Liquid and Illiquid Assets with Fixed Adjustment Costs

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This note describes a continuous-time version of the model in Kaplan and Violante (2014) in which a household optimally splits his wealth between liquid and illiquid assets. Because adjustments are subject to a fixed cost, the model has a stopping-time element. We here describe how such model can be solved numerically using a finite difference method. Based on ongoing research (Kaplan and Moll, 201?). Note that the problem is different from that in Kaplan, Moll and Violante (2018). The problem presented here features a non-convex adjustment cost whereas the problem in Kaplan, Moll and Violante (2018) features a kinked but strictly convex adjustment cost. Both types of cost functions give rise to an inaction region. However, the mathematical structure is fundamentally different: a non-convex adjustment cost results in a stopping-time or impulse-control problem whereas a strictly convex cost function does not.

These impulse control problems can be formulated as so-called Hamilton-Jacobi-Bellman Variational Inequalities (HJBVIs) or Hamilton-Jacobi-Bellman Quasi-Variational Inequalities (HJBQVIs). See Bensoussan and Lions (1982, 1984) and Bardi and Capuzzo-Dolcetta (1997). More recently, Bertucci (2017, 2018) analyzes Mean Field Games with stopping and impulse control, with the prototypical problem featuring a coupled system of an HJBVI or HJBQVI for agents' problems and the corresponding variant of a Kolmogorov Forward equation for the evolution of the distribution of agents.

1 Model Setup

Households can invest in two assets: an illiquid asset a, and a liquid asset b. The liquid asset pays a real return r_b and can be freely traded subject to a borrowing limit. The illiquid asset pays a return $r_a > r_b$. Deposits and withdrawals can be made into and out of the illiquid asset only upon payment of a transaction cost κ. The households receive an income flow z where z can take a finite number of values. Productivity shocks arrive according to a stochastic process which is either Poisson or a diffusion process.

[1]We thank Alberto Polo for help with cleaning up this document and improving the corresponding code.

1.1 Households' Problem

Households solve the following problem

$$
\begin{aligned}
v_k(a,b) &= \max_{\{c_t\},\tau} \ \mathbb{E}_0 \int_0^\tau e^{-\rho t} u(c_t)dt + e^{-\rho\tau}\mathbb{E}_0 v_k^*(a_\tau + b_\tau) \\
\dot{a}_t &= r_a a_t, \quad \dot{b}_t = r_b b_t + w z_t - c_t \\
a_t &\geq 0, \quad b_t \geq 0, \quad (a_0, b_0, z_0) = (a, b, z_k)
\end{aligned} \tag{1}
$$

where $z_t \in \{z_1, z_2\}$ is a two-state Poisson process with intensities λ_1, λ_2, and where

$$
v_k^*(a+b) = \max_{a',b'} \ v_k(a',b') \quad \text{s.t.} \quad a' + b' = a + b - \kappa, \quad a' \geq 0, \ b' \geq 0. \tag{2}
$$

For future reference, denote the optimal adjustment decisions conditional on adjustment by $a_k^*(a,b)$ and $b_k^*(a,b)$. Note that these only depend on the total amount of assets $a+b$ (rather than a and b separately). The HJB equation is

$$
\rho v_k(a,b) = \max_c \ u(c) + \partial_a v_k(a,b) r_a a + \partial_b v_k(a,b)(w z_k + r_b b - c) + \lambda_k (v_{-k}(a,b) - v_k(a,b)) \tag{3}
$$

for $k = 1, 2$, and with state-constraint boundary condition

$$
\partial_b v_k(a,0) \geq u'(w z_k) \tag{4}
$$

and a constraint that

$$
v_k(a,b) \geq v_k^*(a+b) \quad \text{all } a, b. \tag{5}
$$

This can also be written compactly as follows

$$
\begin{aligned}
&\min\{\rho v_k(a,b) - \max_c \ u(c) - \partial_a v_k(a,b) r_a a - \partial_b v_k(a,b)(w z_k + r_b b - c) - \lambda_k (v_{-k}(a,b) - v_k(a,b)), \\
&\quad v_k(a,b) - v_k^*(a+b)\} = 0
\end{aligned} \tag{6}
$$

In mathematics, (6) is called an "HJB variational inequality" (HJBVI for short), or more precisely an "HJB quasi-variational inequality" (HJBQVI). See e.g. Bensoussan and Lions (1982, 1984), Barles, Daher and Romano (1995), Bardi and Capuzzo-Dolcetta (1997) and Tourin (2013).

