be able to do at end of this lecture

Video of convergence back to steady state

https://www.dropbox.com/s/op5u2nlifmmer2o/distribution_tax.mp4?dl=0

## Textbook Heterogeneous Agent Model:
## Aiyagari-Bewley-Huggett

# Households

are heterogeneous in their wealth $a$ and income $y$, solve

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \qquad \text{s.t.}$$

$$da_t = (y_t + r_t a_t - c_t)dt$$

$$y_t \in \{y_1, y_2\} \text{ Poisson with intensities } \lambda_1, \lambda_2$$

$$a_t \geq \underline{a}$$

- $c_t$: consumption
- $u$: utility function, $u' > 0$, $u'' < 0$.
- $\rho$: discount rate
- $r_t$ : interest rate
- $\underline{a} > -\infty$: borrowing limit e.g. if $\underline{a} = 0$, can only save

later: carries over to $y_t$ = general diffusion process.

# Stationary Equilibrium

Bonds in zero net supply (Huggett)

$$0 = S(r) := \int_{\underline{a}}^{\infty} a g_1(a) da + \int_{\underline{a}}^{\infty} a g_2(a) da \tag{EQ}$$

$$\rho v_i(a) = \max_c \; u(c) + v_i'(a)(y_i + ra - c) + \lambda_i (v_j(a) - v_i(a)) \tag{HJB}$$

$$0 = -\frac{d}{da}[s_i(a) g_i(a)] - \lambda_i g_i(a) + \lambda_j g_j(a), \tag{KF}$$

$$s_i(a) = y_i + ra - c_i(a), \quad c_i(a) = (u')^{-1}(v_i'(a)),$$

$$\int_{\underline{a}}^{\infty} (g_1(a) + g_2(a)) da = 1, \quad g_1, g_2 \geq 0$$

- The two PDEs (HJB) and (KF) together with (EQ) fully characterize stationary equilibrium ▸ Derivation of (HJB) ▸ (KF)

# Transition Dynamics

$$0 = S(r,t) := \int_{\underline{a}}^{\infty} a g_1(a,t)da + \int_{\underline{a}}^{\infty} a g_2(a,t)da \tag{EQ}$$

$$\begin{aligned}\rho v_i(a,t) = &\max_c \; u(c) + \partial_a v_i(a,t)(y_i + r(t)a - c) \\ &+ \lambda_i(v_j(a,t) - v_i(a,t)) + \partial_t v_i(a,t),\end{aligned} \tag{HJB}$$

$$\partial_t g_i(a,t) = -\partial_a[s_i(a,t)g_i(a,t)] - \lambda_i g_i(a,t) + \lambda_j g_j(a,t), \tag{KF}$$

$$s_i(a,t) = y_i + r(t)a - c_i(a,t), \quad c_i(a,t) = (u')^{-1}(\partial_a v_i(a,t)),$$

$$\int_{\underline{a}}^{\infty}(g_1(a,t) + g_2(a,t))da = 1, \quad g_1, g_2 \geq 0$$

- Given initial condition $g_{i,0}(a)$, the two PDEs (HJB) and (KF) together with (EQ) fully characterize equilibrium.
- Notation: for any function $f$, $\partial_x f$ means $\frac{\partial f}{\partial x}$

# Borrowing Constraints?

- Q: where is borrowing constraint $a \geq \underline{a}$ in (HJB)?
- A: "in" boundary condition
- Result: $v_i$ must satisfy

$$v_i'(\underline{a}) \geq u'(y_i + r\underline{a}), \quad i = 1, 2 \tag{BC}$$

- Derivation:
  - the FOC still holds at the borrowing constraint

$$u'(c_i(\underline{a})) = v_i'(\underline{a}) \tag{FOC}$$

  - for borrowing constraint not to be violated, need

$$s_i(\underline{a}) = y_i + r\underline{a} - c_i(\underline{a}) \geq 0 \tag{$*$}$$

  - (FOC) and ($*$) $\Rightarrow$ (BC).

