

Appendix for “The Trouble with Rational Expectations in Heterogeneous Agent Models: A Challenge for Macroeconomics”

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A A simple prototype for models requiring global solution methods

To complement the discussion in Section 3.4, this appendix spells out a simple example of a stochastic process that serves as a useful prototype for more complicated macroeconomic models featuring phenomena like state spaces with crisis regions or bimodal ergodic distributions for the aggregate economy. Such models need to be solved using global solution methods because local perturbation methods fail to capture their behavior.

This prototype is the scalar “double-well” or “bistable” diffusion process

$$dX_t = -V'(X_t)dt + \sigma dW_t, \quad (1)$$

where σ parameterizes uncertainty, W_t is a standard Brownian motion, and where $V(x)$ is a “double-well potential” function, that is a function that has two minima at a and c and a local maximum at b in between. For example, $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$ which has minima at -1 and 1 and a maximum at 0 . Equivalently, $-V'(x)$ is S-shaped and intersects zero three times. Good treatments are in [Gardiner \(2009, ch.14\)](#) and [Pavliotis \(2014, ch.7\)](#). An analogous discrete-time formulation is $X_{t+1} = S(X_t) + \sigma\varepsilon_t$ where σ parameterizes uncertainty, ε_t is a random variable, and $S(x)$ is an S-shaped function that intersects the 45-degree line three times. See [Azariadis and Stachurski \(2005\)](#) and [Morelli et al. \(2020\)](#) for discrete-time models with such dynamics.

Without uncertainty $\sigma = 0$ the double-well process (1) has three steady states, two of them stable and one unstable. The two stable steady states are at a and c and the unstable steady state is at b , i.e. the two minima and the maximum of V .

In contrast, with uncertainty $\sigma > 0$, this process has a bimodal stationary distribution¹

$$f(x) \propto \exp(-2V(x)/\sigma^2)$$

in which the two modes (the two maxima of f) are the two minima of V , i.e. points a and c . See for example, Figure 14.1 in [Gardiner \(2009\)](#) or Figure 7.2 in [Pavliotis \(2014\)](#). The system is thus most likely to be found at a and c .

Trajectories of X_t spend most time oscillating around the two stable steady states a and c while occasionally hopping between them – see, for example, Figure 7.2 in [Pavliotis \(2014\)](#). Starting from the high steady state, “the economy” stays close to it most of the time but may be thrown into a “crisis” (low steady state) and get stuck there for a while before ultimately recovering.

¹From the stationary Kolmogorov Forward equation $0 = (V'(x)f(x))' + \frac{\sigma^2}{2}f''(x)$ for all $x \in \mathbb{R}$.

Perturbation methods around $\sigma = 0$ completely miss this type of behavior. See [Gardiner \(2009, ch.7.2.4\)](#) for a good discussion. Intuitively, in the words of [Blanchard \(2014\)](#) cited in the conclusion, a perturbation method would think of the system “*as roughly linear, constantly subject to different shocks, constantly fluctuating, but naturally returning to its steady state over time,*” but this does not reflect the system’s actual dynamics.

B Various concepts of non-rational expectations equilibrium

The economics literature has proposed various concepts of “non-rational expectations equilibrium” or “misspecification equilibrium.” This appendix briefly summarizes a number of these and puts them in relation to each other. As already stated in Section 4.1, using the acronyms defined there or below, the relation between the various equilibrium concepts is summarized in (15) which I restate here for the reader’s convenience:

$$\{REE, RPE, CEE\} \subset SCE \subset IREE \subset TE,$$

where \subset means “is a special case of.”

Self-confirming equilibrium (SCE). See for example [Sargent \(1999\)](#), [Cho and Sargent \(2016\)](#), and [Fudenberg and Levine \(2016\)](#) in the context of games. In a self-confirming equilibrium, actual equilibrium outcomes are statistically consistent with decision makers’ beliefs, i.e. these beliefs are not disappointed. Self-confirming equilibria can be limiting outcomes of adaptive learning processes of the type described in Section 4.3. A rational expectations equilibrium is a self-confirming equilibrium, but not vice versa. Specifically, beliefs may be incorrect for events that are infrequently observed (e.g. events off the equilibrium parts).

