Lectures 11 and 12: Income and Wealth Distribution in the Growth Model

ECO 503: Macroeconomic Theory I

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Plan of Lecture

- Facts on income and wealth distribution
- Heterogeneous agents in the growth model: lifecycle and wealth distribution

Facts on Income and Wealth Distribution

- Will focus on inequality at **top** of income and wealth distribution
  - bottom, middle obviously equally important
- Nice summary of facts on income distribution: Atkinson, Piketty, and Saez (2011), "Top Incomes in the Long Run of History," Journal of Economic Literature
- Also see Piketty (2014) "Capital in the 21st Century", particularly for facts on wealth distribution
- and Piketty and Saez (2014), "Inequality in the Long-Run"
  - 5 page summary of Piketty's 685 page book

But first... back to the roots

- more precisely 1896 and

- In 1896, Vilfredo Pareto examined income and wealth distribution across Europe
  - published "Cours d'économie politique", for whole book see http://www.institutcoppet.org/2012/05/08/cours-deconomie-politique-18
  - relevant part http://www.princeton.edu/~moll/pareto.pdf

poser en ligne droite [1]. Disons immédiatement que nous allons retrouver cette tendance dans les nombreux exemples que nous aurons encore à examiner.

Un autre fait, tout aussi, et même plus remarquable, c'est que les courbes de la répartition des revenus, en Angleterre

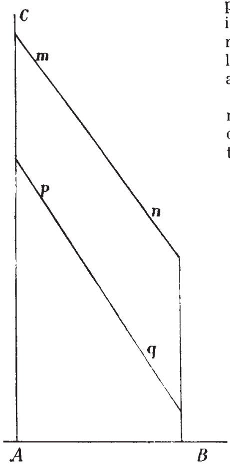


Fig. 47.

Schedule D — Année 1893-94.

x £	N GREAT BRITAIN	N IRELAND
150	400 648	17 717
200	234 185	9 365
300	121 996	4 592
400	74 041	2 684
500	54 419	1 898
600	42 072	1 428
700	34 269	1 104
800	29 311	940
900	25 033	771
1000	22 896	684
2000	9 880	271
3000	6 069	142
4000	4 161	88
5000	3 081	68
10000	1 104	22

et en Irlande, présentent un parallélisme à peu près complet. Ce fait est à rapprocher d'un autre, que nous allons bientôt constater : les inclinaisons des lignes mn, pq obtenues pour dif-

(958) [1] C'est-à-dire que la courbe réelle est interpolée par une droite dont l'équation est

(1) $$\log N = \log A - \alpha \log x.$$

Power Laws

- Pareto (1896): upper-tail distribution of number of people with an income or wealth X greater than a large x is proportional to $1/x^{\zeta}$ for some $\zeta > 0$

$$\Pr(X > x) = kx^{-\zeta}$$

- **Definition:** We say that a variable, x, follows a power law (PL) if there exist $k > 0$ and $\zeta > 0$ such that

$$\Pr(X > x) = kx^{-\zeta}, \quad \text{all } x$$

- x follows a PL $\Leftrightarrow$ x has a Pareto distribution
- Surprisingly many variables follow power laws
    - see Gabaix (2009), "Power Laws in Economics and Finance," very nice, very accessible

Power Laws

- Another way of saying same thing: top inequality is **fractal**
  - ... top 0.01% is M times richer than top 0.1%,... is M times richer than top 1%,... is M times richer than top 10%,...
  - to see this, note that top p percentile x_p satisfies

$$kx_p^{-\zeta} = p/100 \quad \Rightarrow \quad \frac{x_{0.01}}{x_{0.1}} = \frac{x_{0.1}}{x_1} = ... = 10^{1/\zeta}$$

  - average income/wealth above pth percentile is

$$\bar{x}_p = \mathbb{E}[x|x \geq x_p] = \frac{\int_{x_p}^{\infty} x\zeta kx^{-\zeta-1}dx}{kx_p^{-\zeta}} = \frac{\zeta}{\zeta - 1}x_p \quad \Rightarrow$$

$$\frac{\bar{x}_{0.01}}{\bar{x}_{0.1}} = \frac{\bar{x}_{0.1}}{\bar{x}_1} = ... = 10^{1/\zeta}$$

- Related result: if x has a Pareto distribution, then share of x going to top p percent is

$$S(p) = \left(\frac{100}{p}\right)^{1/\zeta-1}$$

Evolution of Top Incomes

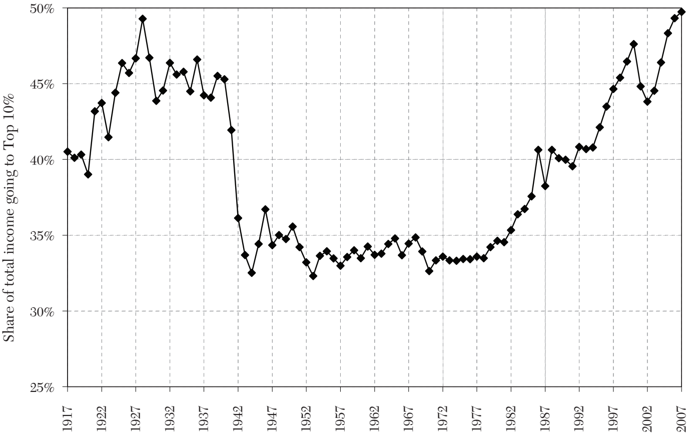


Figure 1. The Top Decile Income Share in the United States, 1917–2007.

Notes: Income is defined as market income including realized capital gains (excludes government transfers). In 2007, top decile includes all families with annual income above $109,600.

Source: Piketty and Saez (2003), series updated to 2007.

