Mean Field Games in Economics Part I

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Larger Project: MFGs in Economics

- Important class of MFGs in economics: "heterogeneous agent models"
- Many interesting questions involve some kind of distribution:
  1. Why are income and wealth so unequally distributed?
  2. Is there a trade-off between inequality and economic growth?
  3. ...
- To theorize about these, need heterogeneous agent models
- Point of my lectures: Potentially high payoff from well-trained mathematicians working on these theories
- based on joint work with Yves Achdou, SeHyoun Ahn, Jiequn Han, Greg Kaplan, Pierre-Louis Lions, Jean-Michel Lasry, Gianluca Violante, Tom Winberry, Christian Wolf
- For lecture notes see `http://www.princeton.edu/~moll/notes.htm`

References

- "PDE Models in Macroeconomics" with Achdou, Buera, Lasry, Lions
  - introduction for mathematicians
  - discuss what we know (not much), pose open questions
  - `http://www.princeton.edu/~moll/PDE_macro.pdf`
- Also see two papers written for economists:
  - "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach" – a benchmark macroecon MFG
    `http://www.princeton.edu/~moll/HACT.pdf`
  - "When Inequality Matters for Macro and Macro Matters for Inequality" – macroeconomic MFGs with common noise
    `http://www.princeton.edu/~moll/WIMM.pdf`

Question for You: What have you covered so far?

- Basic MFG setup?
- Numerics?
- MFGs with common noise?

Plan

Lecture 1

1. A benchmark MFG for macroeconomics: the Aiyagari-Bewley-Huggett (ABH) heterogeneous agent model
2. The ABH model with common noise ("Krusell-Smith")
3. If time: some interesting extensions of the ABH model
    - the "wealthy hand-to-mouth" and marginal propensities to consume (MPCs)
    - present bias and self-control (economics meets psychology)

Lecture 2

1. Numerical solution of MFGs with common noise based on "When Inequality Matters for Macro..."
2. Other stuff...

Benchmark MFG for Macroeconomics

A Benchmark MFG for Macroeconomics

- A prototypical "heterogeneous agent model"
- Based on
  - Rao Aiyagari (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving", Quarterly Journal of Economics
  - Mark Huggett (1993), "The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies"
- Taught in first year of every self-respecting economics PhD program
- Original papers: discrete time; here: continuous time

Households

are heterogeneous in their wealth a and income y, solve

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \qquad \text{s.t.}$$

$$da_t = (y_t + r_t a_t - c_t)dt$$

$$dy_t = \mu(y_t)dt + \sigma(y_t)dW_t$$

$$a_t \geq \underline{a}$$

- c_t: consumption
- u: utility function, $u' > 0$, $u'' < 0$, e.g. $u(c) = \sqrt{c}$ or $c^\theta/\theta, \theta = -1$
- ρ: discount rate
- r_t : interest rate
- W_t: Wiener process, independent across households
- $y_t \in [\underline{y}, \bar{y}]$, reflected at boundaries if it ever reaches them
- $\underline{a} > -\infty$: borrowing limit e.g. if $\underline{a} = 0$, can only save

More general y-processes also possible, e.g. jumps = unemployment

Equilibrium Condition

- Denote joint distribution of (a, y) by $g(a, y, t)$
- Equilibrium: interest rate r_t, $t \geq 0$ determined by

$$0 = \int_0^\infty \int_{\underline{a}}^\infty a g(a, y, t) \mathrm{d}a \mathrm{d}y \quad \text{all } t \geq 0$$

- Interpretation: for everyone who borrows ($a < 0$) there is someone who lends ($a > 0$)
- Econ terminology: "bonds are in zero net supply"

Two Comments

1. Where is the "heterogeneity"? Aren't all households the same???
   - yes, but only "ex ante"
   - "ex post" they experience different histories of income shocks

2. Why infinite horizon? This is stupid, people don't live forever!
   - can interpret as dynasty with altruism towards children

$$V_t^{\text{parent}} = \int_t^{t+T} e^{-\rho(s-t)} u(c_s) ds + e^{-\rho T} V_{t+T}^{\text{child}}$$