Note that we can also write the adjustment value v_k^* as $v_k^* = \mathcal{M} v_k$ where $\mathcal{M}$ is known as the "intervention operator" (see e.g. Oksendal and Sulem, 2002; Azimzadeh, Bayraktar and

Labahn, 2018) so that (suppressing dependence on (a,b)) the HJBQVI equation becomes

$$\min\{\rho v_k - \max_c \; u(c) - r_a a \partial_a v_k - (wz_k + r_b b - c)\partial_b v_k - \lambda_k (v_{-k} - v_k), v_k - \mathcal{M} v_k\} = 0$$

2 Numerical Solution

See `http://www.princeton.edu/~moll/HACTproject/liquid_illiquid_LCP.m`.

2.1 Household's Problem: Linear Complementaritsy Problem + Finite Differences

We use an analogous approach to the simple model of exercising an option as in these notes `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf`. Once discretized the HJBVI (6) becomes

$$\min\{\rho v - u(v) - \mathbf{A}(v)v, v - v^*(v)\} = 0 \tag{7}$$

The main difference to the "exercising an option" problem laid out above is that the HJB equation without adjustment (the left branch of (7)) is non-linear in v and hence some iteration is necessary. Related also the value of having a reoptimized portfolio v^* depends on the value function as can be seen from (2). We proceed as follow:

1. as an initial guess v^0 use the solution to

$$\rho v - u(v) - \mathbf{A}(v)v = 0 \tag{8}$$

   i.e. the problem without adjustment, i.e. in which the fixed cost κ is infinite.

2. Given v^n, find v^{n+1} by solving

$$\min\left\{\frac{v^{n+1} - v^n}{\Delta} + \rho v^{n+1} - u(v^n) - \mathbf{A}(v^n)v^{n+1}, v^{n+1} - v^*(v^n)\right\} = 0$$

   Exactly as in `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf`, this problem can be written as a linear complementarity problem (LCP).

3. Stop when v^{n+1} is sufficiently close to v^n.

2.2 Household's Problem: Alternative (Inferior) Approach

We have also tried another approach to solve the household's problem (6). This approach combines the finite-difference method with a so-called "operator splitting method" to take care of the constraint (5).[2] However, this method seems to be inferior to the LCP-algorithm developed above, particularly in terms of speed.

Denote $v_{i,j,k} = v_k(a_j, b_i)$. The algorithm works as follows. Start with an initial guess $v^0_{i,j,k}$ and for $n = 1, 2, ...$ compute $v^n_{i,j,k}$ as follows

1. given $v^n_{i,j,k}$, obtain $v^{n+\frac{1}{2}}_{i,j,k}$ by solving a discretized HJB equation that is the same as if there were no adjustment decision:

$$\begin{aligned}\frac{v^{n+\frac{1}{2}}_{i,j,k} - v^n_{i,j,k}}{\Delta} + \rho v^{n+\frac{1}{2}}_{i,j,k} =& u(c^n_{i,j,k}) + \partial_a v^{n+\frac{1}{2}}_{i,j,k} r_a a_j + \partial_b v^{n+\frac{1}{2}}_{i,j,k} (w z_k + r_b b_i - c^n_{i,j,k}) \\ &+ \lambda_k (v_{i,j,-k} - v_{i,j,k}), \\ c^n_{i,j,k} =& (u')^{-1}(\partial_b v^n_{i,j,k}).\end{aligned}$$

   Here $\partial_a v_{i,j,k}$ and $\partial_b v_{i,j,k}$ denote the finite-difference approximation of the partial derivative of v with respect to a and b (either forward- or backward-difference approximations), and where Δ is the size of the updating step. In particular, one can write the corresponding system of equations

$$\frac{1}{\Delta}\left(v^{n+\frac{1}{2}} - v^n\right) + \rho v^n = u^n + \mathbf{A} v^{n+\frac{1}{2}} \tag{9}$$

   where v^n is a vector of length $L = I \times J \times 2$ with the stacked value function as its entries, and $\mathbf{A}$ is the $L \times L$ transition matrix corresponding to the discretized process summarizing the evolution of (a_t, b_t, z_t). See `http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf` for details in a similar model (but with one asset only).

