

Lecture 10: Endogenous Growth

ECO 503: Macroeconomic Theory I

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Plan of Lecture

- **Last time:** sustained long-run growth with **exogenous** productivity growth
- **This time:** sustained long-run growth from purely **endogenous** sources

Plan of Lecture

- ① Simplest possible endogenous growth model: AK model
- ② Endogenous growth from human capital accumulation:
Lucas (1988), “On the Mechanics of Economic Development”
- ③ If time (i.e. probably not):
Romer (1990), “Endogenous Technological Change”

AK model

- Originally due to Romer (1986), Rebelo (1991), also see Acemoglu Ch. 11
- Consider standard growth model

- **Preferences:**

$$\int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma}$$

- **Technology:**

$$y = F(k, h), \quad c + i = y, \quad \dot{k} = i - \delta k$$

- **Endowments:** one units of time, $k(0) = k_0$.
- Add one little twist: “AK production function”

$$F(k, h) = Ak$$

- e.g. Cobb-Douglas $y = Ak^{\alpha} h^{1-\alpha}$ with $\alpha = 1$.
- Note: usual Inada condition $\lim_{k \rightarrow \infty} f'(k) = 0$ is no longer satisfied. Instead

AK model

- Usual definition of CE
- Welfare theorems hold \Rightarrow solve as SP's problem

$$\max_{\{c(t)\}} \int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} \quad \text{s.t.}$$

$$\dot{k}(t) = Ak(t) - \delta k(t) - c(t), \quad k(0) = k_0$$

- See e.g. Acemoglu, Chapter 11 for direct analysis of CE

Necessary Conditions

- Solve using Hamiltonian: see Lecture 4
- Necessary conditions

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - A \quad (\text{EE})$$

$$\dot{k} = Ak - \delta k - c \quad (\text{R})$$

$$0 = \lim_{T \rightarrow \infty} e^{-\rho T} k(T) \lambda(T) \quad (\text{TC})$$

$$k(0) = k_0$$

- From (EE) $\lambda(t) = e^{(\rho - \delta - A)t} \lambda(0)$ and hence (TC) is

$$0 = \lim_{T \rightarrow \infty} k(T) e^{-(A - \delta)T}$$

Necessary Conditions

- Alternatively, write in terms of consumption

$$\frac{\dot{c}}{c} = \frac{1}{\sigma} (A - \rho - \delta) \quad (\text{EE}')$$

$$\dot{k} = Ak - \delta k - c \quad (\text{R}')$$

$$0 = \lim_{T \rightarrow \infty} k(T) e^{-(A-\delta)T} \quad (\text{TC}')$$

$$k(0) = k_0$$

- As before, a **balanced growth path (BGP)** is a solution to the SP's problem such that all quantities grow at constant rates

Balanced Growth Path

- As before, let's look for a BGP, i.e. a $k(0)$ such that $\dot{k}(t)/k(t) = g, t \geq 0$
- From (R')

$$g = \frac{\dot{k}}{k} = A - \delta - \frac{c}{k}$$

- \Rightarrow on BGP, need $c/k = \text{constant} \Rightarrow c, k$ grow at same rate

$$\Rightarrow g = \frac{1}{\sigma}(A - \rho - \delta)$$

$$\frac{c(t)}{k(t)} = A - \delta - \frac{1}{\sigma}(A - \rho - \delta), \quad \forall t$$

- **Assumption 1:** $A - \delta > \rho$
 - ensures $g > 0$
 - also ensures $A > \delta \Rightarrow$ (TC') is satisfied
- **Assumption 2:** $(1 - \sigma)(A - \delta) < \rho$
 - ensures $c(t)/k(t) > 0$

Balanced Growth Path

- Q: For which initial conditions k_0 is there a BGP?
- A: There is a BGP for **any** $k(0)$. Proof:

- take an arbitrary k_0
- set

$$c(0) = \left(A - \delta - \frac{1}{\sigma}(A - \rho - \delta) \right) k_0 \quad (*)$$

- $c(t) = e^{gt} c(0), k(t) = e^{gt} k(0)$ satisfies necessary condition
- **Summary:** Under Assumptions 1 and 2, there is a **continuum** of BGPs on which $c(t), y(t), k(t)$ grow at rate

$$g = \frac{1}{\sigma}(A - \rho - \delta)$$

- continuum = one BGP for each k_0
- $c(0)$ pinned down by (*)
- importantly, in the AK model there are **no transition dynamics**. Economy grows at constant rate for any $k(0)$

