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Chapter 5

STATIONARY MONETARY EQUILIBRIUM WITH A CONTINUUM OF INDEPENDENTLY FLUCTUATING CONSUMERS*

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1. Introduction

In this paper, I develop the theme that in the short run consumers may be expected to act as if their marginal utilities of money were constant. This idea was expressed in a previous paper [Bewley (1977)] in terms of a model with one consumer. Here, a general equilibrium model is used. The model is of a pure exchange economy with immortal consumers who hold money in order to offset fluctuations in their endowments and utility functions. It is also assumed that there is a continuum of consumers and that the fluctuations in their utilities and endowments are independent. These assumptions are made so that in the aggregate their fluctuations offset each other and equilibrium prices need not fluctuate. The constancy of prices greatly simplifies the analysis. The main theorem is that if the rate at which a consumer discounts future utility is allowed to go to zero, his marginal utility of money becomes nearly constant. Making the pure rate of time preference small corresponds roughly to speeding up the exogenous random fluctuations.

There are technical difficulties associated with a continuum of independently fluctuating random variables. Let x_a, $a \in [0,1]$, be such a family of random variables. A typical realization of these variables is not a measurable function of a [see Judd (1985)], so that one wonders how to define the integral $\int_0^1 x_a d_a$.

This paper is closely related to several in the literature. The model is similar to that of Lucas (1980), though he includes a Clower constraint, which is not present here. Continua of independent random variables have been used by

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Lucas and Prescott (1974) and Lucas (1975, 1980). Many papers have discussed consumer behavior when the time horizon is distant and the rate of time preference small. These include Yaari (1976), Schechtman (1976) and Schechtman and Escudero (1977). Many of the mathematical techniques used here have been borrowed from the last two papers.

2. The model

2.1. The exogenous stochastic processes

Each consumer is influenced by an exogenous stochastic process. The processes of the various consumers are mutually independent, but are all copies of a single stochastic process $\{s_t\}_{t=-\infty}^{\infty}$. That is, all the exogenous processes have the same distribution as $\{s_t\}$, even though they are independent.

The process $\{s_t\}$ is a stationary Markov chain on a finite set S. I assume that the process $\{s_t\}$ has a unique stationary distribution. The s_t are distributed according to this distribution. Elements of S are denoted by s and are referred to as states of the environment.

The sample space of the process $\{s_t\}$ is $\Sigma = \{(\dots, s_{-1}, s_0, s_1, \dots) | s_t \in S$, for all $t\}$. Elements of Σ are denoted by $\underset{\sim}{s}$ and are referred to as states of the world. s_t is the tth component of $\underset{\sim}{s}$. The measurable subsets of Σ are noted by $\mathscr{S}$, and P denotes the probability on $\mathscr{S}$. $\mathscr{S}$ is the smallest complete σ-field such that all the random variables s_t are measurable with respect to $\mathscr{S}$. $\mathscr{S}_t$ denotes the smallest complete σ-field with respect to which the random variables s_n, $n \leqq t$, are measurable. $\mathscr{S}_t$ is contained in $\mathscr{S}$.

An event $A \in \mathscr{S}$ is said to occur almost surely or for almost every $\underset{\sim}{s}$ if $P(A) = 1$.

σ: $\Sigma \to \Sigma$ denotes the shift operator defined by the formula $(\sigma \underset{\sim}{s})_t = s_{t+1}$, for all t.

2.2. The space of consumer characteristics

There are L commodities, where L is a positive integer. R^L denotes L-dimensional Euclidean space and $R_+^L = \{x \in R^L | x_k \geqq 0$, for all $k\}$. R_+^L is the consumption possibility set of each consumer.

The characteristics of a consumer are defined by (u, ω, ρ). u: $R_+^L \times S \to (-\infty, \infty)$ is the utility function, ω: $S \to R_+^L$ describes the endowment, and $\rho > 0$ is the pure rate of time preference. If the state of the environment is s_t at time t, then the consumer's endowment at that time is $\omega(s_t)$ and his utility for a bundle $x \in R_+^L$ is $u(x, s_t)$. ρ is the interest rate at which future utility is discounted.