Evolution of Top Incomes

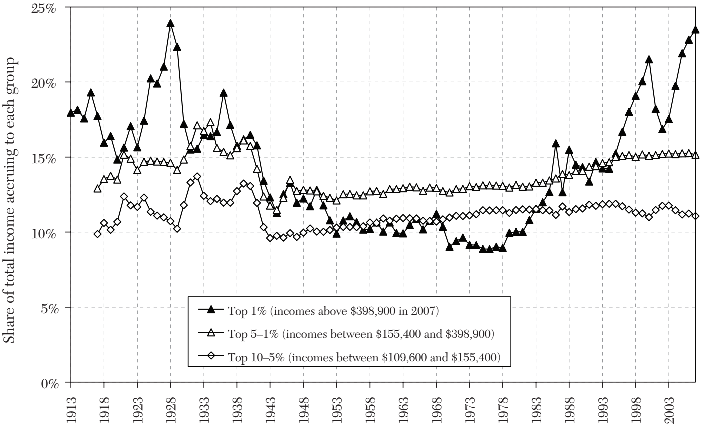


Figure 2. Decomposing the Top Decile US Income Share into three Groups, 1913–2007

Notes: Income is defined as market income including capital gains (excludes all government transfers).
Top 1 percent denotes the top percentile (families with annual income above $398,900 in 2007).
Top 5–1 percent denotes the next 4 percent (families with annual income between $155,400 and $398,900 in 2007).
Top 10–5 percent denotes the next 5 percent (bottom half of the top decile, families with annual income between $109,600 and $155,400 in 2007).

Evolution of Top Incomes

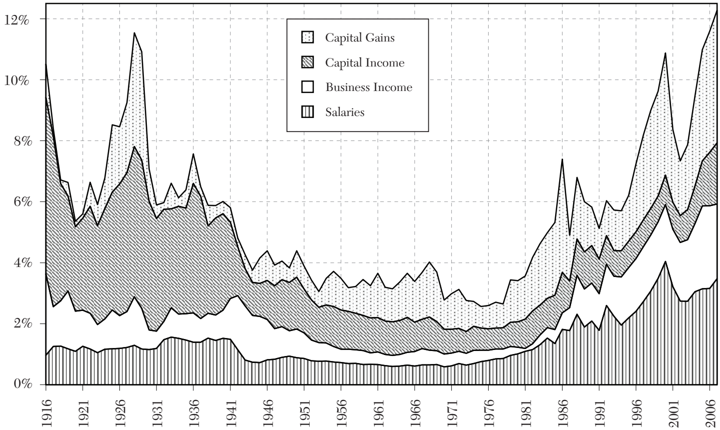


Figure 3. The Top 0.1 Percent Income Share and Composition, 1916–2007

Notes: The figure displays the top 0.1 percent income share and its composition. Income is defined as market income including capital gains (excludes all government transfers). Salaries include wages and salaries, bonus, exercised stock-options, and pensions. Business income includes profits from sole proprietorships, partnerships, and S-corporations. Capital income includes interest income, dividends, rents, royalties, and fiduciary income. Capital gains includes realized capital gains net of losses.

Cross-Country Evidence

- In practice, quite often don't have data on share of top 1%
- Use Pareto interpolation. Assume income has CDF

$$F(y) = 1 - \left(\frac{y}{y_0}\right)^{-\zeta}$$

- Useful property of Pareto distribution: average above threshold proportional to threshold

$$\mathbb{E}[\tilde{y}|\tilde{y} \geq y] = \frac{\int_y^\infty zf(z)dz}{1 - F(y)} = \frac{\zeta}{\zeta - 1}y$$

- Estimate and report $\beta \equiv \zeta/(\zeta - 1)$.
- Example: $\beta = 2$ means average income of individuals with income above \$100,000 is \$200,000 and average income of individuals with income above \$1 million is \$2 million.
- Obviously imperfect, but useful because ζ or β is exactly what our theories generate.

Cross-Country Evidence

TABLE 6
Comparative Top Income Shares

	Around 1949			Around 2005		
	Share of top 1%	Share of top 0.1%	β coefficient	Share of top 1%	Share of top 0.1%	β coefficient
Indonesia	19.87	7.03	2.22			
Argentina	19.34	7.87	2.56	16.75	7.02	2.65
Ireland	12.92	4.00	1.96	10.30		2.00
Netherlands	12.05	3.80	2.00	5.38	1.08	1.43
India	12.00	5.24	2.78	8.95	3.64	2.56
Germany	11.60	3.90	2.11	11.10	4.40	2.49
United Kingdom	11.47	3.45	1.92	14.25	5.19	2.28
Australia	11.26	3.31	1.88	8.79	2.68	1.94
United States	10.95	3.34	1.94	17.42	7.70	2.82
Canada	10.69	2.91	1.77	13.56	5.23	2.42
Singapore	10.38	3.24	1.98	13.28	4.29	2.04
New Zealand	9.98	2.42	1.63	8.76	2.51	1.84
Switzerland	9.88	3.23	2.06	7.76	2.67	2.16
France	9.01	2.61	1.86	8.73	2.48	1.83
Norway	8.88	2.74	1.96	11.82	5.59	3.08
Japan	7.89	1.82	1.57	9.20	2.40	1.71
Finland	7.71		1.63	7.08	2.65	2.34
Sweden	7.64	1.96	1.69	6.28	1.91	1.93
Spain			1.99	8.79	2.62	1.90
Portugal		3.57	1.94	9.13	2.26	1.65
Italy				9.03	2.55	1.82
China				5.87	1.20	1.45