2. Compute

$$(v^*_{i,j,k})^{n+\frac{1}{2}} = \max_{a',b'} \; v^{n+\frac{1}{2}}_k(a', b') \quad \text{s.t.} \quad a' + b' = a_j + b_i - \kappa, \quad a' \geq 0, \; b' \geq 0.$$

   where $v^{n+\frac{1}{2}}_k(a', b')$ is $v^{n+\frac{1}{2}}_{i,j,k}$ interpolated at points (a', b').

[2]Note that this splitting method has nothing to do with the "splitting the drift" trick in `http://www.princeton.edu/~moll/HACTproject/two_asset_nonconvex.pdf`.

3. given $v_{i,j,k}^{n+\frac{1}{2}}$ by setting

$$v_{i,j,k}^{n+1} = \max\{v_{i,j,k}^{n+\frac{1}{2}}, (v_{i,j,k}^{*})^{n+\frac{1}{2}}\}. \tag{10}$$

4. If $v_{i,j,k}^{n+1}$ is "close to" $v_{i,j,k}^{n}$, stop.

The algorithm is called a "splitting algorithm" because the "operator" on v defined by (6) is split into two steps: that of going from v^n to $v^{n+\frac{1}{2}}$ and that of going from $v^{n+\frac{1}{2}}$ to v^{n+1}. It can be shown that the algorithm converges if the discretization of the HJB equation in step 1 satisfies the monotonicity, consistency and stability conditions of Barles and Souganidis (1991). See Achdou et al. (2017) for a discussion in the context of a model without a stopping-time decision. See e.g. Barles, Daher and Romano (1995) and Tourin (2013) for a discussion of convergence of numerical schemes for models with a stopping-time decision.

2.3 Kolmogorov Forward Equation

Without adjustment the KF equation is

$$0 = -\partial_a(s_k^a(a,b)g_k) - \partial_b(s_k^b(a,b)g_k) - \lambda_k g_k + \lambda_{-k} g_{-k}$$

for all (a,b) and $k = 1,2$. Here s_k^a and s_k^b are the illiquid and liquid saving policy functions. With adjustment there are extra terms. The mathematical formulation of Kolmogorov Forward equations with stopping and/or impulse control is not straightforward. See Bertucci (2017, 2018) for a treatment. However, this is not an obstacle for the numerical solution. In particular, it turns out to be quite easy to work with the discretized process as captured by the matrix $\mathbf{A}$ constructed in section 2.1 (or 2.2). See section 2 here `http://www.princeton.edu/~moll/HACTproject/HACT_numerical_appendix.pdf` for a more detailed explanation of this logic in a model without an adjustment decision.

If there were no adjustment, things would be very simple. In particular, the vector of stacked finite difference of the distribution, g_ℓ for $\ell = 1, ..., L$ where $L = I \times J \times 2$, would satisfy a linear system

$$0 = \mathbf{A}^{\mathrm{T}} g$$

where $\mathbf{A}^{\mathrm{T}}$ is the transpose of the transition matrix $\mathbf{A}$ from the HJB equation (8).

We now explain how to deal with the fact that there is adjustment. We first introduce some additional notation. Denote by $a_{i,j,k}^* = a_k^*(a_j, b_i)$ and $b_{i,j,k}^* = b_k^*(a_j, b_i)$ the optimal "adjustment targets" conditional on adjustment. Switching notation to the stacked and discretized state space, $\ell = 1, ..., L$, denote by $k^*(\ell)$ the grid point $k = 1, ..., L$ that is reached from point $\ell = 1, ..., L$ upon adjustment. Finally, denote by $\mathcal{I}$ the set of grid points in the inaction region. This set is defined by the requirement that $v_\ell > v_\ell^*$ for all $\ell \in \mathcal{I}$, whereas $v_\ell = v_\ell^*$ for all $\ell \notin \mathcal{I}$ (i.e. $v_\ell = v_\ell^*$ for points in the adjustment region).

