Lecture 1:

Overview, Hamiltonians and Phase Diagrams

ECO 521: Advanced Macroeconomics I

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Course Overview

Two Parts:

(1) Substance: income and wealth distribution in macroeconomics

(2) Tools: continuous time methods

- Everything is flexible, feedback very useful!

Substance: Where I'm going

- Income and wealth distribution in macroeconomics...
- ... or models in which relevant state variable is a distribution
- Want to get you started on building these kind of models
- Why should you be interested in this?
  - Fertile area of research, excellent dissertation topics!
  - Many open questions
  - Hard – high entry barriers

Substance: Where I'm going

- Some questions we will try to answer:
    - Why are income and wealth so unequally distributed?
    - Does inequality affect macroeconomic aggregates?
    - Does inequality affect how the macroeconomy responds to shocks?
    - What are the forces that lead to the concentration of economic activity in a few very large firms?
    - How important is firm heterogeneity and reallocation for the aggregate economy?

- More generally, want to give you an idea of some open questions in macro and get you started with your research!

Put in a Nice Way

"The most important discovery was the evidence on the pervasiveness of heterogeneity and diversity in economic life."

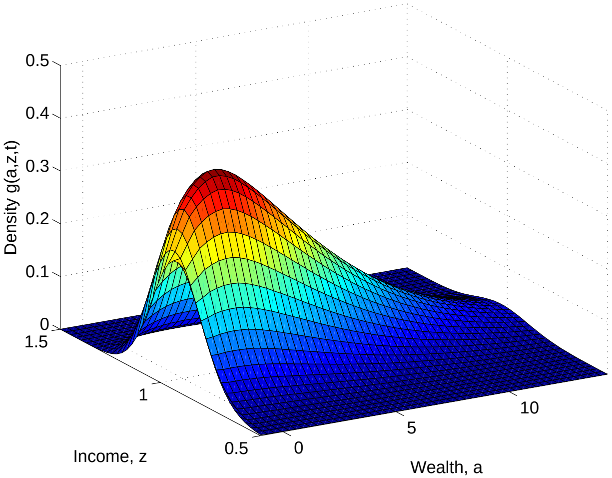
(Heckman, 2001)

"While we often must focus on aggregates for macroeconomic policy, it is impossible to think coherently about national well-being while ignoring inequality and poverty, neither of which is visible in aggregate data. Indeed, and except in exceptional cases, macroeconomic aggregates themselves depend on distribution."

(Deaton, 2016)

What you'll be able to do at end of this course

- Joint distribution of income and wealth in Aiyagari model



What you'll be able to do at end of this course

- Experiment: effect of one-time redistribution of wealth

What you'll be able to do at end of this course

Video of convergence back to steady state

https://www.dropbox.com/s/op5u2nlifmmer2o/distribution_tax.mp4?dl=0

Plan of Lecture

(1) Hamiltonians

(2) Phase diagrams

(3) Finite difference methods and shooting algorithm

Hamiltonians

- Pretty much all deterministic optimal control problems in continuous time can be written as

$$v(x_0) = \max_{\{\alpha(t)\}_{t \geq 0}} \int_0^{\infty} e^{-\rho t} r(x(t), \alpha(t))\, dt$$

subject to the law of motion for the state

$$\dot{x}(t) = f(x(t), \alpha(t)) \quad \text{and} \quad \alpha(t) \in A$$

for $t \geq 0$, $x(0) = x_0$ given.

- $\rho \geq 0$: discount rate
- $x \in X \subseteq \mathbb{R}^N$: state vector
- $\alpha \in A \subseteq \mathbb{R}^M$: control vector
- $r : X \times A \to \mathbb{R}$: instantaneous return function

Example: Neoclassical Growth Model

$$v(k_0) = \max_{\{c(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c(t))dt$$

subject to

$$\dot{k}(t) = F(k(t)) - \delta k(t) - c(t)$$

for $t \geq 0$, $k(0) = k_0$ given.