Cross-Country Evidence

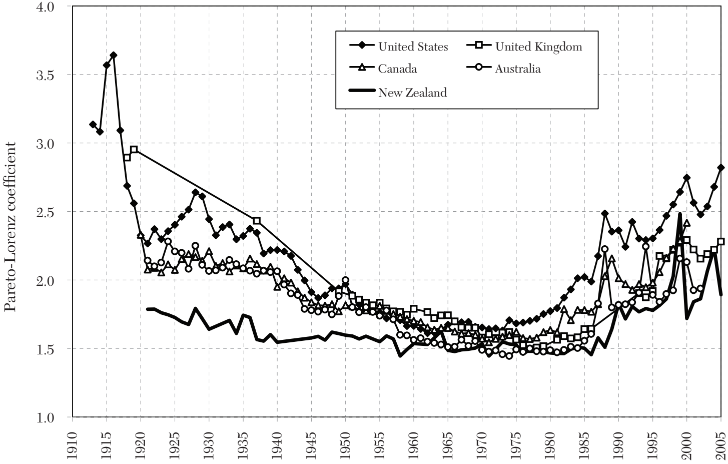


Figure 12. Inverted-Pareto β Coefficients: English-Speaking Countries, 1910–2005

Cross-Country Evidence

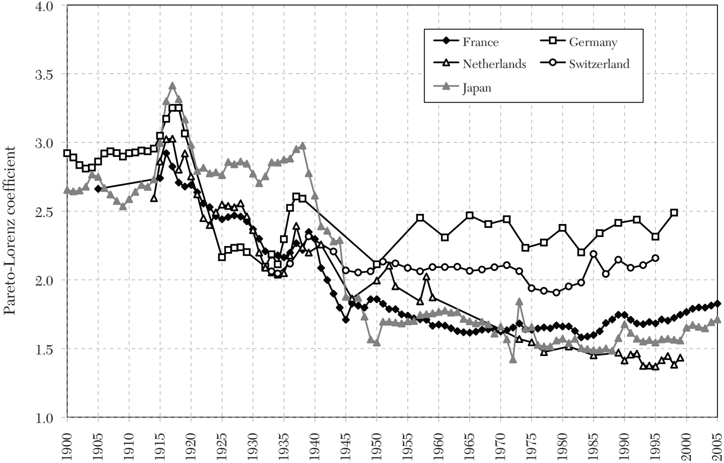


Figure 13. Inverted-Pareto β Coefficients, Middle Europe and Japan, 1900–2005

Cross-Country Evidence

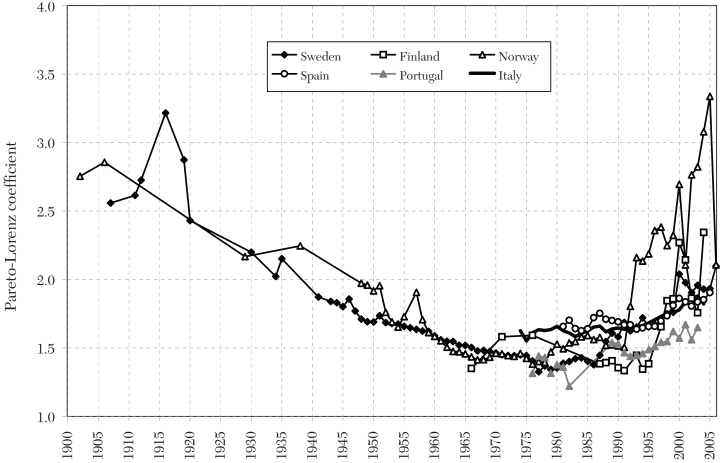


Figure 14. Inverted-Pareto β Coefficients, Nordic and Southern Europe, 1900–2006

Cross-Country Evidence

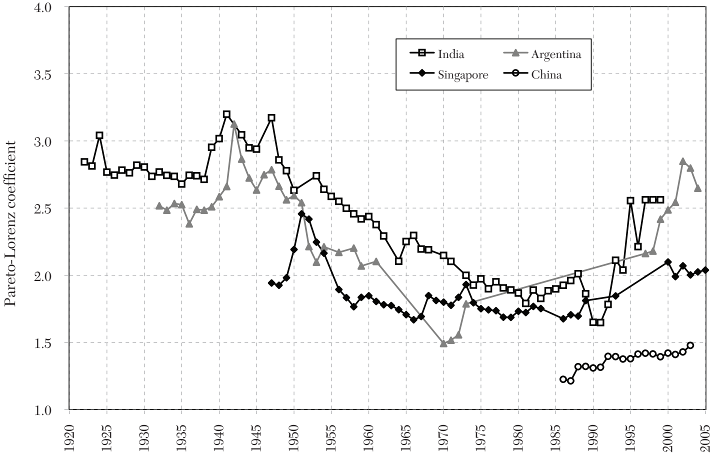


Figure 15. Inverted-Pareto β Coefficients, Developing Countries: 1920–2005

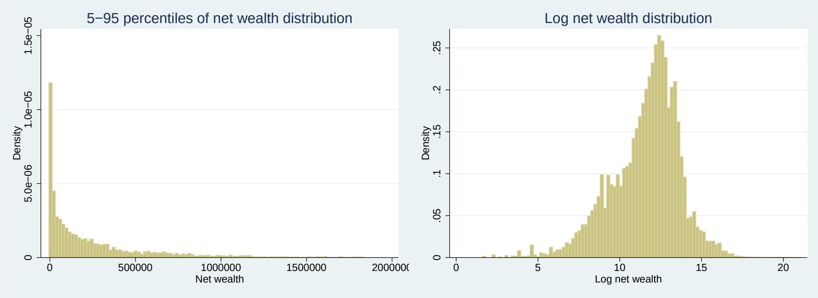
Income Inequality and Growth

from Atkinson, Piketty, Saez (2011)

- Growth of average real incomes 1975-2006 in US vs. France:
  - US: 32.2 %
  - France: 27.1 %
- Growth of average real incomes 1975-2006 in US vs. France **excluding top 1%**:
  - US: 17.9 %
  - France: 26.4 %
- Footnote: "It is important to note that such international growth comparisons are sensitive to the exact choice of years compared, the price deflator used, the exact definition of income in each country, and hence are primarily illustrative."