The problem with using transition matrix $\mathbf{A}$ alone is that it does not capture adjustment. To introduce adjustment, we now define a binary matrix $\mathbf{M}$ which we term the "intervention matrix". The elements of $\mathbf{M}$, denoted by $M_{\ell,k}$ for $\ell = 1, ..., L$ and $k = 1, ..., L$ are given by

$$M_{\ell,k} = \begin{cases} 1, & \text{if } \ell \in \mathcal{I} \text{ and } \ell = k \\ 1, & \text{if } \ell \notin \mathcal{I} \text{ and } k^*(\ell) = k \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

This matrix moves points that are in the adjustment region to their corresponding adjustment targets. For instance, note that for points in the adjustment region the outside option $v^*(v)$ in the discretized HJBVI equation (7) satisfies $v^*(v) = \mathbf{M}v$. Therefore, the intervention matrix is the natural discretization of the intervention operator $\mathcal{M}$ discussed in Section 1.

To see how we use $\mathbf{M}$ to solve the Kolmogorov Forward equation with adjustment, consider a time-dependent KF equation but with fixed policy rules given by $\mathbf{A}$ and $\mathbf{M}$. Denoting $g^n = g(t^n), n = 1, ..., N$, the goal is to find a mapping from g^n to g^{n+1}. Motivated by the "operator splitting method" in section 2.2 and using the same notation as there, we split the step of finding g^{n+1} given g^n into two sub-steps:

1. Given g^n find $g^{n+\frac{1}{2}}$ from
$$g^{n+\frac{1}{2}} = \mathbf{M}^{\mathrm{T}} g^n \tag{12}$$

2. Given $g^{n+\frac{1}{2}}$ find g^{n+1} from[3]
$$\frac{g^{n+1} - g^{n+\frac{1}{2}}}{\Delta t} = (\mathbf{A}\mathbf{M})^{\mathrm{T}} g^{n+1} \tag{13}$$

Adjustment introduces two related but distinct questions: (i) how should we treat the density at grid points in the *adjustment* region? (ii) how should we treat the density at grid points in the *inaction* region but from which the stochastic process for idiosyncratic state variables ends up in the *adjustment* region?

The two parts of the operator splitting scheme show how we answer these questions. Step 1 answers question (i) by simply moving any mass from the adjustment region to the inaction region. Step 2 answers question (ii). Instead of using matrix $\mathbf{A}$ as the transition matrix as in the case without adjustment, we now use matrix $\mathbf{AM}$ as the transition matrix.[4] To

[3]This is an implicit scheme. The analogous explicit scheme is $\frac{g^{n+1} - g^{n+\frac{1}{2}}}{\Delta t} = (\mathbf{AM})^{\mathrm{T}} g^{n+\frac{1}{2}}$.
[4]Without adjustment (i.e., $\mathcal{I} = \{1, ..., L\}$), matrix $\mathbf{M}$ would be an identity matrix.

understand this, recall that $\mathbf{A}$ is a Poisson transition matrix with rows corresponding to the starting position of a Poisson process and columns corresponding to the finishing position. For each row ℓ, the purpose of matrix $\mathbf{M}$ is to take the entries of $\mathbf{A}_{\ell,k}$ that finish in the adjustment region (i.e. columns $k \notin \mathcal{I}$) and move them to columns $k^*(\ell)$ corresponding to the adjustment target. So, whereas transition matrix $\mathbf{A}$ can switch a process into the adjustment region, our updated transition matrix $\mathbf{AM}$ instead switches the process immediately to its corresponding adjustment target. Note that $\mathbf{AM}$ is still a valid Poisson transition matrix for all rows $\ell \in \mathcal{I}$. In particular, the rows sum to zero and diagonal elements are non-positive (capturing outflows) whereas off-diagonal elements are non-negative (capturing inflows).[5]