Restricted perceptions equilibrium (RPE). See for example [Sargent \(1991\)](#) and [Branch \(2006\)](#). An RPE is a SCE in which decision makers use restricted forecasting models that do not nest rational expectations, i.e. mis-specified perceived laws of motion. As a result, a RPE is never a REE. Learning processes that update restricted forecasting models may settle down to a RPE (and never a REE). As discussed in Section 3.3, to actually simplify computations in heterogeneous-agent economies (Criterion 1), candidate alternative approaches should *not* start from rational expectations as a special case and so RPE is a natural equilibrium concept. See [Baley and Turén \(2024\)](#) for a recent application of RPE in a model of forecasters’ heterogeneous inflation expectations.

Consistent expectations equilibrium (CEE). See [Hommes and Sorger \(1998\)](#). In a CEE decision makers use autoregressive forecasting models such as an $AR(p)$ process, with a consistency condition for the autocorrelations as for SCE. A CEE is thus a special case of a SCE. In principle, a CEE could coincide with the REE if the RE solution happened to be exactly of the same autoregressive structure as the assumed forecasting model. But RE solutions typically do not take this form (and certainly RE equilibrium prices in heterogeneous-agent models do

not) so the CEE forecasting model is typically mis-specified. Thus a CEE is typically a special case of a RPE (Branch, 2006) so that (15) could also write $CEE \subset RPE$.

Another equilibrium concept not included in (15). *Oblivious equilibrium* is an equilibrium concept for stochastic dynamic games like dynamic industry models of imperfect competition (Weintraub et al., 2005, 2008, 2010). These games suffer from a curse of dimensionality similar to the one in heterogeneous-agent models: in a Markov Perfect Equilibrium, the distribution of other players' states is a state variable in player's dynamic programming problem. Oblivious equilibrium restricts players' policies to be "oblivious" to changes in the distribution of other players' states and instead restricts these policies to be functions of the constant long-run average of this distribution.²

C The Morgenstern (1935) critique of perfect foresight in general equilibrium

In an article written ninety (!) years ago, Morgenstern (1935) criticized the assumption of perfect foresight in general equilibrium. This article is relevant because perfect foresight is the precursor to – or, indeed, special case of – rational expectations in models without uncertainty. The article is in German and is entitled "Vollkommene Voraussicht und wirtschaftliches Gleichgewicht" which translates to "Perfect Foresight and General Equilibrium." To make it more accessible to English readers, I am including a translation of some key passages. The same passages in the original German are further below. The editor kindly pointed me to an existing English translation of the whole text (Morgenstern, 1976) but, at least for the key passages below, my translation is more literal and closer to the original German text.

English Translation. I translated the passages using ChatGPT and then edited for clarity.

Morgenstern (1935), p.337: *"The pride of theoretical economics is the theory of general economic equilibrium, which has been developed in various forms. [...] [The following] remarks are intended to draw attention to a problem of equilibrium theory – and thus of every variety of theoretical economics [...]. This concerns the assumption of what is here used synonymously as 'full foresight' or 'perfect foresight,' which is allegedly one of the preconditions of equilibrium."*

p.342: *"It is 'economic' things and events that are to be foreseen. On the admissible assumption that it is exactly known what is meant by this (for example, prices, production yields, etc.), one finds that, owing to the interdependence of all economic processes and conditions with one another, and of these with all other facts, no matter how small a segment of events could be specified whose foresight would not at the same time entail foresight of the entire remainder. [...] The most important and ultimately decisive elements of this kind are the individual acts of behavior from which the complex magnitudes arise."*

²The standard formulation only considers models without aggregate shocks and so is not relevant for our purposes. Weintraub et al. (2010) extend the approach to aggregate shocks by restricting strategies to depend only on a low-dimensional vector that is a function of the history of the aggregate shock (e.g. a truncation of the history).

“The forward-looking individual must therefore know not only exactly the influence of his own actions on prices, but also that of all other individuals, and that of his own future behavior on the behavior of others – especially those who are personally relevant to him. The circle of these relevant individuals is extraordinarily large, since all indirect effects must also be foreseen exactly. [Perfect foresight] also leads to the result that individuals must possess complete insight into theoretical economics – which is only to be supplied by equilibrium theory itself; for how else should they be able to foresee long-range effects?”