Takeaway from AK model

- With a linear production function $y = F(k, h) = Ak$, standard growth model features **endogenous growth**
 - no need for exogenous growth in A
 - g affected by model parameters $(\sigma, A, \rho, \delta)$
- Important take-away: **linearity**
 - to get endogenous growth, (almost) always need to make some sort of linearity assumption
 - i.e. there is some linear equation $\dot{X} = gX$
 - if it doesn't jump out at you, it's hidden somewhere

Lucas (1988): “On the Mechanics of Economic Development”

- One of most cited papers in economics

<https://ideas.repec.org/top/top.item.nbcites.html>

- Famous quote (p.5):
- *“I do not see how one can look at figures like these without seeing them representing possibilities. Is there some action a government of India could take that would lead the Indian economy to grow like Indonesia’s or Egypt’s? If so, what exactly? If not, what is it about the “nature of India” that makes it so? The consequences for human welfare involved in questions like these are simply staggering: **once one starts to think about them, it is hard to think about anything else.**”*

Plan for Lucas (1988)

- (1) Model with physical, human capital
- (2) Reduce to one state variable: human capital
- (3) External production effects
- (4) External learning effects

Growth Model with Human Capital

- Standard growth with growing human capital

- **Preferences:**

$$\int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma}$$

- **Technology:**

$$y = F(k, h), \quad c + i = y, \quad \dot{k} = i - \delta k$$
$$\frac{\dot{h}}{h} = \kappa$$

- **Endowments:** $k(0) = k_0$
- In contrast to previous lecture, h grows directly, rather than $y = F(k, Ah)$ with A growing
 - mathematically equivalent
 - somewhat different economics

Growth Model with Human Capital

- As in previous lecture, work with **detrended** variables

$$\tilde{y} = ye^{-\kappa t}, \quad \tilde{k} = ke^{-\kappa t}$$

- Technology summarized by

$$\begin{aligned}\tilde{c} + \dot{\tilde{k}} &= f(\tilde{k}) - (\delta + \kappa)\tilde{k} \\ \dot{h} &= h\kappa\end{aligned}$$

where $f(k) = F(k, 1)$

Growth Model with Human Capital

- Solve as planner's problem (dropping \sim 's for simplicity)

$$\max_{\{c(t)\}} \int_0^{\infty} e^{-\tilde{\rho}t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \quad \text{s.t.}$$

$$\dot{k}(t) = f(k(t)) - (\delta + \kappa)k(t) - c(t)$$

with $\tilde{\rho} = \rho + g(\sigma - 1)$

- Model has BGP on which all variables grow at rate κ
 - unique and globally stable from any k_0

(1) Both Physical and Human Capital

- Modify to make growth of h depend on the allocation of time
- Have one unit of time: split between production and human capital accumulation.

- **Preferences:**

$$\int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma}$$

- **Technology:**

$$y = F(k, uh), \quad c + i = y, \quad \dot{k} = i - \delta k$$
$$\frac{\dot{h}}{h} = g(1 - u)$$

where $g(\cdot)$ is an increasing function

- **Endowments:** one unit of time, $k(0) = k_0$
- Controls are pair (u, i) . State is (h, k) .

Both Physical and Human Capital

- For best complete development, see
 - Caballe and Santos, JPE, 1993

For earlier versions, see

- Uzawa, 1965, IER
- Razin, 1972, REStud
- Lucas, 1988, JME
- Rebelo, 1990, JPE

(2) Model with Human Capital Only

- Interesting, but complicated literature.
- And want to add further complication of external effects.
- Reduce to one state variable by dropping physical capital.
- An “ Ak ” model – for us, “ Ah ”

Model with Human Capital Only

- Retain CRRA preferences

$$\int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

- Technology

$$c = hu$$

$$\dot{h} = hg(1 - u)$$

Model with Human Capital Only

- Hamiltonian and FOCs

$$\mathcal{H}(h, \mu, u) = \frac{(hu)^{1-\sigma}}{1-\sigma} + \mu hg(1-u)$$

$$\begin{aligned}\dot{\mu} &= \rho\mu - \frac{\partial \mathcal{H}}{\partial h} \\ &= \rho\mu - h^{-\sigma} u^{1-\sigma} - \mu g(1-u)\end{aligned}$$

$$h^{1-\sigma} u^{-\sigma} = \mu hg'(1-u)$$

- This model does not have a st.st.