- See slides on viscosity solutions for more rigorous discussion

`http://www.princeton.edu/~moll/viscosity_slides.pdf`

# Plan

1. Consumption, saving and inequality in partial equilibrium
2. General equilibrium
3. Computations

# MPCs and Speed of Hitting Borrowing Constraint

Behavior near borrowing constraint depends on two factors

1. tightness of constraint
2. properties of $u$ as $c \to 0$

Assumption 1:
*The coefficient of absolute risk aversion $R(c) = -u''(c)/u'(c)$ when wealth $a$ approaches the borrowing limit $\underline{a}$ is finite, that is*

$$\underline{R} = -\lim_{a \to \underline{a}} \frac{u''(y_1 + ra)}{u'(y_1 + ra)} < \infty$$

- sufficient condition for A1: borrowing constraint is tighter than "natural borrowing constraint" $\underline{a} > -y_1/r$
- e.g. with CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \Rightarrow \quad \underline{R} = \frac{\gamma}{y_1 + r\underline{a}}$$

- but weaker: e.g. A1 satisfied by $\underline{a} = -y_1/r$, $u(c) = -\gamma e^{-\gamma c}$

# MPCs and Speed of Hitting Borrowing Constraint

## Proposition

*Assume $r < \rho$, $y_1 < y_2$ and that A1 holds. The solution to (HJB) has following properties:*

1. $s_1(\underline{a}) = 0$ *but* $s_1(a) < 0$ *all* $a > \underline{a}$*: only households exactly at the borrowing constraint are constrained.*
2. *Saving and consumption policy functions close to* $a = \underline{a}$ *satisfy*

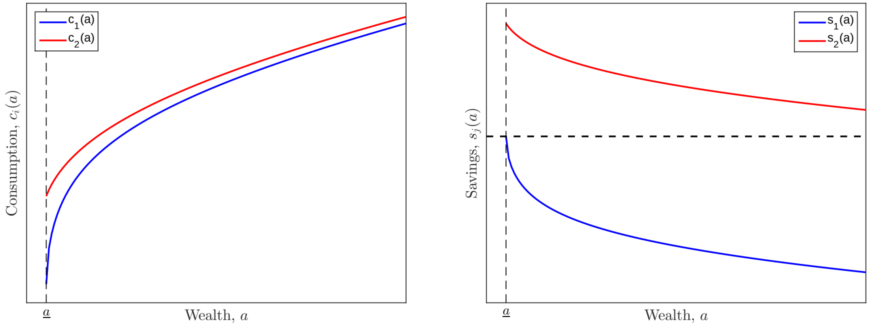
$$s_1(a) \approx -\nu\sqrt{a - \underline{a}}$$

$$c_1(a) \approx y_1 + ra + \nu\sqrt{a - \underline{a}}$$

$$c_1'(a) \approx r + \frac{1}{2}\frac{\nu}{\sqrt{a - \underline{a}}}$$

$$\nu = \sqrt{2\frac{(\rho - r)u'(\underline{c}_1) + \lambda_1[u'(\underline{c}_1) - u'(\underline{c}_2)]}{-u''(\underline{c}_1)}} > 0$$

# Consumption, Savings at Borrowing Constraint



# Consumption, Savings at Borrowing Constraint

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$$\nu = \sqrt{2\frac{(\rho-r)u'(\underline{c}_1)+\lambda_1[u'(\underline{c}_1)-u'(\underline{c}_2)]}{-u''(\underline{c}_1)}} > 0\sqrt{\frac{2\underline{c}_1}{\gamma}\left(\rho - r + \lambda_1\left[1 - (\underline{c}_2/\underline{c}_1\right.\right.}$$

# Saving Behavior at Borrowing Constraint

## Corollary

*The wealth of worker who keeps $y_1$ converges to borrowing constraint in finite time at speed governed by $\nu$:*

$$a(t) - \underline{a} \approx \left[ \left( \sqrt{a_0 - \underline{a}} - \frac{\nu}{2} t \right)^{+} \right]^2$$

Derivation: integrate $\dot{a}(t) = -\nu \sqrt{a(t) - \underline{a}}$

Note: similarity to stopping time problems

# Stationary Wealth Distribution

- Recall equation for stationary distribution

$$0 = -\frac{d}{da}[s_i(a)g_i(a)] - \lambda_i g_i(a) + \lambda_j g_j(a) \tag{KF}$$