- Here the state is $x = k$ and the control $\alpha = c$
- $r(x, \alpha) = u(\alpha)$
- $f(x, \alpha) = F(x) - \delta x - \alpha$

Hamiltonian: General Formulation

- Consider the general optimal control problem two slides back
- Can obtain necessary and sufficient conditions for an optimum using the following procedure ("cookbook")
- Current-value Hamiltonian

$$\mathcal{H}(x, \alpha, \lambda) = r(x, \alpha) + \lambda f(x, \alpha)$$

- $\lambda \in \mathbb{R}^N$: "co-state" vector

Hamiltonian: General Formulation

- Necessary and sufficient conditions:

$$\mathcal{H}_\alpha \left(x\left(t\right), \alpha\left(t\right), \lambda\left(t\right)\right) = 0$$

$$\dot{\lambda}\left(t\right) = \rho\lambda\left(t\right) - \mathcal{H}_x\left(x\left(t\right), \alpha\left(t\right), \lambda\left(t\right)\right)$$

$$\dot{x}\left(t\right) = f\left(x\left(t\right), \alpha\left(t\right)\right)$$

  for all $t \geq 0$

- Initial value for state variable(s): $x(0) = x_0$
- Boundary condition for co-state variable(s) $\lambda\left(t\right)$, called "transversality condition"

$$\lim_{T\to\infty} e^{-\rho T}\lambda\left(T\right)x\left(T\right) = 0$$

- `http://www.princeton.edu/~moll/ECO503Web/Lecture2_ECO503.pdf` (Slide 26 ff)
- Note: initial value of the co-state variable $\lambda\left(0\right)$ not predetermined

Example: Neoclassical Growth Model

- Recall: $r(x, \alpha) = u(\alpha)$ and $f(x, \alpha) = F(x) - \delta x - \alpha$
- Using the "cookbook"

$$\mathcal{H}(k, c, \lambda) = u(c) + \lambda[F(k) - \delta k - c]$$

- We have

$$\mathcal{H}_c(k, c, \lambda) = u'(c) - \lambda$$

$$\mathcal{H}_k(k, c, \lambda) = \lambda(F'(k) - \delta)$$

- Therefore conditions for optimum are:

$$\dot{\lambda} = \lambda(\rho + \delta - F'(k))$$

$$\dot{k} = F(k) - \delta k - c \qquad \text{(ODE)}$$

$$u'(c) = \lambda$$

with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T}\lambda(T)k(T) = 0$.

Example: Neoclassical Growth Model

- Interpretation: continuous time Euler equation

- In discrete time

$$\lambda_t = \beta\lambda_{t+1}(F'(k_{t+1}) + 1 - \delta)$$

$$k_{t+1} = F(k_t) + (1-\delta)k_t - c_t$$

$$u'(c_t) = \lambda_t$$

- (ODE) is continous-time analogue

Phase Diagrams

- How analyze (ODE)? In one-dimensional case (scalar x): use phase-diagram

- Two possible phase-diagrams:
  (i) in (λ, k)-space: more general strategy
  (ii) in (c, k)-space: nicer in terms of the economics

- For (i), use $u'(c) = \lambda$ or $c = (u')^{-1}(\lambda)$ to write (ODE) as

$$\begin{aligned} \dot{\lambda} &= \lambda(\rho + \delta - F'(k)) \\ \dot{k} &= F(k) - \delta k - (u')^{-1}(\lambda) \end{aligned} \tag{ODE'}$$

  with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T}\lambda(T)k(T) = 0$.

- Homework 1: draw phase-diagram in (λ, k)-space.

Phase Diagrams

- For (ii), note that

$$\dot{\lambda} = u''(c)\dot{c}$$

and substitute into equation for $\dot{\lambda}$:

$$u''(c)\dot{c} = u'(c)(\rho + \delta - F'(k))$$

- Or define the "coefficient of relative risk aversion"

$$\sigma(c) := -\frac{u''(c)c}{u'(c)} > 0$$

and write (ODE) as

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\sigma(c)}(F'(k) - \rho - \delta) \\ \dot{k} &= F(k) - \delta k - c \end{aligned} \qquad \text{(ODE")}$$

with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T} u'(c(T))k(T) = 0$.