Balanced Growth Path (BGP)

- Seek solution with u constant, independent, of h . If so

$$\frac{\dot{h}}{h} = g(1 - u) = \kappa, \text{ say,}$$

and

$$\frac{\dot{\mu}}{\mu} = \rho - ug'(1 - u) - \kappa$$

- FOC for u then implies

$$h^{-\sigma} u^{-\sigma} = \mu g'(1 - u)$$

and since u is constant

$$-\sigma \frac{\dot{h}}{h} = \frac{\dot{\mu}}{\mu}$$

- Then optimal u satisfies

$$-\sigma\kappa = \rho - ug'(1-u) - \kappa$$

$$ug'(1-u) = \rho - (1-\sigma)\kappa$$

- **Example:** Log utility, $g(1-u) = \delta(1-u)$ where δ is a constant.

$$u = \frac{\rho}{\delta}, \quad \kappa = \delta \left(1 - \frac{\rho}{\delta}\right)$$

- **Example:** Log utility, $g(1-u) = \delta(1-u)^\gamma$
- Unique solution $u \in (0, 1)$ as long as $\gamma < 1$.

(3),(4) Externalities

- Many people argue: important feature of human capital = externalities
- Here: two versions
 - external production effect
 - external learning effect
- Also **useful tool**: solving models with externalities
 - externalities \Rightarrow welfare theorems fail
 - here: trick so that can still solve CE with externalities as planning problem
 - “ (k, K) models”. Here: “ (h, H) model”
 - same trick also applies in other cases, e.g. models with externalities

(3) An External Production Effect

- Let h be human capital of individual producer, and H be everyone else's
- Individual's technology is

$$c = uhH^\xi$$

- (h, H) strategy for solving CE with externalities as planner's problem
 - ① solve problem (take FOCs) **taking as given** H
 - ② impose that in equilibrium $h = H$
- In contrast, problem of planner **internalizing** effect on H
 - ① impose $h = H$
 - ② solve problem (take FOCs)
- Order is crucial

(3) An External Production Effect

- Step 1 of (h, H) solution strategy

$$\begin{aligned} \max_{\{u(t)\}} \int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \quad \text{s.t.} \\ c = uhH^{\xi} \\ \dot{h} = hg(1-u) \end{aligned}$$

taking as given time path for $H(t)$, $t \geq 0$

- Hamiltonian for individual is

$$\mathcal{H}(h, H, \mu, u) = \frac{(hH^{\xi}u)^{1-\sigma}}{1-\sigma} + \mu hg(1-u)$$

An External Production Effect

- FOCs are

$$\dot{\mu} = \rho\mu - H^{\xi(1-\sigma)} h^{-\sigma} u^{1-\sigma} - \mu g(1-u)$$

$$(hH^{\xi})^{1-\sigma} u^{-\sigma} = \mu hg'(1-u)$$

- Step 2: in equilibrium, $H = h$

$$\dot{\mu} = \rho\mu - h^{\xi(1-\sigma)-\sigma} u^{1-\sigma} - \mu g(1-u)$$

$$h^{(1+\xi)(1-\sigma)} u^{-\sigma} = \mu hg'(1-u)$$

- Instead for central planner (first set $H = h$, then maximize, i.e. take FOCs)

$$\dot{\mu} = \rho\mu - (1 + \xi) h^{\xi(1-\sigma)-\sigma} u^{1-\sigma} - \mu g(1-u)$$

- For equilibrium (planner too) seek solution with u constant,
 $g(1 - u) = \kappa$

$$\begin{aligned}\frac{\dot{\mu}}{\mu} &= \rho - \kappa - \frac{1}{\mu} h^{\xi(1-\sigma)-\sigma} u^{1-\sigma} \\ &= \rho - \kappa - u g'(1 - u)\end{aligned}$$

- FOC and constant u imply

$$\frac{\dot{\mu}}{\mu} = (\xi - \xi\sigma - \sigma)\kappa$$

- Equilibrium u solves

$$\rho - \kappa - u g'(1 - u) = (\xi - \xi\sigma - \sigma)\kappa$$

- Log utility case

$$\rho - ug'(1 - u) = 0$$

- Contrast to planner (with log utility)

$$\rho - (1 + \xi)ug'(1 - u) = 0$$

- Planner puts higher value on human capital, invests more

- Log utility, linear g : $g(1 - u) = \delta(1 - u)$:
- Equilibrium

$$u = \frac{\rho}{\delta}, \quad \kappa = \delta \left(1 - \frac{\rho}{\delta}\right)$$

- Planner

$$u = \frac{\rho}{\delta(1 + \xi)}, \quad \kappa = \delta \left(1 - \frac{\rho}{\delta(1 + \xi)}\right)$$

Cross-country income differences?