Wealth Distribution

U.S., Source: Survey of Consumer Finances

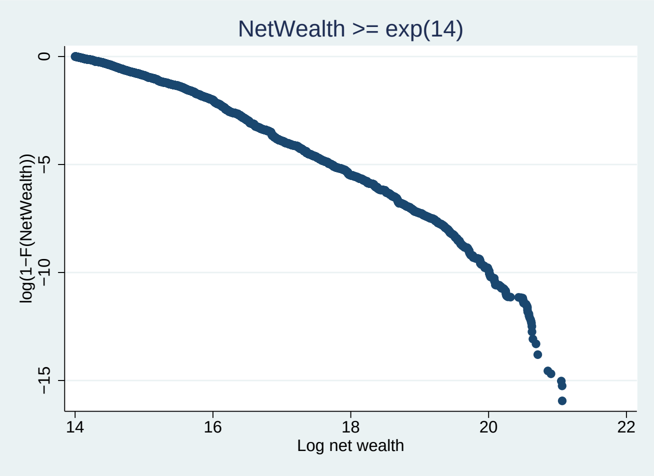


Features of U.S. Wealth Distribution:

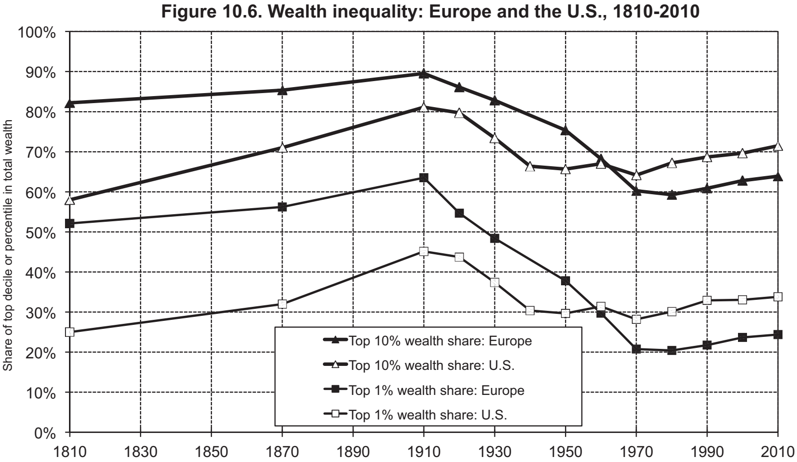
- right skewness
- heavy upper tail
- Pareto distribution

U.S. Wealth Distribution

Upper Tail, Source: Survey of Consumer Finances



Wealth Distribution: U.S. vs. Europe



Until the mid 20th century, wealth inequality was higher in Europe than in the United States.
Sources and series: see piketty.pse.ens.fr/capital21c.

Income and Wealth Distribution in the Growth Model

1. Preliminaries: perpetual youth model
2. Heterogeneous agents in the growth model: lifecycle and wealth distribution
   - relatively general formulation
   - special case with analytic solution, Pareto distribution of wealth (= simplest possible heterogeneous agent model)
   - later courses: richer models ("Aiyagari-Bewley-Huggett")

Preliminaries: Perpetual Youth Model

- Assumption we made so far: individuals live forever, $T = \infty$

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Of course, a "bad" assumption: people don't live forever
  - descriptive realism is not the objective of modeling in (macro)economics
  - $T = \infty$ is innocuous for thinking about many issues
- But for **some** issues, individuals' lifecyle is at heart of issue
- Problem: lifecycle usually complicates things considerably
- Here: how to introduce lifecycle without complicating things
  - modeling trick due to Yaari (1965), Blanchard (1985)
  - "Blanchard-Yaari perpetual youth model"
  - also see Acemoglu, Ch. 9.7, 9.8
  - life-cycle but don't have to keep track of individuals' age

Preliminaries: Perpetual Youth Model

- Each period, an individual faces a constant probability of death p, with $0 < p < 1$
  - "perpetual youth": age has no effect on future longevity
- Let T=random variable denoting time of death
- Define "survival function"

$$S(t) = \Pr(T > t) = (1 - p)^t$$

- Expected lifespan

$$p + 2(1-p)p + 3(1-p)^2 p + ... = \frac{1}{p}$$

- Individual preferences

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t [(1-p)^t u(c_t) + (1-p)^{t-1} p \times u(\text{death})]$$

Preliminaries: Perpetual Youth Model

- Normalizing $u(\text{death}) = 0$, preferences are

$$\sum_{t=0}^{\infty} (\beta(1-p))^t u(c_t)$$

- Convenient property: just like infinite-horizon model with adjusted discount factor $\hat{\beta} = \beta(1-p)$
- Mortality simply decreases the discount factor
- Strong assumption: p independent of age

Preliminaries: Perpetual Youth Model

- Often convenient to do this in continuous time
- Let T=random variable denoting time of death
- Define "survival function" $S(t) = \Pr(T > t)$
- Assumption: "survival function" is exponential

$$S(t) = e^{-pt}, \quad 0 < p < \infty$$

- Probability of death in interval $(t, t + \Delta)$

$$\begin{aligned} \Pr(t < T < T + \Delta) &= S(t) - S(t + \Delta) \\ &= e^{-pt} - e^{-p(t+\Delta)} \\ &\approx (1 - pt) - (1 - p(t + \Delta)) \\ &= p\Delta \end{aligned}$$

where the $\approx$ uses $e^{-pt} \approx 1 - pt$ for pt small

- equivalent interpretation: death = Poisson process with rate p

Preliminaries: Perpetual Youth Model

- Preferences

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c(t))dt = \int_0^\infty e^{-\rho t} S(t) u(c(t))dt$$
$$= \int_0^\infty e^{-(\rho+p)t} u(c(t))dt$$

- As in discrete time: like infinite-horizon model with adjusted discount rate $\hat{\rho} = \rho + p$
- Mortality simply increases the discount rate
- Finally, as in discrete time, expected lifetime is

$$\int_0^\infty tpe^{-pt}dt = \frac{1}{p}$$

Preliminaries: Perpetual Youth Model

- Now think of an entire population of individuals subject to constant death rate p
- when an individual dies, one child is born, so that population is constant
- Q: what's the cross-sectional age distribution?
- A: it's also exponential

$$\Pr(\text{age} > x) = e^{-px}$$

- For future reference, denote corresponding density by

$$\pi(x) = \Pr(\text{age} = x) = pe^{-px}$$

Description of Economy

- **Preferences and Demographics:** continuum of individuals indexed by i
  - constant death rate p, i.e. "perpetual youth"
  - when an individual dies, one child is born
  - identical preferences

$$\int_0^\infty e^{-(\rho+p)t} u(c_i(t))dt, \quad u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$$

- **Technology:** representative firm with technology

$$Y = F(K,H), \quad \dot{K} = I - \delta K$$

- **Endowments:** individuals have one unit of time, initial wealth a_0
- Convention:
  - small letters = individual variables
  - capital letters = aggregate variables