   - avoids dependence of results on assumptions about terminal condition

Stationary MFG

Functions v and g on $(\underline{a}, \infty) \times (\underline{y}, \bar{y})$ and scalar r satisfy

$$\rho v = \max_c \; u(c) + (y + ra - c)\partial_a v + \mu(y)\partial_y v + \frac{\sigma^2(y)}{2}\partial_{yy} v \qquad \text{(HJB)}$$

with state constraint $a \geq \underline{a}$ and $0 = \partial_y v(a, \underline{y}) = \partial_y v(a, \bar{y})$ all a

$$0 = -\,\partial_a(s(a,y)g) - \partial_y(\mu(y)g) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g) \qquad \text{(FP)}$$

$$s(a,y) := y + ra - c(a,y), \quad c(a,y) = (u')^{-1}(\partial_a v(a,y)),$$

$$1 = \int_0^\infty \int_{\underline{a}}^\infty g \mathrm{d}a\mathrm{d}y, \quad g \geq 0$$

$$0 = \int_0^\infty \int_{\underline{a}}^\infty a g \mathrm{d}a\mathrm{d}y \qquad \text{(EQ)}$$

- A Mean Field Game! Coupling through scalar r determined by (EQ)

Stationary MFG: More Standard, Compact Notation

Define Hamiltonian

$$H(p) := \max_{c \geq 0}\{u(c) - pc\}$$

Functions v and g on $(\underline{a}, \infty) \times (\underline{y}, \bar{y})$ and scalar r satisfy

$$\rho v = H(\partial_a v) + (y + ra)\partial_a v + \mu(y)\partial_y v + \frac{\sigma^2(y)}{2}\partial_{yy} v \qquad \text{(HJB)}$$

with state constraint $a \geq \underline{a}$ and $0 = \partial_y v(a, \underline{y}) = \partial_y v(a, \bar{y})$ all a

$$0 = -\partial_a\big((y + ra + H'(\partial_a v))g\big) - \partial_y(\mu(y)g) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g) \qquad \text{(FP)}$$

$$1 = \int_0^\infty \int_{\underline{a}}^\infty g \mathrm{d}a \mathrm{d}y, \quad g \geq 0$$

$$0 = \int_0^\infty \int_{\underline{a}}^\infty a g \mathrm{d}a \mathrm{d}y \qquad \text{(EQ)}$$

Remark: More Complicated than Alessio's MFG

- Alessio had

$$-\partial_t u - \Delta u + H(x, Du) = F(m)$$

$$\partial_t m - \Delta m - \text{div}(m H_p(x, Du)) = 0$$

$$u(T) = G(m(T)), \quad m(0) = m_0$$

with $t \in (0, T), x \in \Omega, H : \Omega \times \mathbb{R}^N \to \mathbb{R}$, C^1 and convex in p

- Some differences:
    1. don't have separation between $H(x, Du)$ and $F(m)$. Instead HJB equation features $r(m)x_1 \times \partial_{x_1} u$
    2. state constraint $x \in \bar{X} \subset \Omega$. Note: it will actually bind
    3. no second-order term for x_1, only for x_2
    4. ...

Time-Dependent MFG

Functions v and g on $(\underline{a}, \infty) \times (\underline{y}, \bar{y}) \times (0, T)$ satisfy

$$\rho v = \max_c \; u(c) + (y + r(t)a - c)\partial_a v + \mu(y)\partial_y v + \frac{\sigma^2(y)}{2}\partial_{yy} v + \partial_t v \quad \text{(HJB)}$$

with state constraint $a \geq \underline{a}$ and $0 = \partial_y v(a, \underline{y}, t) = \partial_y v(a, \bar{y}, t)$ all a, t

$$\partial_t g = -\partial_a(sg) - \partial_y(\mu(y)g) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g) \quad \text{(FP)}$$

$$s := y + r(t)a - c, \quad c = (u')^{-1}(\partial_a v),$$

$$1 = \int_0^\infty \int_{\underline{a}}^\infty g \, \mathrm{d}a \, \mathrm{d}y, \quad g \geq 0$$

$$0 = \int_0^\infty \int_{\underline{a}}^\infty a g \, \mathrm{d}a \, \mathrm{d}y \quad \text{(EQ)}$$

$$g(a, y, 0) = g_0(a, y) \quad v(a, y, T) = v_\infty(a, y)$$

Coupling through $r(t)$, $t \geq 0$ determined by (EQ)