Some readers may wonder why the first step is necessary? To see this, consider an initial distribution g^0 with mass in the adjustment region. Essentially, without step 1, the distribution at all future points in time $g^n, n = 1, ..., N$ would always keep mass in the adjustment region. This is because matrix $\mathbf{M}$ is specifically constructed so that the columns of $\mathbf{AM}$ contain only zeros for $k \notin \mathcal{I}$. Therefore the corresponding rows of its transpose $(\mathbf{AM})^{\mathrm{T}}$ will contain only zeros and hence $\frac{g^{n+1}-g^{n+\frac{1}{2}}}{\Delta t} = (\mathbf{AM})^{\mathrm{T}} g^{n+1} = 0$ for all points in the adjustment region, meaning that any mass that starts there will stay there.

How can we find a stationary distribution g? The simplest strategy is to simply run (12) and (13) forward in time until convergence. This works well in practice. Alternatively, a stationary distribution satisfies

$$\begin{aligned} (i) \quad & g = \mathbf{M}^{\mathrm{T}} g, \quad \text{and} \\ (ii) \quad & 0 = (\mathbf{AM})^{\mathrm{T}} g \end{aligned} \tag{14}$$

or

$$\begin{aligned} (i) \quad & g_\ell = 0 \text{ if } \ell \notin \mathcal{I}, \quad \text{and} \\ (ii) \quad & 0 = (\mathbf{AM})^{\mathrm{T}} g \end{aligned}$$

Condition (i) simply ensures g has no mass in the adjustment region. This additional condition is needed because the rows of matrix $(\mathbf{AM})^{\mathrm{T}}$ corresponding to points in the adjustment region will contain only zeros.[6]

[5] For rows $\ell \notin \mathcal{I}$, matrix $\mathbf{AM}$ may have negative off-diagonal elements. However, this will not affect calculations using $\mathbf{AM}$ so long as the density g has no mass at points $\ell \notin \mathcal{I}$.

[6] To implement this numerically, we define matrix $\mathbf{D}$ whose elements $D_{\ell,k}$ are given by

$$D_{\ell,k} = \begin{cases} 1, & \text{if } \ell \notin \mathcal{I} \text{ and } \ell = k \\ 0, & \text{otherwise} \end{cases}$$

Matrix $\mathbf{D}$ contains zeros everywhere except for the elements of the diagonal corresponding to points in the adjustment region, where it contains a 1 (any constant will work). Matrix $\mathbf{D}$ is used to ensure that condition

Finally, consider the case where the policy rules change over time, that is $\mathbf{A}(t)$ $\mathbf{M}(t)$ are time-dependent. In this case, the time dependent KF equation can be solved by solving the fully time-dependent analogue of (12) and (13), namely

1. Given g^n find $g^{n+\frac{1}{2}}$ from $g^{n+\frac{1}{2}} = (\mathbf{M}^n)^{\mathrm{T}} g^n$
2. Given $g^{n+\frac{1}{2}}$ find g^{n+1} from $\frac{g^{n+1}-g^{n+\frac{1}{2}}}{\Delta t} = (\mathbf{A}^n \mathbf{M}^n)^{\mathrm{T}} g^{n+1}$

where $\mathbf{A}^n = \mathbf{A}(t^n)$ and $\mathbf{M}^n = \mathbf{M}(t^n), n = 1, ..., N$.

3 Results

Figure 1 plots the "adjustment region", i.e. the region of the state space (a, b) in which individuals adjust their portfolios. The adjustment region is in yellow and the non-adjustment region is in blue. Figures 2 and 3 plot the adjustment targets for liquid assets b and illiquid assets a conditional on adjusting. Finally Figure 4 plots the stationary distribution.