Cohort Structure

- Economy has a non-trivial cohort structure/age distribution
- Need some notation to keep track of this:
  - t: calendar time
  - τ: date of birth, will use to index cohorts
  - x: age $= t - \tau$
  - e.g. in $t = 2010$, a member of the $\tau = 1980$ cohort is $x = 30$ years old
- Since all households within a cohort are identical, sufficient to keep track of cohort: "representative household of cohort τ"
- Notation: for any variable y, $y(t|\tau)$ means y at t for those born at τ
- Next slides: elements of CE with cohort structure
  - HH maximization, firm maximization, market clearing

HH problem

- The representative household of cohort τ takes $\{w(t), r(t)\}$ as given and solves

$$\max_{\{c(t|\tau)\}} \int_{\tau}^{\infty} e^{-(\rho+p)(t-\tau)} u(c(t|\tau))dt \quad \text{s.t.}$$

$$\dot{a}(t|\tau) = w(t) + r(t)a(t|\tau) - c(t|\tau) \tag{HH}$$

$$a(\tau|\tau) = a_0(\tau) > 0, \quad a(t|\tau) \geq 0$$

- again, $a(t|\tau)$ means wealth at t for those born at τ
- Note: borrowing constraint
    - cannot borrow at all, can only save
    - in general not innocuous, may want to explore other possibilities (not today)

Wealth at Death

- Because death is random, households will generally die with positive wealth: “accidental bequests”
  - would not happen with finite horizon T
  - households don’t like dying with positive wealth = wasted consumption
  - would like to insure against that, but cannot
- This is a “friction”: **incomplete insurance markets**
  - welfare theorems do not hold anymore
  - will encounter richer forms of **“market incompleteness”** later in semester/year: uninsurable income risk also during individual lifetimes (Aiyagari-Bewley-Huggett)
- Original Blanchard-Yaari model introduced form of insurance: “annuities”/“life insurance”
  - see Acemoglu, Ch.9.7 and 9.8
  - annuities $\Rightarrow$ wealth distribution less interesting

Wealth at Death

- Aggregate **flow** of accidental bequests in interval $(t, t + \Delta)$
$$p\Delta \int_0^\infty a(t|t-x)\pi(x)dx$$
where you should recall $\pi(x) = \Pr(\text{age} = x)$ and $x = t - \tau$
- Raises question: what happens to those accidental bequests?
- **Assumption:** they get redistributed **equally** to all newborns, fraction $\omega \geq 0$ may get wasted in bequest process
    - possible interpretation of equal redistribution: 100% estate tax
    - ω may capture lawyers (or government, church)
    - later will require $\omega > 0$, could be relaxed in richer model
    - assumptions somewhat extreme, mainly for simplicity
- in $(t, t + \Delta)$, $p\Delta$ people die, $p\Delta$ people born (total mass $= 1$)
$\Rightarrow p\Delta\times$ starting wealth $= p\Delta\times$ sum of accidental bequests
- Hence starting wealth of individuals born at date t is:
$$a_0(t) = (1-\omega)\int_0^\infty a(t|t-x)\pi(x)dx$$

Firm Problem

- Firms take $\{w(t), r(t)\}$ as given and solve

$$\max_{\{K(t),H(t)\}} \int_0^\infty e^{-\int_0^t r(s)ds}(F(K(t),H(t)) - w(t)H(t) - I(t))dt$$

$$\dot{K}(t) = I(t) - \delta K(t) \tag{F}$$

Equilibrium and Market Clearing

- **Definition**: A competitive equilibrium (SOMCE) for the growth model with heterogeneous agents are time paths for prices $\{w(t), r(t)\}$ and quantities $\{K(t), c(t|\tau), a(t|\tau)\}$, all t, τ such that

1. (HH max) Taking $\{w(t), r(t)\}$ as given and $a_0(\tau) = (1-\omega)K(\tau)$, individuals solve (HH)
2. (Firm max) Taking $\{w(t), r(t)\}$ as given, firms solve (F)
3. (Market clearing) For each t:

$$K(t) = \int_0^\infty a(t|t-x)\pi(x)dx \qquad (*)$$

where you should recall $\pi(x) = \Pr(\text{age} = x)$ and $x = t - \tau$

Characterizing CE

- Necessary conditions for individuals

$$
\begin{aligned}
\frac{\dot{c}(t|\tau)}{c(t|\tau)} &= \frac{1}{\sigma}(r - \rho - p) \\
\dot{a}(t|\tau) &= w(t) + r(t)a(t|\tau) - c(t|\tau) \\
0 &= \lim_{T \to \infty} a(T|\tau)c(T|\tau)^{-\sigma} \\
a(\tau|\tau) &= a_0(\tau) = (1 - \omega)K(\tau)
\end{aligned}
$$

- Necessary conditions for firms

$$
F_H(K(t), 1) = w(t), \quad F_K(K(t), 1) = r(t) + \delta
$$

- $+$ market clearing $(*)$

Comments

- $F_K(K,1) = r + \delta$ can be derived from (F) as follows
- Substitute $\dot{K}$ equation into constraint and use "integration by parts" (http://en.wikipedia.org/wiki/Integration_by_parts)

$$\int_0^\infty e^{-\int_0^t r(s)ds}\dot{K}(t)dt = K(t)e^{-\int_0^t r(s)ds}\Big|_0^\infty + \int_0^\infty e^{-\int_0^t r(s)ds} r(t)K(t)dt$$
$$= \int_0^\infty e^{-\int_0^t r(s)ds} r(t)K(t)dt$$

where need to impose a TVC. Hence (F) becomes

$$\max_{\{K(t),H(t)\}} \int_0^\infty e^{-\int_0^t r(s)ds}(F(K(t),H(t)) - w(t)H(t) - (r+\delta)K(t))dt$$

i.e. a sequence of static problems

- Or use Hamiltonian: $\mathcal{H} = F(K,H) - wH - I + \lambda(I - \delta K)$

$$\Rightarrow \quad \dot{\lambda} = r\lambda - F_K(K,H) + \lambda\delta, \quad \lambda = 1$$
$$\Rightarrow \quad \dot{\lambda} = 0 \quad \Rightarrow \quad F_K(K,H) = r + \delta$$