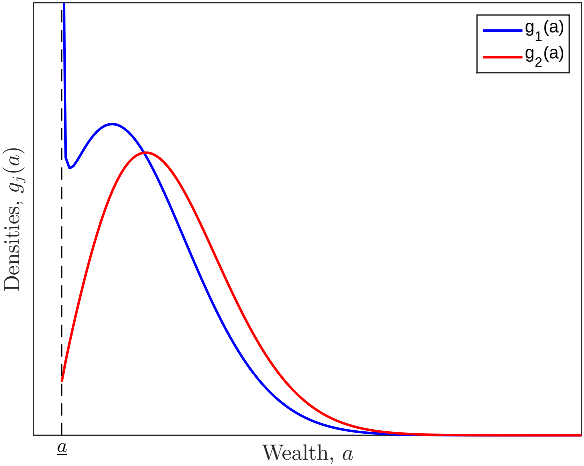
- Lemma: the solution to (KF) is

$$g_i(a) = \frac{\kappa_i}{s_i(a)} \exp\left(-\int_{\underline{a}}^{a}\left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}dx\right)\right)$$

  with $\kappa_1, \kappa_2$ pinned down by $g_i$'s integrating to one

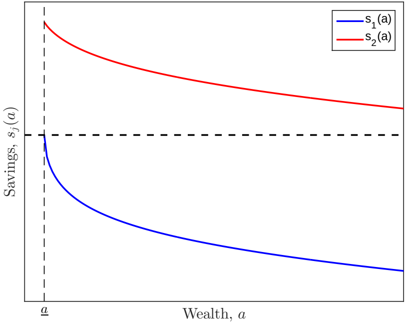
- Corollary: Dirac point mass of type $y_1$ individuals at constraint $\lim_{a\to\underline{a}} g_1(a) = \infty$

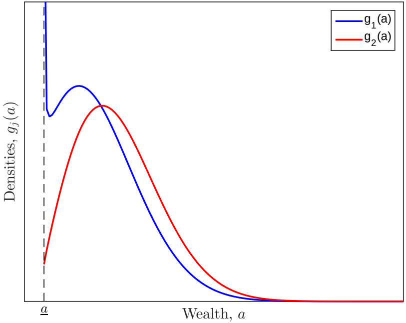
# Stationary Wealth Distribution



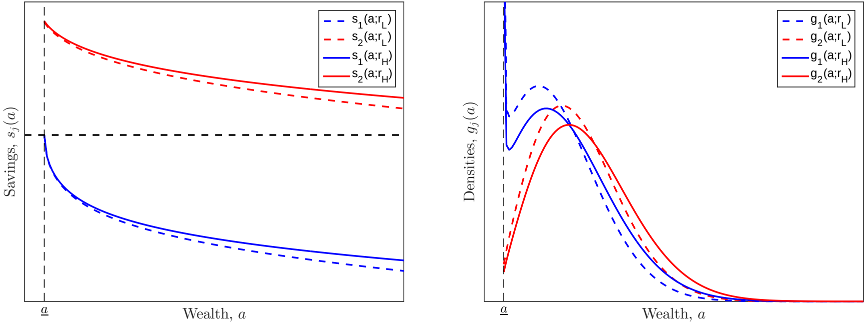

Note: in numerical solution, Dirac mass = finite spike in density

# General Equilibrium

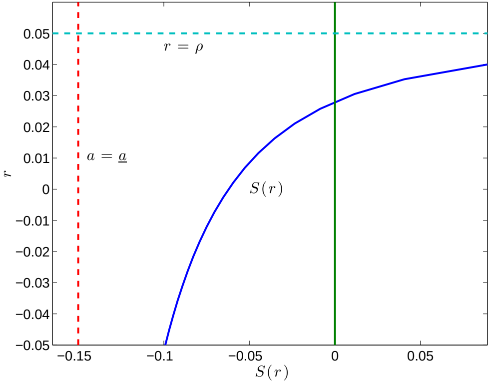


# Increase in $r$ from $r_L$ to $r_H > r_L$



# Stationary Equilibrium




$$\text{Asset Supply} \quad S(r) = \int_{\underline{a}}^{\infty} a g_1(a; r) da + \int_{\underline{a}}^{\infty} a g_2(a; r) da$$

- Proposition: a stationary equilibrium exists
- Big open question: uniqueness. Any ideas? Need to find conditions s.t. $S'(r) \geq 0$.