$\mathscr{U} = \{u\colon R_+^L \times S \to R|$ for each s, $u(\cdot, s)\colon R_+^L \to R$ is continuously differentiable$\}$ is the set of possible utility functions. Each ω may be thought of as belonging to $R_+^{L|S|}$, where $|S|$ denotes the cardinality of S. $\mathscr{U} \times R_+^{L|S|} \times (0, \infty)$ is the space of consumer characteristics (u, ω, ρ).

I give $\mathscr{U}$ the topology of uniform $\mathscr{C}^1$-convergence on compacts. That is, $\lim_{n\to\infty} u_n = u$ if and only if for each $r > 0$, $\lim_{n\to\infty} u_n(x, s) = u(x, s)$ and $\lim_{n\to\infty} \partial u_n(x, s)/\partial x_k = \partial u(x, s)/\partial x_k$, uniformly for $\|x\| \leqq r$, $s \in S$ and $k = 1, \ldots, L$.

2.3. The economy

An economy is defined by a measurable function $(u, \omega, \rho)\colon [0, 1] \to \mathscr{U} \times R_+^{L|S|} \times (0, \infty)$. (u_a, ω_a, ρ_a) denotes the value of this function at $a \in [0, 1]$. When I say that (u, ω, ρ) is measurable, I mean that there is a sequence of simple functions, $(u^n, \omega^n, \rho^n)\colon [0, 1] \to \mathscr{U} \times R_+^{L|S|} \times (0, \infty)$ such that $\lim_{n\to\infty}(u_a^n, \omega_a^n, \rho_a^n) = (u_a, \omega_a, \rho_a)$, for almost every $a \in [0, 1]$. A function is simple if it takes on only finitely many values.

Each agent $a \in [0, 1]$ observes $s_{at} \in S$. The stochastic processes $\{s_{at}\}_{t=-\infty}^{\infty}$, for $a \in [0, 1]$, are mutually independent and are identically distributed according to P.

2.4. Programs

A consumption program for a single consumer is of the form $\underset{\sim}{x} = (x_0, x_1, \ldots)$, where $x_t\colon \Sigma \to R_+^L$ is measurable with respect to $\mathscr{S}_t$, for all t. It is stationary if $x_t(\underset{\sim}{s}) = x_0(\sigma^t \underset{\sim}{s})$ almost surely, for all t.

An initial endowment function $\omega\colon \mathscr{S} \to R_+^L$ determines a stationary program $\underset{\sim}{\omega} = (\omega_0, \omega_1, \ldots)$, where $\omega_t(\underset{\sim}{s}) = \omega(s_t)$, for all t.

2.5. Allocations

An allocation for the economy is a set of stationary consumption programs indexed by $a \in [0, 1]$, call it $(\underset{\sim}{x}_a)_{a\in[0,1]}$. We require that $\mathrm{E}x_{a0}$ be a measurable function of a, where $\mathrm{E}x_{a0} = \int x_{a0}(\underset{\sim}{s})P(\mathrm{d}\underset{\sim}{s})$.

The total consumption associated with the allocation $(\underset{\sim}{x}_a)_{a\in[0,1]}$ is defined to be $\int_0^1 \mathrm{E}x_{a0}\,\mathrm{d}a$.

An allocation is feasible if $\int_0^1 \mathrm{E}x_{a0}\,\mathrm{d}a = \int_0^1 \mathrm{E}\omega_a(s_0)\,\mathrm{d}a$.

2.6. Demand

A price vector is an element of $\operatorname{int}\Delta^{L-1} = \{p \in R_+^L | \sum_{k=1}^L p_k = 1$ and $p_k > 0$, for all $k\}$. I now define the stationary demand associated with a price vector p and a fixed set of consumer characteristics (u, ω, ρ).