Comments

- Solving for a time-varying equilibrium is hard
- Challenges
  - have to keep track of evolution of entire distribution of $a(t|\tau)$
  - find time paths of prices $\{w(t), r(t)\}$ that solve $(*)$ for each t

Stationary Equilibrium

- **Definition**: a **stationary equilibrium** or steady state equilibrium for the growth model with heterogeneous agents are prices w^*, r^* and quantities $\{K^*, c(x), a(x)\}$, such that

1. (HH max) Taking w^*, r^* as given, individuals solve

$$\max_{\{c(x)\}} \int_0^\infty e^{-(\rho+p)x} u(c(x))dx \quad \text{s.t.}$$

$$\dot{a}(x) = w^* + r^* a(x) - c(x), \quad a(0) = (1-\omega)K^*, \quad a(x) \geq 0$$

2. (Firm max) Taking w^*, r^* as given, firms solve (F) with constant prices and hence

$$F_K(K^*, 1) = r^* + \delta, \quad F_H(K^*, 1) = w^*$$

3. (Market clearing)

$$K^* = \int_0^\infty a^*(x)\pi(x)dx$$

where you should recall $\pi(x) = \Pr(\text{age} = x)$ and $x = t - \tau$

Stationary Equilibrium: Comments

- In HHs' problem, keep track of age x rather than calendar time t and birth cohort τ, making use of fact that

$$a(t|t-x) = a(x), \quad \text{all } t$$

- Importantly: **aggregates constant** (just like in steady state in growth model)...
- but **rich dynamics at individual level**
  - individuals/cohorts "churning around" in stationary distribution
- Typically, no analytic solutions for stationary equilibrium
- $\Rightarrow$ solve for stationary equilibrium **numerically**
  - will learn how to do this later this semester/year for related models (Aiyagari-Bewley-Huggett)
  - challenge: have to find **stationary wealth distribution**
  - much easier than time-varying equilibrium because **prices w^*, r^* are just scalars**

Stationary Equilibrium

- Steps for solving stationary equilibrium
  1. given w^*, r^* solve individuals' problem
  2. compute stationary wealth distribution and aggregate supply of capital. Here simple: given solution to individuals' problem $a^*(x)$, aggregate capital supply is $K^{s*} = \int_0^\infty a^*(x)\pi(x)dx$
  3. given r^*, w^*, compute capital and labor demand satisfying

$$F_K(K^{d*}, H^{d*}) = r^* + \delta, \quad F_H(K^{d*}, H^{d*}) = w^*$$

  4. find r^*, w^* such that markets clear $K^{d*} = K^{s*}$, $H^{d*} = 1$

- Main difference from st. st. in rep. agent model: step (2)
- Steps above motivate computational algorithm: iterate on prices until markets clear
- Next: special case with analytic solution

Analytic Solution: Capitalists and Workers

- Consider again a model where we split households into "capitalists" and "workers" (as in Lecture 8)
- Further assume capitalists have log utility
  - later: generalize to CRRA utility
- Will show: stationary equilibrium can be solved analytically
  - stationary wealth distribution = Pareto distribution
  - based on Jones (2014) "The Macroeconomics of Piketty", see
    `http://web.stanford.edu/~chadj/piketty.pdf`
    `http://web.stanford.edu/~chadj/SimplePareto.pdf`

Capitalists and Workers

- **Preferences and Demographics:**
  - continuum of capitalists indexed by i
  - constant death rate p, i.e. "perpetual youth". When a capitalist dies, one "child capitalist" is born
  - logarithmic utility (CRRA with $\sigma = 1$)

$$\int_0^\infty e^{-(\rho+p)t} U(C_i(t))dt, \quad U(C) = \log C$$

  - representative worker

$$\int_0^\infty e^{-\rho t} u(c(t))dt$$

- **Technology:** representative firm with technology

$$Y = F(K, H), \quad \dot{K} = I - \delta K$$

- **Endowments:** workers have one unit of time, capitalists have initial wealth a_0

Capitalists' and workers' problem

- The representative capitalist of cohort τ takes $\{r(t)\}$ as given and solves

$$\max_{\{C(t|\tau)\}} \int_{\tau}^{\infty} e^{-(\rho+p)(t-\tau)} \log C(t|\tau) dt \quad \text{s.t.}$$

$$\dot{a}(t|\tau) = r(t)a(t|\tau) - C(t|\tau)$$

$$a(\tau|\tau) = a_0(\tau) > 0, \quad a(t|\tau) \geq 0$$

- The representative worker cannot save, simply supplies his entire labor endowment, consumes his income ("hand to mouth")

$$c(t) = w(t)$$

- Firms same as before
- Market clearing conditions also analogous

CE with Capitalists and Workers

- Necessary conditions for capitalists with log utility

$$\frac{\dot{C}(t|\tau)}{C(t|\tau)} = r(t) - \rho - p \tag{1}$$

$$\dot{a}(t|\tau) = r(t)a(t|\tau) - C(t|\tau) \tag{2}$$

$$0 = \lim_{T\to\infty} a(T|\tau)/C(T|\tau)$$

$$a(\tau|\tau) = a_0(\tau) = (1-\omega)K(\tau)$$

- Necessary conditions for workers

$$c(t) = w(t), \quad h(t) = 1$$

- Necessary conditions for firms

$$F_H(K,1) = w, \quad F_K(K,1) = r + \delta$$

- + market clearing

Log Utility for Capitalists

- **Lemma**: with log utility capitalists consume a constant fraction $(\rho + p)$ of their wealth

$$C(t|\tau) = (\rho + p)a(t|\tau)$$
$$\dot{a}(t|\tau) = (r - \rho - p)a(t|\tau)$$

- **Proof**: Guess and verify. Guess $C(t|\tau) = \theta a(t|\tau)$ for unknown θ. Substituting into (1)

$$\frac{\dot{C}(t|\tau)}{C(t|\tau)} = \frac{\dot{a}(t|\tau)}{a(t|\tau)} = r(t) - \rho - p$$