# Computations for Heterogeneous Agent Model

# Computations for Heterogeneous Agent Model

- Hard part: HJB equation. But already know how to do that
- Easy part: KF equation. Once you solved HJB equation, get KF equation "for free"
- System to be solved

$$\rho v_1(a) = \max_c \; u(c) + v_1'(a)(y_1 + ra - c) + \lambda_1(v_2(a) - v_1(a))$$

$$\rho v_2(a) = \max_c \; u(c) + v_2'(a)(y_2 + ra - c) + \lambda_2(v_1(a) - v_2(a))$$

$$0 = -\frac{d}{da}[s_1(a)g_1(a)] - \lambda_1 g_1(a) + \lambda_2 g_2(a)$$

$$0 = -\frac{d}{da}[s_2(a)g_2(a)] - \lambda_2 g_2(a) + \lambda_1 g_1(a)$$

$$1 = \int_{\underline{a}}^{\infty} g_1(a)da + \int_{\underline{a}}^{\infty} g_2(a)da$$

$$0 = \int_{\underline{a}}^{\infty} ag_1(a)da + \int_{\underline{a}}^{\infty} ag_2(a)da := S(r)$$

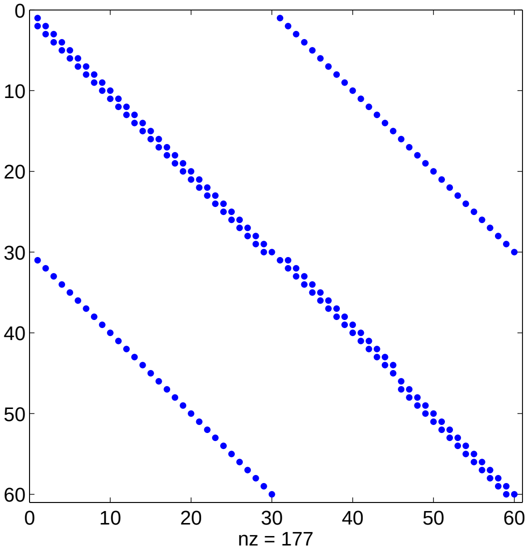
# Computations for Heterogeneous Agent Model

- As before, discretized HJB equation is

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v})\mathbf{v} \tag{HJBd}$$

- $\mathbf{A}$ is $N \times N$ transition matrix
    - here $N = 2 \times I$, $I$=number of wealth grid points
    - $\mathbf{A}$ depends on $\mathbf{v}$ (nonlinear problem)
    - solve using implicit scheme

# Visualization of **A** (output of `spy(A)` in Matlab)



# Computing the FK Equation

- Equations to be solved

$$0 = -\frac{d}{da}[s_1(a)g_1(a)] - \lambda_1 g_1(a) + \lambda_2 g_2(a)$$

$$0 = -\frac{d}{da}[s_2(a)g_2(a)] - \lambda_2 g_2(a) + \lambda_1 g_1(a)$$

  with $1 = \int_{\underline{a}}^{\infty} g_1(a)da + \int_{\underline{a}}^{\infty} g_2(a)da$
- Actually, super easy: discretized version is simply

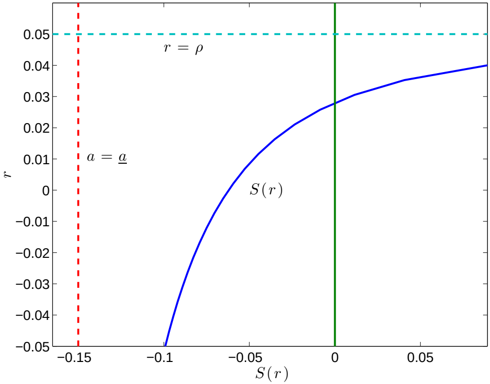
$$0 = \mathbf{A}(\mathbf{v})^{\mathrm{T}}\mathbf{g} \qquad \text{(KFd)}$$

  - eigenvalue problem
  - get KF for free, one more reason for using implicit scheme
- Why transpose? See lectures 6 and 7
  - operator in (HJB) is "adjoint" of operator in (KF)
  - "adjoint" = infinite-dimensional analogue of matrix transpose
- In principle, can use similar strategy in discrete time