A Variant with a Different Coupling

- Model on previous slides is Huggett's (1993) version of ABH model
- Aiyagari (1994) proposed variant with slightly different coupling
- Households identical except that there income is $w_t y_t$ rather than y_t where w_t is the wage rate
- In addition to households, there is a large number of identical firms that solve

$$\max_{K_t, L_t} \{ZF(K_t, L_t) - r_t K_t - w_t L_t\}$$

  - Z: productivity, constant for now
  - K_t: capital, L_t: labor
  - F: production function, increasing and concave, e.g. $F(K, L) = \sqrt{KL}$
- New equilibrium conditions:

$$L_t = \int_0^\infty \int_{\underline{a}}^\infty y g(a, y, t) \mathrm{d}a\mathrm{d}y, \quad K_t = \int_0^\infty \int_{\underline{a}}^\infty a g(a, y, t) \mathrm{d}a\mathrm{d}y$$

Stationary MFG, Aiyagari's Variant

Functions v and g on $(\underline{a}, \infty) \times (\underline{y}, \bar{y})$ and scalar r satisfy

$$\rho v = \max_c \ u(c) + (wy + ra - c)\partial_a v + \mu(y)\partial_y v + \frac{\sigma^2(y)}{2}\partial_{yy} v \qquad \text{(HJB)}$$

with state constraint $a \geq \underline{a}$ and $0 = \partial_y v(a, \underline{y}) = \partial_y v(a, \bar{y})$ all a

$$0 = -\partial_a(s(a,y)g) - \partial_y(\mu(y)g) + \frac{1}{2}\partial_{yy}(\sigma^2(y)g) \qquad \text{(FP)}$$

$$s(a,y) := wy + ra - c(a,y), \quad c(a,y) = (u')^{-1}(\partial_a v(a,y)),$$

$$1 = \int_0^\infty \int_{\underline{a}}^\infty g \, da dy, \quad g \geq 0$$

$$r = Z\partial_K F(K,L) = \frac{1}{2} Z\sqrt{L/K}, \quad w = Z\partial_L F(K,L) = \frac{1}{2} Z\sqrt{K/L},$$
$$K = \int_0^\infty \int_{\underline{a}}^\infty a g \, da dy, \quad L = \int_0^\infty \int_{\underline{a}}^\infty y g \, da dy \qquad \text{(EQ)}$$

- Coupling through scalars r and w (prices) determined by (EQ)
- Here: prices only depend on 1st moments. Also other possibilities.

What We Know and What We Don't Know

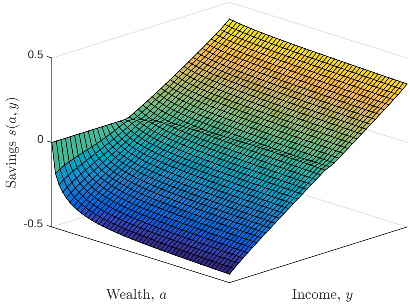
- Known:
  1. existence of solution to stationary MFG
  2. uniqueness of solution to stationary MFG under some assumptions (mostly $-u'(c)/(u''(c)c) \geq 1$ for all c)
- Not known:
  3. existence of solution to time-dependent MFG (though PL says it's "trivial"!)
  4. uniqueness of solution to time-dependent MFG
- Note: 2. still listed as open question in "PDE Models in Macroeconomics" (2014) – not true anymore

Other Theoretical Properties

State constraint $a \geq \underline{a}$ has some interesting implications

1. Savings $s(a,y) = y + ra - c(a,y) = y + ra + H'(\partial_a v(a,y))$ satisfy

   As $a \to \underline{a}$ $\quad s(a,y) \sim -\nu(y)\sqrt{a-\underline{a}}, \quad \nu(y) > 0$ for $y \leq y^*, y^* > \underline{y}$



2. Implies that g has a Dirac point mass at $\underline{a}$ for $y \leq y^*$
3. Implications for "marginal propensity to consume" (MPCs) ...