- Note: $\frac{1}{\sigma(c)}$ = "intertemporal elasticity of substitution" (IES)

Steady State

- In steady state $\dot{k} = \dot{c} = 0$. Therefore

$$F'(k^*) = \rho + \delta$$

$$c^* = F(k^*) - \delta k^*$$

- Same as in discrete time with $\beta = 1/(1+\rho)$.

- For example, if $F(k) = Ak^\alpha, \alpha < 1$. Then

$$k^* = \left(\frac{\alpha A}{\rho + \delta}\right)^{\frac{1}{1-\alpha}}$$

Phase Diagram

- See graph that I drew in lecture by hand or Figure 8.1 in Acemoglu's textbook
- Obtain saddle path
- Prove stability of steady state
- Important: saddle path is not a "knife edge" case in the sense that the system only converges to steady state if $(c(0), k(0))$ happens to lie on the saddle path and diverges for all other initial conditions
- In contrast to the state variable $k(t)$, $c(t)$ is a "jump variable." That is, $c(0)$ is free and always adjusts so as to lie on the saddle path

Violations of Transversality Condition

- Question: how do you know that trajectories with $c(0)$ off the saddle path violate the transversality condition?

- See Acemoglu, chapter 8 “The Neoclassical Growth Model” section 5 “Transitional Dynamics”

  - if $c(0)$ below saddle path, $k(t) \to k_{\text{max}}$ and $c(t) \to 0$

  - if $c(0)$ above saddle path, $k(t) \to 0$ in finite time while $c(t) > 0$. Violates feasibility.

  - local analysis/linearization gives same answer
    `http://www.princeton.edu/~moll/ECO503Web/Lecture4_ECO503.pdf`

  - notes that most rigorous and straightforward way is to use that concave problems have unique solution (his Theorem 7.14)

Numerical Solution: Finite-Difference Method

- By far the simplest and most transparent method for numerically solving differential equations.
- Approximate $k(t)$ and $c(t)$ at N discrete points in the time dimension, t^n, $n = 1, ..., N$. Denote distance between grid points by Δt.
- Use short-hand notation $k^n = k(t^n)$.
- Approximate derivatives

$$\dot{k}(t^n) \approx \frac{k^{n+1} - k^n}{\Delta t}$$

- Approximate (ODE") as

$$\frac{c^{n+1} - c^n}{\Delta t} \frac{1}{c^n} = \frac{1}{\sigma(c^n)} (F'(k^n) - \rho - \delta)$$

$$\frac{k^{n+1} - k^n}{\Delta t} = F(k^n) - \delta k^n - c^n$$

Finite-Difference Method/Shooting Algorithm

- Or

$$
\begin{aligned}
c^{n+1} &= \Delta t c^n \frac{1}{\sigma(c^n)}(F'(k^n) - \rho - \delta) + c^n \\
k^{n+1} &= \Delta t(F(k^n) - \delta k^n - c^n) + k^n
\end{aligned}
\tag{FD}
$$

  with $k^0 = k_0$ given.

- Homework 2: draw phase diagram/saddle path in MATLAB.
- Assume $F(k) = Ak^\alpha$, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $A = 1$, $\alpha = 0.3$, $\sigma = 2$, $\rho = \delta = 0.05$, $k_0 = \frac{1}{2}k^*$, $\Delta t = 0.1$, $N = 700$.
- Algorithm:
  (i) guess c^0
  (ii) obtain (c^n, k^n), $n = 1, ..., N$ by running (FD) forward in time.
  (iii) If the sequence converges to (c^*, k^*), then you have obtained the correct saddle path. If not, back to (i) and try different c^0.
- This is called a "shooting algorithm"