- Differences in initial, individual h persist forever
- Virtue? Defect? Depends on view on convergence/divergence.
- Migration incentives? Whatever is my h , I am more productive in economy where H is highest.

(4) An External Learning Effect

- no external effect in production

$$c = hu$$

- Instead, external effect appears in the law of motion for h

$$\dot{h} = h^{1-\theta} H^\theta g(1-u)$$

- Can work this out as before, results similar.

Models of Long Run Growth

Always ask yourselves: What $\dot{X} = g X$ equation is the "engine" of growth?

$$Y(t) = K(t)^\alpha [A(t) h(t) u L(t)]^{1-\alpha}$$

- Solow: $\alpha < 1$, $u = 1$, exogenous $\dot{A} = g A$ (also growth model with growth)
- Rebelo: $\alpha = 1$, $u = 1$, $\dot{K} = s A K - \delta K$, endogenous $s = I/Y$
- Lucas: $\dot{h} = g(1 - u) h$, endogenous $0 < u < 1$
- Romer: $\dot{A} = (1 - u) h L A$, endogenous $0 < u < 1$

Endog. Growth with R&D: Romer (1990)

- Nice discussion: “ideas” are non-rival \Rightarrow cannot support equilibrium with price-taking behavior.
- Output: $F(A, X)$ where X : rival inputs, A : non-rival inputs
- By replication $F(A, \lambda X) = \lambda F(A, X)$

$$\Rightarrow F(\lambda A, \lambda X) > \lambda F(A, X)$$

- Similarly, $F(A, X) = X \cdot \frac{\partial F(A, X)}{\partial X}$

$$\Rightarrow F(A, X) < A \cdot \frac{\partial F(A, X)}{\partial A} + X \cdot \frac{\partial F(A, X)}{\partial X}$$

- Implication: if all inputs were paid their value marginal products, the firm would suffer losses \Rightarrow Cannot survive as a price-taker!
- Romer's solution: monopolistic competition.

Model

- Constant population, no capital
- Intermediate goods $x(i)$, $0 \leq i \leq h$
- Consumers maximize

$$\int_0^{\infty} e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} dt,$$

where

$$c = \left(\int_0^h x(i)^\nu di \right)^{1/\nu}, \quad 0 < \nu < 1$$

- Labor endowment: one unit
- One unit of labor needed to produce each unit of $x(i)$, all i .
- Total labor used to produce intermediate goods: u
- Labor used to produce new ideas – R&D : $1 - u$

$$\frac{dh}{dt} = \delta h(1 - u)$$

- Each new idea – each $i > h$ – yields a permanent, exclusive right to produce the intermediate good i . Each old idea – $i \leq h$ – produced by a monopolist

- Strategy: We're headed for an Ak -type equilibrium.
- Let's take knowledge level h and labor allocation u as given.
- Solve for static equilibrium quantities $x(i)$, goods prices $p(i)$, wage rate w , profit per firm π
- Then we'll look at the dynamics

- Given total goods spending M and prices $p(i)$, consumers solve

$$\max_x \left(\int_0^h x(i)^\nu di \right)^{1/\nu}, \quad 0 < \nu < 1$$

subject to

$$\int_0^h p(i)x(i)di \leq M.$$

- Demand functions are

$$x(i) = \left(\frac{p(i)}{P} \right)^{-\frac{1}{1-\nu}} \frac{M}{P}, \quad P \equiv \left(\int_0^h p(i)^{\frac{\nu}{\nu-1}} di \right)^{\frac{\nu-1}{\nu}}$$