Marginal propensity to consume (MPC)

- MPC answers question: if I give you 1$, how much of it will you save and how much will you consume?
- Data: poorer people have higher MPCs
- Attractive feature of ABH model: can generate this
- MPC closely related to derivative of consumption function $c(a, y) = -H'(\partial_a v(a, y))$ with respect to a
- ABH model: as $a \to \underline{a}$

$$c(a, y) \sim y + ra + \nu(y)\sqrt{a - \underline{a}}$$

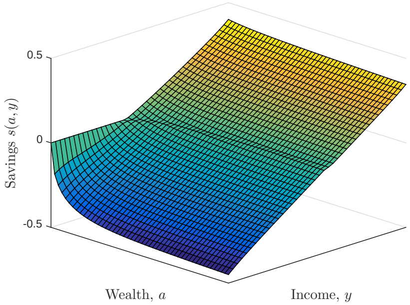
$$\partial_a c(a, y) \sim r + \frac{1}{2}\frac{\nu(y)}{\sqrt{a - \underline{a}}}$$

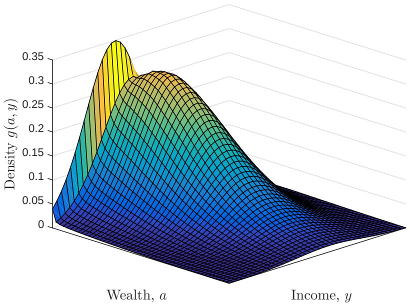
and so poorer people (closer to $\underline{a}$) have higher MPCs

Numerical Solution

- Finite difference method based on Achdou and Capuzzo-Dolcetta (2010) and Achdou (2013)
- See "Income and Wealth Distribution in Macroeconomics" and Matlab codes on my website

  `http://www.princeton.edu/~moll/HACTproject.htm`





Macroeconomic MFGs with Common Noise

Macroeconomic MFGs with Common Noise

- **This is where the money is!**
- Can fit 90% of macroeconomics into this apparatus so any progress would be extremely valuable
- To understand setup consider Aiyagari (1994) with stochastic aggregate productivity, Z, common to all firms
- First studied by
    - Per Krusell and Tony Smith (1998), "Income and Wealth Heterogeneity in the Macroeconomy", J of Political Economy
    - Wouter Den Haan (1996), "Heterogeneity, Aggregate Uncertainty, and the Short-Term Interest Rate", Journal of Business and Economic Statistics
- Language: instead of "common noise" economists say "aggregate shocks" or "aggregate uncertainty"

Macroeconomic MFGs with Common Noise

- Households:

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \qquad \text{s.t.}$$

$$da_t = (w_t y_t + r_t a_t - c_t) dt$$

$$dy_t = \mu(y_t) dt + \sigma(y_t) dW_t$$

$$a_t \geq \underline{a}$$

- Firms:

$$\max_{K_t, L_t} \{Z_t F(K_t, L_t) - r_t K_t - w_t L_t\}$$

$$dZ_t = \mu^Z(Z_t) dt + \sigma^Z(Z_t) dB_t, \quad \text{common } B_t \text{ for all firms}$$

$$\Rightarrow r_t = Z_t \partial_K F(K_t, L_t), \quad w_t = Z_t \partial_L F(K_t, L_t)$$

- Equilibrium:

$$L_t = \int_0^\infty \int_{\underline{a}}^\infty y g(a, y, t) \mathrm{d}a \mathrm{d}y, \quad K_t = \int_0^\infty \int_{\underline{a}}^\infty a g(a, y, t) \mathrm{d}a \mathrm{d}y$$

Macroeconomic MFGs with Common Noise

- Households:

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \qquad \text{s.t.}$$

$$da_t = (w_t y_t + r_t a_t - c_t) dt$$

$$dy_t = \mu(y_t) dt + \sigma(y_t) dW_t$$

$$a_t \geq \underline{a}$$

- Firms:

$$\max_{K_t, L_t} \{Z_t F(K_t, L_t) - r_t K_t - w_t L_t\}$$

$$dZ_t = \mu^Z(Z_t) dt + \sigma^Z(Z_t) dB_t, \quad \text{common } B_t \text{ for all firms}$$

$$\Rightarrow r_t = Z_t \partial_K F(K_t, L_t), \quad w_t = Z_t \partial_L F(K_t, L_t)$$