- From (2), then

$$C(t|\tau) = r(t)a(t|\tau) - \dot{a}(t|\tau) = (\rho + p)a(t|\tau). \square$$

Stationary Equilibrium

with capitalists and workers

- **Definition**: a **stationary equilibrium** or steady state equilibrium for the growth model with workers and heterogeneous capitalists are scalars w^*, r^* and quantities $\{K^*, c^*, C^*(x), a^*(x)\}$, such that

  1. capitalists maximize taking as given r^* and $a_0 = (1-\omega)K^*$
  2. workers maximize taking as given w^*
  3. firms maximize taking as given w^*, r^*
  4. markets clear

$$K^* = \int_0^\infty a^*(x)\pi(x)dx$$

Stationary Equilibrium

with capitalists and workers

- Follow similar steps as before:

1. given w^* solve workers' problem ($c^* = w^*$)
2. given r^* solve capitalists' problem
3. compute stationary wealth distribution and aggregate supply of capital. Here simple: given solution to individuals' problem $a^*(x)$, aggregate capital supply is $K^{s*} = \int_0^\infty a^*(x)\pi(x)dx$
4. given r^*, w^*, compute capital and labor demand satisfying
$$F_K(K^{d*}, H^{d*}) = r^* + \delta, \quad F_H(K^{d*}, H^{d*}) = w^*$$
5. find r^*, w^* such that markets clear $K^{d*} = K^{s*}$, $H^{d*} = 1$

Aggregate Capital Supply

with capitalists and workers

- With log utility

$$\dot{a}^*(x) = (r^* - \rho - p)a^*(x), \quad a^*(0) = a_0$$
$$a^*(x) = e^{(r^* - \rho - p)x} a_0$$

- Recall stationary age distribution $\pi(x) = pe^{-px}$
- Hence stationary capital supply is

$$\begin{aligned} K^{s*} &= \int_0^\infty a^*(x)\pi(x)dx \\ &= \int_0^\infty e^{(r^* - \rho - p)x} a_0 p e^{-px} dx \\ &= \frac{p}{2p + \rho - r^*} a_0 \end{aligned}$$

- for given a_0, stationary capital supply increasing in r^*

Aggregate Capital Supply

with capitalists and workers

- But recall a_0 is also endogenous

$$a_0 = (1-\omega)K^*$$

- Hence

$$K^{s*} = \frac{p}{2p+\rho-r^*}(1-\omega)K^{s*}$$

- Hence stationary capital supply is infinitely elastic at interest rate

$$r^* = \rho + p(1+\omega)$$

- Steady state capital stock can be found from demand

$$F_K(K^*,1) = \rho + p(1+\omega) + \delta$$

- Given K^*, w^*, c^*, C^* can be found as usual
- This concludes characterization of aggregates

Steady State Wealth Distribution

- But we're also interested in **distribution** of income and wealth
- For wealth distribution, proceed in two steps
  - wealth distribution in **partial equilibrium** (i.e. given fixed r^*)
  - wealth distribution in **general equilibrium** (i.e. at eq. r^*)

Steady State Wealth Distribution

in partial equilibrium

- Recall $a^*(x) = e^{(r^*-\rho-p)x}a_0$
- Convenient property: wealth and age are perfectly correlated and know age distribution

$$\Pr(\text{age} > x) = e^{-px}$$

- Define "age it takes to reach wealth level a" (inverse of $a^*(x)$)

$$x(a) = \frac{1}{r^* - \rho - p}\log\left(\frac{a}{a_0}\right)$$

- Therefore wealth distribution is

$$\begin{aligned}\Pr(\text{wealth} > a) &= \Pr(\text{age} > x(a)) = e^{-px(a)} \\ &= e^{-\frac{p}{r^*-\rho-p}\log(a/a_0)} \\ &= \left(\frac{a}{a_0}\right)^{-\zeta}, \quad \zeta = \frac{p}{r^* - \rho - p}\end{aligned}$$

- Formula for Pareto distribution!

Steady State Wealth Distribution

in partial equilibrium

- Wealth distribution is Pareto with CDF

$$G(a) = \Pr(\text{wealth} \leq a) = 1 - \left(\frac{a}{a_0}\right)^{-\zeta}, \quad \zeta = \frac{p}{r^* - \rho - p}$$

- Corresponding density is $g(a) = \zeta a_0^{\zeta} a^{-\zeta-1}$
- Wealth inequality governed by

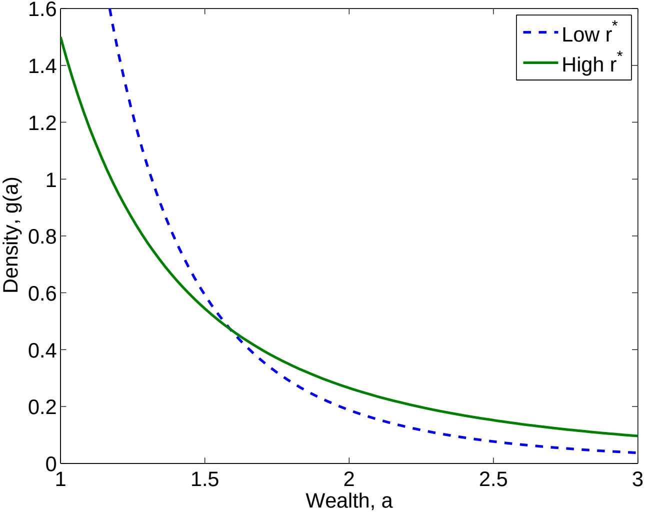
$$\eta = \frac{1}{\zeta} = \frac{r^* - \rho - p}{p}$$

- Wealth inequality is
    - increasing in r^* (in extension with growth this would be $r^* - g$, i.e. Piketty's "fundamental law of capitalism")
    - decreasing in p and ρ
- Aside: can check that get same formula for K^* as before

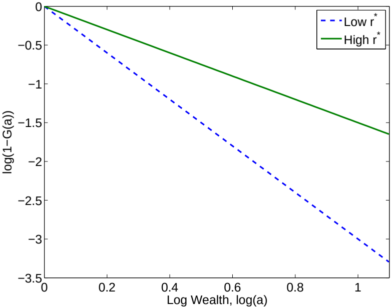
$$K^{s*} = \int_{a_0}^{\infty} a g(a) da = \frac{\zeta}{\zeta - 1} a_0 = \frac{p}{2p + \rho - r^*} a_0$$