# Finding the Equilibrium Interest Rate

Use bisection method

- increase $r$ whenever $S(r) < 0$
- decrease $r$ whenever $S(r) > 0$



# A Model with a Continuum of Income Types

- Assume idiosyncratic income follows diffusion process

$$dy_t = \mu(y_t)dt + \sigma(y_t)dW_t$$

- Reflecting barriers at $\underline{y}$ and $\bar{y}$

$$\rho v(a,y) = \max_c \; u(c) + \partial_a v(a,y)(y + ra - c) + \mu(y)\partial_y v(a,y) + \frac{\sigma^2(y)}{2}\partial_{yy} v(a,y)$$

$$0 = -\partial_a[s(a,y)g(a,y)] - \partial_y[\mu(y)g(a,y)] + \frac{1}{2}\partial_{yy}[\sigma^2(y)g(a,y)]$$

$$1 = \int_0^\infty \int_{\underline{a}}^\infty g(a,y)dady$$

$$0 = \int_0^\infty \int_{\underline{a}}^\infty ag(a,y)dady := S(r)$$

- Borrowing constraint: $\partial_a v(\underline{a}, y) \geq u'(y + r\underline{a})$, all $y$
- reflecting barriers (see e.g. Dixit “Art of Smooth Pasting”)

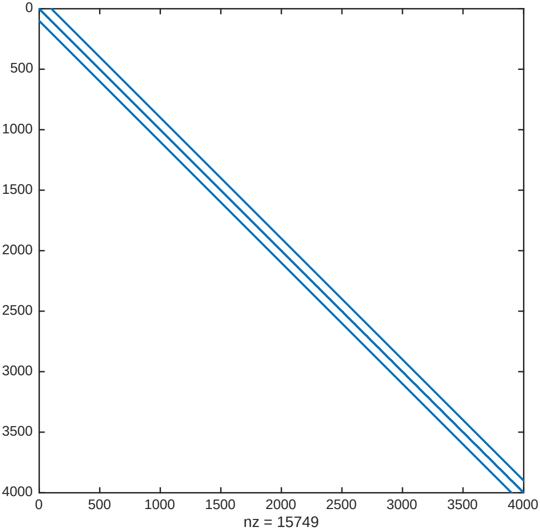
$$0 = \partial_y v(a, \underline{y}) = \partial_y v(a, \bar{y})$$

It doesn't matter whether you solve ODEs or PDEs

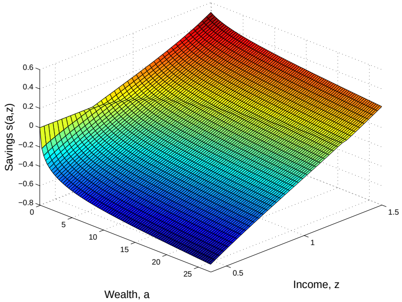
$\Rightarrow$ everything generalizes

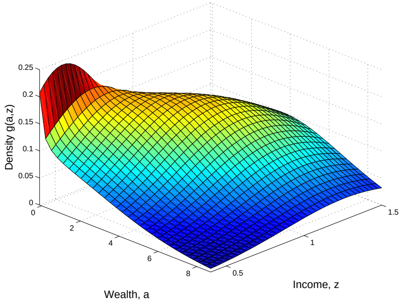
`http://www.princeton.edu/~moll/HACTproject/huggett_diffusion_partialeq.m`

# Visualization of **A** (output of `spy(A)` in Matlab)



# Saving Policy Function and Stationary Distribution



# Transition Dynamics/MIT Shocks

# Transition Dynamics

Do Aiyagari version of the model

$$r(t) = F_K(K(t),1) - \delta, \qquad w(t) = F_L(K(t),1) \tag{P}$$

$$K(t) = \int a g_1(a,t)da + \int a g_2(a,t)da \tag{K}$$

$$\begin{aligned}\rho v_i(a,t) = &\max_c \; u(c) + \partial_a v_i(a,t)(w(t)z_i + r(t)a - c) \\ &+ \lambda_i(v_j(a,t) - v_i(a,t)) + \partial_t v_i(a,t),\end{aligned} \tag{HJB}$$

$$\partial_t g_i(a,t) = -\partial_a[s_i(a,t)g_i(a,t)] - \lambda_i g_i(a,t) + \lambda_j g_j(a,t), \tag{KF}$$

$$s_i(a,t) = w(t)z_i + r(t)a - c_i(a,t), \quad c_i(a,t) = (u')^{-1}(\partial_a v_i(a,t))$$

- Given initial condition $g_{i,0}(a)$, the two PDEs (HJB) and (KF) together with (P) and (K) fully characterize equilibrium.