- Equilibrium if restrict to stationary y-process with 1st moment = 1:

$$L_t = 1, \quad K_t = \int_0^\infty \int_{\underline{a}}^\infty a g(a, y, t) \mathrm{d}a \mathrm{d}y$$

Macroeconomic MFGs with Common Noise

- Question: What is the appropriate state variable in HJB equation?
- Answer: it's really the entire joint distribution of income and wealth (plus aggregate productivity Z)

$$\text{state} = (g, Z)$$

- Problem: g is infinite-dimensional object
- Why not enough to keep track of first moment K_t? Answer: evolution dK_t depends on entire g, not just K_t (unless all individual da_t's are linear in a_t).
- Nicely explained in Victor Rios-Rull (1997), "Computation of Equilibria in Heterogeneous Agent Models"

Macroeconomic MFGs with Common Noise

- Krusell-Smith get around this with an approximation: households care only about some finite set of moments of wealth distribution (in practice, only first moment):

$$v(a, y; g, Z) \approx \tilde{v}(a, y; K, Z)$$

- $\Rightarrow$ convert infinite-dimensional problem into finite-dimensional one
- Makes some sort of sense:
    - interaction only through prices (r, w)
    - assumption is that households only take into account future price movements that are due to movements of the aggregate capital stock (and Z)
    - bounded rationality interpretation.
- May be ok assumption for environment on previous slide
- Furthermore approach fails in more general environments

Tomorrow's Lecture

- A computational method for MFGs with common noise, based on "When Inequality Matters for Macro..."
- Idea: linearize MFG with common noise Z_t around MFG without common noise $Z_t = 1$
- Works beautifully in practice and in many different applications
- But we have no idea about the underlying mathematics!
- $\Rightarrow$ Great problem for mathematicians

Extensions of the ABH Model

Rest of Today's Lecture

Some interesting extensions of the ABH model

- the "wealthy hand-to-mouth" and marginal propensities to consume (MPCs)

- present bias and self-control (economics meets psychology)

MPCs depends on balance sheet composition

	3 HtM groups			2 HtM groups	
	P-HtM	W-HtM	N-HtM	HtM-NW	N-HtM-NW
MPC out of transitory income shock	0.243	0.301	0.127	0.229	0.201
	(0.065)	(0.048)	(0.036)	(0.054)	(0.030)

Table from Kaplan, Violante and Weidner (2014) using PSID

- Poor HtM: no liquid wealth and no illiquid wealth
- Wealthy HtM: no liquid wealth but positive illiquid wealth
- Non HtM: positive liquid wealth

See also: Broda-Parker, Misra-Surico, Jappelli-Pistaferri, Baker

The "wealthy hand-to-mouth" and MPCs

$$\max_{\{c_t, d_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \quad \text{s.t.}$$

$$\dot{b}_t = y_t + r^b b_t - d_t - \chi(d_t, a_t) - c_t$$

$$\dot{a}_t = r^a a_t + d_t$$

$$dy_t = \mu(y_t) dt + \sigma(y_t) dW_t$$

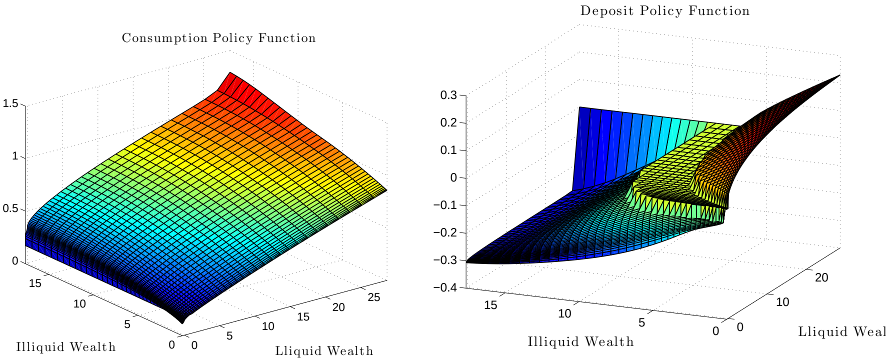
$$a_t \geq \underline{a}, \quad b_t \geq \underline{b}$$

- a_t: illiquid assets
- b_t: liquid assets
- c_t: consumption
- χ: transaction cost function
- y_t: individual income
- d_t: deposits into illiquid acc