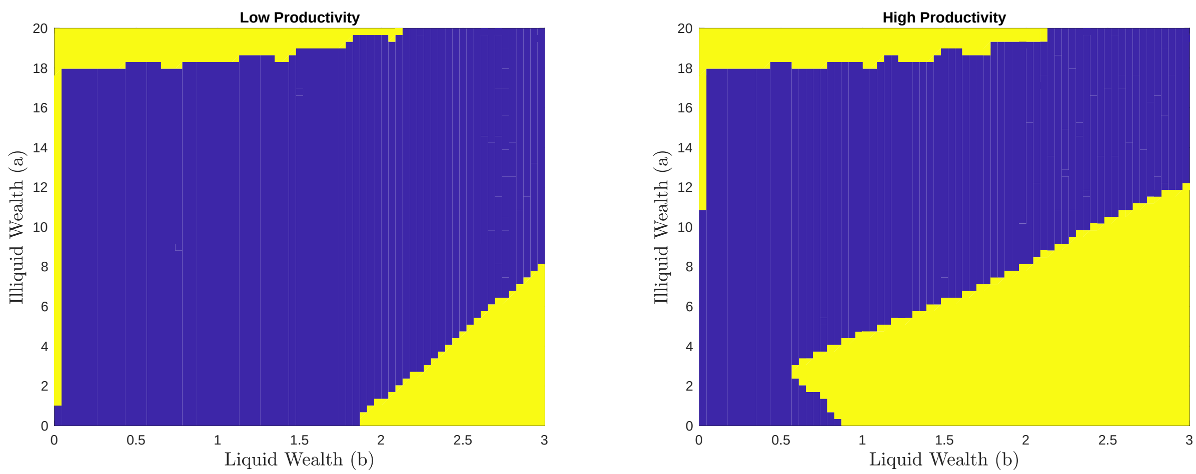

Figure 1: Adjustment and Non-Adjustment Regions

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$g_\ell = 0$ if $\ell \notin \mathcal{I}$, or equivalently $g = \mathbf{M}^{\mathrm{T}} g$, is met. Specifically, let $\tilde{\mathbf{A}}^{\mathrm{T}} = \mathbf{D} + (\mathbf{AM})^{\mathrm{T}}$. One can now solve $0 = \tilde{\mathbf{A}}^{\mathrm{T}} g$ exactly as one would solve $0 = \mathbf{A}^{\mathrm{T}} g$ when there is no adjustment.