# Transition Dynamics

- Recall discretized equations for stationary equilibrium

$$\rho \mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v})\mathbf{v}$$
$$0 = \mathbf{A}(\mathbf{v})^{\top}\mathbf{g}$$

- Transition dynamics
  - denote $v_{i,j}^n = v_i(a_j, t^n)$ and stack into $\mathbf{v}^n$
  - denote $g_{i,j}^n = g_i(a_j, t^n)$ and stack into $\mathbf{g}^n$

$$\rho \mathbf{v}^n = \mathbf{u}(\mathbf{v}^{n+1}) + \mathbf{A}(\mathbf{v}^{n+1})\mathbf{v}^n + \frac{1}{\Delta t}(\mathbf{v}^{n+1} - \mathbf{v}^n)$$
$$\frac{\mathbf{g}^{n+1} - \mathbf{g}^n}{\Delta t} = \mathbf{A}(\mathbf{v}^n)^{\top}\mathbf{g}^{n+1}$$

- Terminal condition for $\mathbf{v}$: $\mathbf{v}^N = \mathbf{v}_\infty$ (steady state)
- Initial condition for $\mathbf{g}$: $\mathbf{g}^1 = \mathbf{g}_0$.

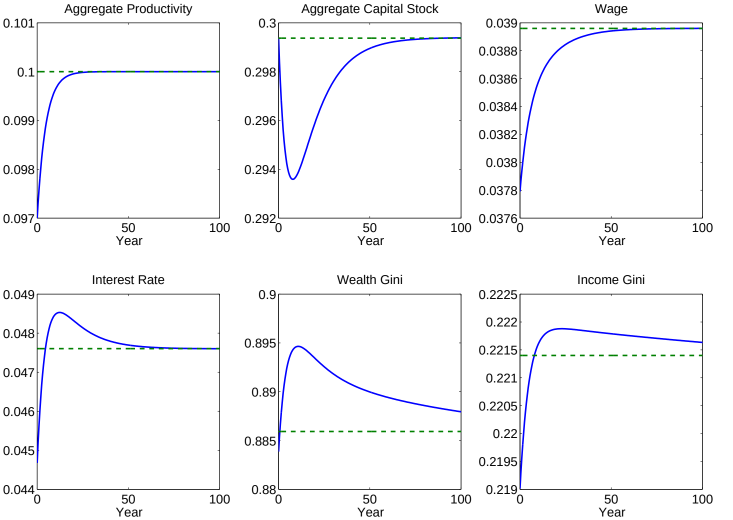
# Transition Dynamics

- (HJB) looks forward, runs backwards in time
- (KF) looks backward, runs forward in time
- Algorithm: Guess $K^0(t)$ and then for $\ell = 0, 1, 2, \ldots$
  1. find prices $r^\ell(t)$ and $w^\ell(t)$
  2. solve (HJB) backwards in time given terminal cond'n $v_{i,\infty}(a)$
  3. solve (KF) forward in time given given initial condition $g_{i,0}(a)$
  4. Compute $S^\ell(t) = \int a g_1^\ell(a,t)da + \int a g_2^\ell(a,t)da$
  5. Update $K^{\ell+1}(t) = (1-\xi)K^\ell(t) + \xi S^\ell(t)$ where $\xi \in (0,1]$ is a relaxation parameter

# An MIT Shock

Modification: $Y_t = F_t(K, L) = A_t K^\alpha L^{1-\alpha}$, $dA_t = \nu(\bar{A} - A_t)dt$

`http://www.princeton.edu/~moll/HACTproject/aiyagari_poisson_MITshock.m`



# Open Questions

- Title of course/lecture "Income and Wealth Distribution in Macro"
- Aiyagari-Bewley-Huggett model = rich theory of wealth distribution
    - caveat: ability to match data? See problem set
    - either way, important building block for richer models
- ... but no deep theory of income distribution
    - labor income $= w \times z$, $z =$ exogenous process
    - capital income $= r \times a$, i.e. proportional to wealth
- Can we do better?
    - idea: marry with assignment model $\Rightarrow$ income = $w(z)$, $w'' \neq 0$
- References:
    - Sattinger (1979), "Differential Rents and the Distribution of Earnings"
    - these Acemoglu lecture notes `http://economics.mit.edu/files/10480`
    - Gabaix and Landier (2008), "Why has CEO Pay Increased so Much?"
    - Acemoglu and Autor (2011), "Skills, Tasks and Technologies"