“The improbably high demands placed on the intellectual capacity of economic agents show at the same time that the equilibrium systems do not encompass ordinary human beings, but at least demigods who are exactly alike among themselves, if indeed the requirement of full foresight is to be met. This, then, is of no use whatsoever. If “full” or “perfect” foresight – in the strictly definable sense evidently intended by the economic authors, namely of unlimited foresight – is to be taken as a premise of equilibrium theories, then it is a completely nonsensical assumption.”

p.345: *“The necessity that, with perfect foresight, each individual must grasp all economic interrelations – that is, must master economic theory – leads to a scientifically and logically curious fact. If perfect foresight were an indispensable condition for formulating general equilibrium theory, it would result in the further paradox that science is already presupposed in the very object it is meant to investigate. [...] This logical-scientific issue is most clearly illustrated by comparison with the natural sciences. In physics or chemistry, there is absolutely no assumption that physical or chemical laws are known by the very objects these sciences aim to explain – for example, that atoms need to make assumptions about the behavior and states of other atoms.”*

Original German. In the original German, the same passages are as follows:

Morgenstern (1935), S.337: *“Den Stolz der theoretischen Ökonomie bildet die Theorie des allgemeinen wirtschaftlichen Gleichgewichtes, die in verschiedenen Formen entwickelt worden ist. [Die nachfolgenden] Ausführungen bezwecken auf ein Problem der Gleichgewichtstheorie – und damit jeder Abart von theoretischer Ökonomie – hinzuweisen [...]. Es handelt sich um die Annahme der (hier synonym gebrauchten) ‘vollen Voraussicht’ oder ‘vollkommenen Voraussicht’, die angeblich eine der Vorbedingungen des Gleichgewichtes ist.”*

S.342: *“Vorausgesehen werden sollen ‘wirtschaftliche’ Dinge und Ereignisse. Unter der zulässigen Annahme, es sei genau bekannt, was darunter zu verstehen ist (z.B. Preise, Produktionserträge usw.), findet man, daß infolge der Interdependenz aller wirtschaftlichen Prozesse und Gegebenheiten untereinander und dieser mit allen anderen Tatsachen kein noch so kleiner Ausschnitt aus dem Geschehen angegeben werden könnte, dessen Voraussicht nicht zugleich die Voraussicht des gesamten Restes bedeutete. [...] Die wichtigsten und letztlich entscheidenden Elemente dieser Art sind die individuellen Verhaltensakte aus denen die komplexen Größen hervorgehen.”*

“Das vorausschauende Individuum muß also nicht nur genau den Einfluß seines eigenen Handelns auf die Preise kennen, sondern auch den aller anderen Individuen und den seines eigenen zukünftigen Verhaltens auf das der anderen, namentlich der für ihn persönlich relevanten. Der Kreis dieser relevanten Individuen ist außerordentlich groß, da doch auch alle indirekten Wirkungen genau mitvorausgesehen werden müssen. [Die ‘vollkommene Voraussicht’] führt übrigens dazu, daß die Individuen

auch eine vollständige Einsicht in die – erst durch die Gleichgewichtstheorie zu liefernde – theoretische Ökonomie haben müssen, denn wie anders sollten sie sonst die Fernwirkungen voraussehen können?”

“Die unwahrscheinlich hohen Ansprüche, die an die intellektuelle Leistungsfähigkeit der Wirtschaftssubjekte gestellt werden, beweisen zugleich, daß in den Gleichgewichtssystemen keine gewöhnlichen Menschen erfaßt werden, sondern mindestens untereinander genau gleiche Halbgötter, falls eben die Forderung voller Voraussicht erfüllt sein soll. Damit ist also nichts anzufangen. Wenn ‘volle’ oder ‘vollkommene’ Voraussicht im streng angebbaren und von den ökonomischen Autoren offenbar gemeinten Sinne einer unbeschränkten Voraussicht den Gleichgewichtstheorien zugrunde gelegt werden soll, so handelt es sich um eine völlig sinnlose Annahme.”

S.345: “Die Notwendigkeit, dass jedes Individuum bei völliger Voraussicht alle wirtschaftlichen Zusammenhänge überschauen, also die theoretische Ökonomie beherrschen muss, führt zu einer wissenschaftslogisch merkwürdigen Tatsache. Ware völlige Voraussicht eine unerlässliche Bedingung für die Aufstellung der Gleichgewichtstheorie, so ergäbe sich das weitere Paradox, dass die Wissenschaft bei dem Objekt, das sie erst erforschen soll, schon vorausgesetzt wird [...] Die wissenschaftslogische Situation ist am klarsten gegenüber den Naturwissenschaften darzutun. Bei der Physik oder Chemie wird in gar keiner Weise die Präexistenz physikalischer oder chemischer Lehrsätze bei den von diesen Wissenschaften zu erklärenden Objekten – z.B. den Atomen und Elementen – vorausgesetzt, derart, daß die Atome Annahmen über das Verhalten und die Zustände der anderen Atome machen müßten.”

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