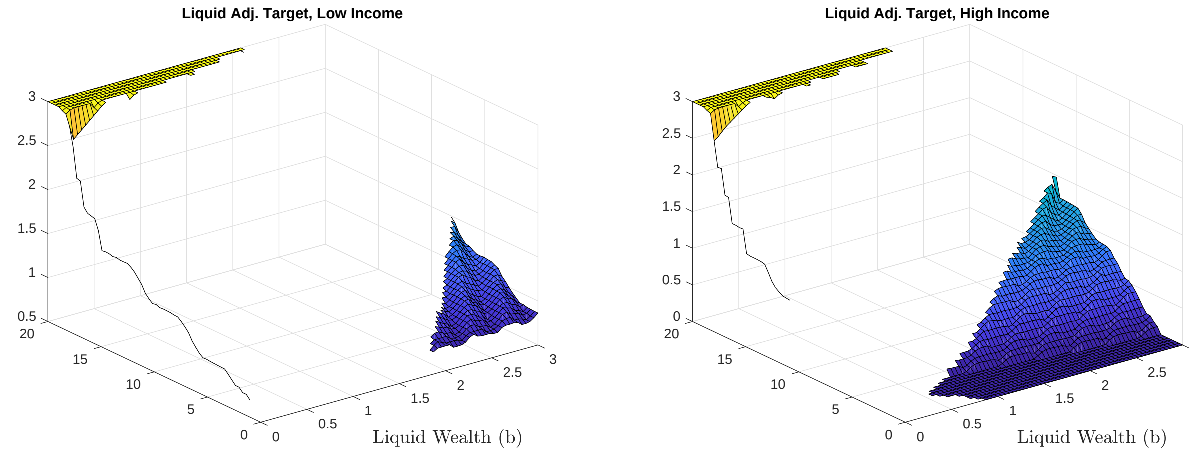

Figure 2: Liquid Wealth Adjustment Targets, $b_k^*(a,b)$

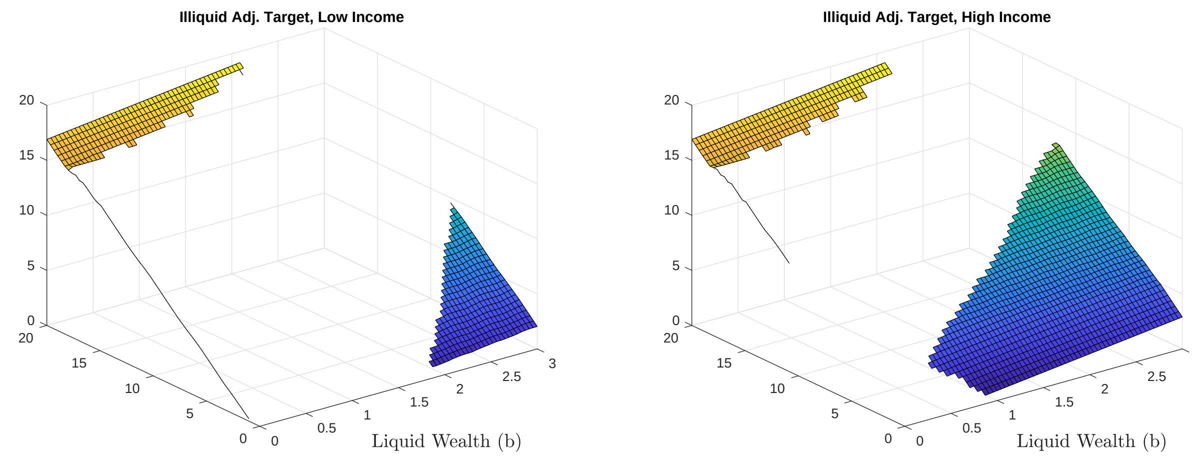

Figure 3: Illiquid Wealth Adjustment Targets, $a_k^*(a,b)$

Azimzadeh, P., E. Bayraktar, and G. Labahn. 2018. "Convergence of Implicit Schemes for Hamilton–Jacobi–Bellman Quasi-Variational Inequalities." *SIAM Journal on Control and Optimization*, 56(6): 3994–4016.

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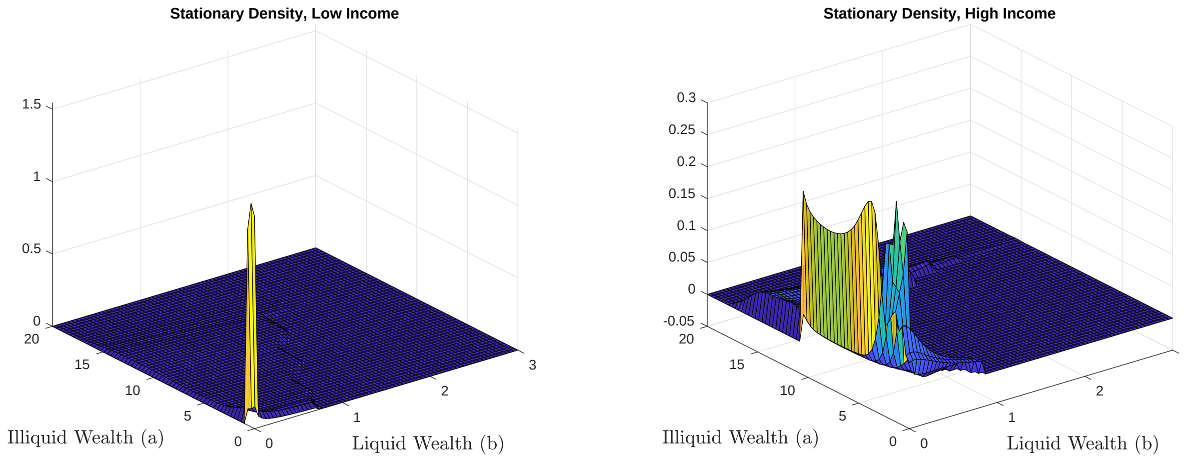

Figure 4: Stationary Distributions, $g_k(a, b)$

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