# Open Question: Less Restrictive Assignment Models?

- Sattinger setup, notation in `http://economics.mit.edu/files/10480`
- Workers with skill $s$, CDF $H(s)$
- Firms with productivity $x$, CDF $G(x)$
- One-to-one matching, output $f(x, s)$
- Result: if $f_{xs}(x, s) > 0$ all $(x, s)$ ($f$ is supermodular), then "positive assortative matching" (PAM), assignment equation is

$$x = \phi(s) \quad \text{with} \quad \phi' > 0$$

- Wage function $w(s)$ found from $w'(s) = f_s(\phi(s), s) \Rightarrow w''(s) > 0$
- Open question:
  - supermodularity = strong, sufficient condition for obtaining assignment equation $x = \phi(s)$
  - possible to obtain assignment equation under weaker assumptions than supermodularity, still able to say something?

# Appendix

# Derivation of Poisson KF Equation ▸ Back

- Work with CDF (in wealth dimension)

$$G_i(a,t) := \Pr(\tilde{a}_t \leq a, \tilde{y}_t = y_i)$$

- Income switches from $y_i$ to $y_j$ with probability $\Delta\lambda_i$
- Over period of length $\Delta$, wealth evolves as $\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta s_i(\tilde{a}_t)$
- Similarly, answer to question "where did $\tilde{a}_{t+\Delta}$ come from?" is

$$\tilde{a}_t = \tilde{a}_{t+\Delta} - \Delta s_i(\tilde{a}_{t+\Delta})$$

- Momentarily ignoring income switches and assuming $s_i(a) < 0$

$$\Pr(\tilde{a}_{t+\Delta} \leq a) = \underbrace{\Pr(\tilde{a}_t \leq a)}_{\text{already below } a} + \underbrace{\Pr(a \leq \tilde{a}_t \leq a - \Delta s_i(a))}_{\text{cross threshold } a} = \Pr(\tilde{a}_t \leq a - \Delta s_i(a))$$

- Fraction of people with wealth below $a$ evolves as

$$\begin{aligned}\Pr(\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_i) = (1-\Delta\lambda_i)\Pr(\tilde{a}_t \leq a - \Delta s_i(a), \tilde{y}_t = y_i) \\ + \Delta\lambda_j \Pr(\tilde{a}_t \leq a - \Delta s_j(a), \tilde{y}_t = y_j)\end{aligned}$$

- Intuition: if have wealth $< a - \Delta s_i(a)$ at $t$, have wealth $< a$ at $t + \Delta$

# Derivation of Poisson KF Equation

- Subtracting $G_i(a,t)$ from both sides and dividing by $\Delta$

$$\frac{G_i(a,t+\Delta)-G_i(a,t)}{\Delta}=\frac{G_i(a-\Delta s_i(a),t)-G_i(a,t)}{\Delta} \\ -\lambda_i G_i(a-\Delta s_i(a),t)+\lambda_j G_j(a-\Delta s_j(a),t)$$

- Taking the limit as $\Delta \to 0$

$$\partial_t G_i(a,t)=-s_i(a)\partial_a G_i(a,t)-\lambda_i G_i(a,t)+\lambda_j G_j(a,t)$$

where we have used that

$$\lim_{\Delta\to 0}\frac{G_i(a-\Delta s_i(a),t)-G(a,t)}{\Delta}=\lim_{x\to 0}\frac{G(a-x,t)-G(a,t)}{x}s_i(a) \\ =-s_i(a)\partial_a G_i(a,t)$$

- Intuition: if $s_i(a)<0$, $\Pr(\tilde{a}_t \leq a, \tilde{y}_t = y_i)$ increases at rate $g_i(a,t)$
- Differentiate w.r.t. $a$ and use $g_i(a,t)=\partial_a G_i(a,t) \Rightarrow$

$$\partial_t g_i(a,t)=-\partial_a[s_i(a,t)g_i(a,t)]-\lambda_i g_i(a,t)+\lambda_j g_j(a,t)$$