- Derivation: slide 31 of

http://www.crei.cat/people/gali/pdf_files/monograph/slides-ch3.pdf

- Write demand functions and revenues as

$$x(i) = Ap(i)^{-\frac{1}{1-\nu}}, \quad A \equiv P^{\frac{1}{1-\nu}} \frac{M}{P}$$

$$p(i)x(i) = A^{1-\nu} x(i)^\nu$$

- Each intermediate good monopolist solves

$$\max_{x(i)} [A^{1-\nu} x(i)^\nu - wx(i)]$$

- Follows that everyone produces same output x

$$x(i) = x = A \left(\frac{\nu}{w} \right)^{\frac{1}{1-\nu}}$$

- Prices are constant markup $1/\nu$ over marginal costs:

$$p(i) = p = \frac{w}{\nu}$$

- Normalize prices by setting total spending on goods M equal to one
- Then the constant A satisfies

$$\begin{aligned} A &= \left(\int_0^h p(i)^{\nu/(\nu-1)} di \right)^{-1} M \\ &= \left(\int_0^h p(i)^{\nu/(\nu-1)} di \right)^{-1} \\ &= \left(h \left(\frac{\nu}{w} \right)^{\frac{\nu}{1-\nu}} \right)^{-1} \end{aligned}$$

- Total units of goods produced equals labor in goods-producing sector

$$\begin{aligned}u &= hx \\&= hA \left(\frac{\nu}{w} \right)^{\frac{1}{1-\nu}} \\&= h \left(h \left(\frac{\nu}{w} \right)^{\frac{\nu}{1-\nu}} \right)^{-1} \left(\frac{\nu}{w} \right)^{\frac{1}{1-\nu}} \\&= \frac{\nu}{w}\end{aligned}$$

- Profit per firm is

$$\begin{aligned}\pi &= A \left(\frac{w}{\nu}\right)^{\nu/(\nu-1)} - wA \left(\frac{w}{\nu}\right)^{1/(\nu-1)} \\ &= (1 - \nu)A \left(\frac{w}{\nu}\right)^{\nu/(\nu-1)} \\ &= (1 - \nu) \left(h \left(\frac{w}{\nu}\right)^{\nu/(\nu-1)}\right)^{-1} \left(\frac{w}{\nu}\right)^{\nu/(\nu-1)} \\ &= \frac{1 - \nu}{h}\end{aligned}$$

- Sum up: Given h and u , all prices, quantities solved for.

- Now determine labor allocation u and thus

$$\frac{dh}{dt} = \delta h(1 - u)$$

- Suppose at date t person withdraw one unit of labor for period $(t, t + \varepsilon)$ and devotes this time to R&D.
- Gives up $w\varepsilon$ in wage income. Discovers $\delta h\varepsilon$ new products.
- Each product is patented and yields an infinite stream of π profits

$$\int_0^{\infty} \pi e^{-rt} dt = \frac{\pi}{r}$$

- Value of each patent is π/r , so in equilibrium

$$\begin{aligned}w &= \delta h \frac{\pi}{r} \\ &= \frac{\delta h}{r} \frac{1 - \nu}{h} \quad (\text{use } \pi \text{ from slide 38}) \\ &= \frac{\delta}{r} (1 - \nu) \\ \frac{\nu}{u} &= \frac{\delta}{r} (1 - \nu) \quad (\text{use } w \text{ from slide 37})\end{aligned}$$

- Solve for labor allocation

$$u = \frac{\nu}{1 - \nu} \frac{r}{\delta}$$

- Last step is to find interest rate r . Euler equation:

$$r = \rho + \sigma \frac{\dot{c}}{c}$$

- From c on slide 29, have

$$c = h^{1/\nu} x, \quad \text{with } x \text{ constant} \quad \Rightarrow \quad \frac{\dot{c}}{c} = \frac{1}{\nu} \frac{\dot{h}}{h}$$

- Goal: find $\kappa = \dot{h}/h$. Interest rate is

$$r = \rho + \frac{\sigma}{\nu} \kappa \tag{*}$$

- And growth rate is

$$\kappa = \delta(1 - u) = \delta \left(1 - \frac{\nu}{1 - \nu} \frac{r}{\delta} \right) \tag{**}$$

- Solve (*) and (**) for

$$\kappa = \frac{\delta(1 - \nu) - \nu\rho}{1 - \nu + \sigma}$$

- For log utility and linear g , compare

$$\kappa = \frac{\delta(1 - \nu) - \nu\rho}{2 - \nu}$$

- to economy with production externality (first case in Lucas (1988) notes)

$$\kappa = \delta \left(1 - \frac{\rho}{\delta} \right)$$

-and economy with learning externality (second case in Lucas (1988) notes)

$$\kappa = \frac{\delta - \rho}{1 + \theta}$$