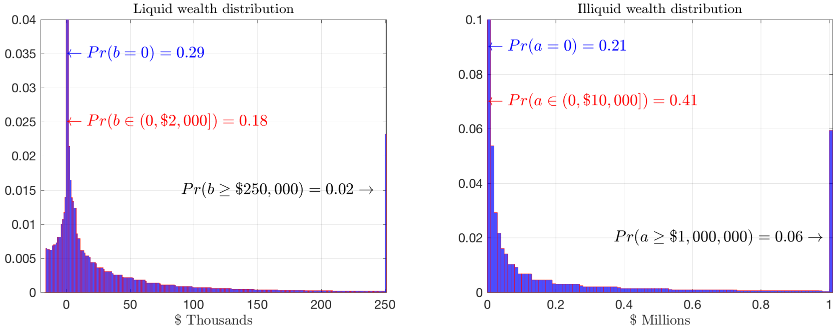
$$\chi(d, a) = \chi_0 |d| + \frac{\chi_1}{2} \left| \frac{d}{a} \right|^2 a$$

- kinked component $\Rightarrow$ inaction, quadratic component $\Rightarrow |d| < \infty$

Policy Functions



Model matches key feature of U.S. wealth distribution



	Data	Model
Mean illiquid assets (rel to GDP)	2.920	2.920
Mean liquid assets (rel to GDP)	0.260	0.263
Poor hand-to-mouth	10%	10%
Wealthy hand-to-mouth	20%	19%

Implications

- The presence of the “wealthy hand-to-mouth” has important implications for monetary and fiscal policy
- If add common noise, Krusell-Smith approach fails – see “When Inequality Matters for Macro...”

Present Bias and Self Control

- Last 40 years have seen explosion of “behavioral economics”
- So far, not much of this in macroeconomics
- Irving Fisher (1930s) “a small income, other things being equal, tends to produce a high rate of impatience, partly from the thought that provision for the present is necessary both for the present itself and for the future as well, and partly from lack of foresight and self-control.”
- like the “wealthy hand-to-mouth” this results in MPC heterogeneity and could have important implications for monetary and fiscal policy
- a model of this based on Harris and Laibson (2010), Cao and Werning (2015)

Key idea

Model a game where a sequence of decision makers will not be in power forever

- once out of power objectives change.
- focus on a stationary setting with a Poisson arrival rate for switching decision makers
- Markov equilibrium of game of “current selves” against “future selves”

Present Bias and Self Control

- continuation lifetime utility at time t of the decision maker is:

$$V_t := \mathbb{E}_t \left[\int_0^{\tau} e^{-\rho s} U_1(c_{t+s}) ds + e^{-\rho \tau} W_{t+\tau} \right]$$

  - $\rho > 0$: discount rate
  - τ: random time at which the agent currently making decision is removed, distributed exponentially $1 - e^{-\lambda \tau}$

- for agent out of decision making

$$W_t := \int_0^{\infty} e^{-\rho s} U_0(c_{t+s}) ds$$

Present Bias and Self Control

$$\rho V(a) = \max_{c\geq 0}\{U_1(c) + V'(a)(ra - c) + \lambda(W(a) - V(a))\},$$

$$\rho W(a) = U_0(\hat{c}(a)) + W'(a)(ra - \widehat{c}(a))$$

where $\widehat{c}(a)$= solution to maximization in first equation.

Open questions

- existence
- uniqueness (reason to expect multiplicity)

Some Rules for Doing Economics

1. Always start with the data
    - a model is only useful if it is consistent with some data
2. If you can't solve a problem, then change it
    - economics is not like physics, you can make any assumption you want (subject to rule # 1)
3. Don't underestimate language barriers
    - for example, economists don't know ∇, Δ!
4. It's ok to use imprecise notation as long as people know what you mean

Conclusion of Lecture 1

- Mean field games extremely useful in economics...
- ... lots of exciting questions involve mean field type interactions...
- ... but mathematics often pretty challenging, at least for the average economist.
- what economists ultimately care about: computations
    - more complicated extensions of standard ABH model
    - e.g. two assets, fixed costs, ...
- Potentially high payoff from mathematicians working on this!