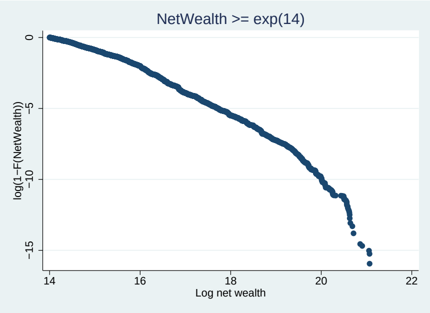
Steady State Wealth Distribution

in partial equilibrium



Steady State Wealth Distribution
in partial equilibrium





Steady State Wealth Distribution

General equilibrium

- Recall steady state interest rate $r^* = \rho + p(1+\omega)$
- In general equilibrium, wealth distribution is Pareto with

$$\eta = \frac{1}{\zeta} = \frac{r^* - \rho - p}{p} = \frac{\omega}{p}$$

- **Assumption:** $\omega > 0$, i.e. some waste in bequest process
  - if $\omega = 0$, r^* would adjust so that $K^* = a_0$, i.e. no wealth growth at individual level and perfect equality $\omega = 0$
  - Jones (2014) considers model with population growth in which case can set $\omega = 0$
  - intuitively, distribute accidental bequests over growing population $\Rightarrow$ looks just like a fraction of wealth is lost
- In general equilibrium, wealth distribution completely pinned down from demographics/waste. Return to this momentarily.

Income Inequality

- In this model, two types of income inequality
  - between groups: capitalists vs. workers
  - within groups: inequality among capitalists
- **Between-group inequality**
  - worker income $w^* = F_H(K^*, 1)$
  - average income of capitalists $r^* K^* = F_K(K^*, 1) K^*$
  - note: $w^* + r^* K^* = Y^* - \delta K^*$
  - analogues in data: capital and labor shares $RK/Y, wH/Y$
  - "capitalists vs. workers" view of the world is big part of the reason why people care so much about declining labor share
- **Within-group inequality**
  - all workers identical
  - distribution of capital income $r^* a$ inherits Pareto shape of wealth distribution (same tail exponent)

Some Interesting Extensions

- Jones (2014) adds
  - wealth taxation at rate τ, after tax return is $r^* - \tau$ (capital taxation $r^*(1-\tau)$ is isomorphic)
  - population growth at rate n
  - AK production $\Rightarrow$ economy grows at rate g
  - more general consumption rules $c = \alpha a$, e.g. can show that with CRRA utility for capitalists

$$\alpha = \frac{\rho + p - (1-\sigma)(r^* - \tau)}{\sigma}$$

Some Interesting Extensions

- Law of motion for individual wealth accumulation is

$$\dot{a}(x) = (r^* - \tau - \alpha)a(x)$$

- Following similar steps as above, can derive formulas for
  - wealth inequality in partial equilibrium

$$\eta = \frac{1}{\zeta} = \frac{r^* - \tau - g - \alpha}{n + p}$$
$$= \frac{r^* - \tau - g - \rho - p}{\sigma(n + p)} \quad \text{in CRRA case}$$

  - equilibrium interest rate

$$r^* = n + g + \tau + \alpha$$

  - wealth inequality in general equilibrium

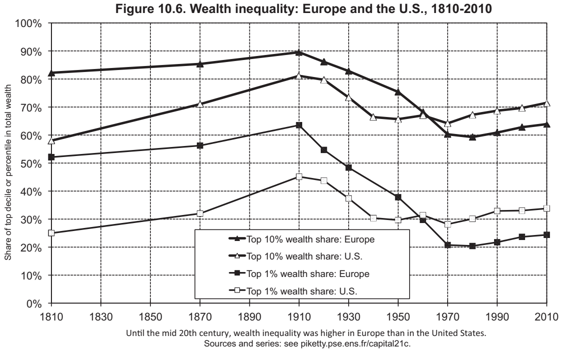
$$\eta = \frac{1}{\zeta} = \frac{n}{n + p}$$

Observations

- In **partial equilibrium**, wealth inequality $1/\zeta$ is increasing in gap between after-tax rate of return capital $r^* - \tau$ and growth rate g
- However, in **general equilibrium**, wealth distribution completely pinned down from demographics, **independent** of wealth tax rate τ
  - result relies heavily on perfectly elastic supply of capital
  - not true in richer models
  - but interesting extreme case

Observations

- To think about graphs like the one below, need to analyze model's **transition dynamics**, steady states not enough
- This is much harder...

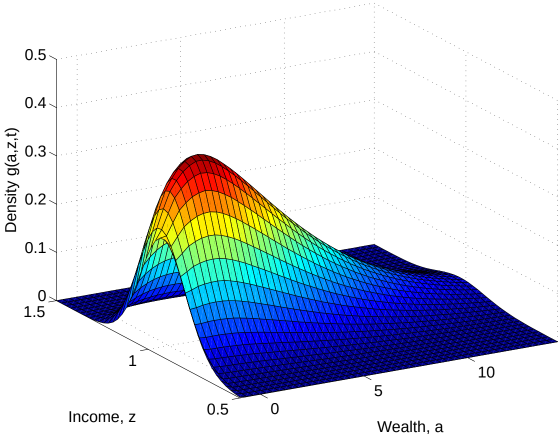


Some Concluding Observations

- Analytic solutions are a bit of an "analytic straitjacket"
- Changing any of the assumptions $\Rightarrow$ analytic solution out the window
  - effects of **progressive** rather than linear taxation?
  - transition dynamics for arbitrary initial wealth distribution?
  - other sources of randomness than death, e.g. labor income risk (breaks perfect correlation between wealth and age)?
- Need more general methods for solving heterogeneous agent models
  - computational methods for handling problems with realistic heterogeneity
  - second half of semester or next semester

Some Concluding Observations

- Example of computational solution of richer model



Some Concluding Observations

- Example: effect of one-time redistribution of wealth