The consumer uses money. His initial balances at the beginning of period zero are described by the function $M_{-1}: \Sigma \to [0, \infty)$. M_{-1} is measurable with respect to $\mathscr{S}_{-1}$. His money balance at the end of period t when he follows program $\underset{\sim}{x}$ is denoted by

$$m_t(p, \underset{\sim}{x}, M_{-1}, \underset{\sim}{s}) = M_{-1}(\underset{\sim}{s}) + \sum_{n=0}^{t} p \cdot (\omega(s_n) - x_n(\underset{\sim}{s})).$$

The budget set of the consumer is

$$\beta(p, M_{-1}) = \{\underset{\sim}{x} | \underset{\sim}{x} \text{ is a consumption program and } m_t(p, x, M_{-1}, \underset{\sim}{s}) \geqq 0 \text{ almost surely, for all } t\}. \tag{1}$$

The consumer's problem is

$$\max\left\{ \mathrm{E} \sum_{t=0}^{\infty} (1+\rho)^{-t} u(x_t(\underset{\sim}{s}), s_t) | \underset{\sim}{x} \in \beta(p, M_{-1}) \right\}. \tag{2}$$

$\xi(p, M_{-1})$ denotes the set of solutions to this problem. When a solution exists, it is unique.

If the initial money balance function, M_{-1}, is chosen correctly, then the program $\xi(p, M_{-1})$ is stationary. The initial balance function, M_{-1}, is itself called stationary if it gives rise to a stationary demand, $\xi(p, M_{-1})$. M_{-1} is stationary if and only if $M_{-1}(\sigma\underset{\sim}{s}) = m_0(p, \underset{\sim}{x}, M_{-1}, \underset{\sim}{s})$ almost surely, where $\underset{\sim}{x} = \xi(p, M_{-1})$. Under the conditions assumed in this paper, there exists a unique stationary distribution of money corresponding to each price vector p, call it $M_{-1}(p)$. $\xi(p) \equiv \xi(p, M_{-1}(p))$ is the stationary demand determined by p.

$\xi_a(p)$ and $M_{a,-1}(p)$ denote, respectively, the stationary demand and initial balances of consumer $a \in [0,1]$, where consumer a has characteristics (u_a, ω_a, ρ_a).

2.7. Equilibrium

A stationary equilibrium is defined by $(p, \underset{\sim}{x}_a)_{a \in [0,1]}$, where p is a price vector, $(\underset{\sim}{x}_a)$ is a feasible allocation and $\underset{\sim}{x}_a = \xi_a(p)$, for all a.

2.8. The supply of money

The aggregate money balance determined by a stationary equilibrium $(p,(x_a)_{a\in[0,1]})$ is $\int_0^1 \mathrm{E}M_{a,-1}(p)\,\mathrm{d}a$.

2.9. The marginal utility of money

Associated with any optimal program, there is a marginal utility of money function. Let $\lambda_{a0}(p)$: $\Sigma \to (0,\infty)$ be the marginal utility of money for consumer $a \in [0,1]$ at time zero when he uses the stationary optimal program $\xi_a(p)$ and has the stationary initial balance $M_{a,-1}(p)$. $\lambda_{a0}(p,\underset{\sim}{s})$ is his marginal utility of money at time zero when the state of the world is $\underset{\sim}{s}$. His marginal utility of money at time t is $\lambda_{a0}(p,\sigma^t\underset{\sim}{s})$.

I prove that if the consumer's rate of time preference is sufficiently small, then $\lambda_{a0}(p,s_0)$ is nearly constant. It is in fact nearly equal to a constant $\lambda_{a\infty}(p)$ defined as follows. For each $\lambda > 0$, there exists a unique vector $x \in R^L_+$ such that $\partial u_a(x,s)/\partial x_k \leqq \lambda p_k$, for all k, with equality if $x_k > 0$. Denote this vector by $G_a(p,\lambda,s)$. There is also a unique value of λ such that $\mathrm{E}p \cdot G_a(p,\lambda,s_0) = \mathrm{E}p \cdot \omega(s_0)$. This value of λ is denoted $\lambda_{a\infty}(p)$.

3. *Assumptions*

We here list the assumptions made. Some have already been mentioned.

Assumption 1. $\{s_t\}$ is a Markov chain with stationary transition probabilities and with state space S.

Assumption 2. The process $\{s_t\}$ is ergodic and is distributed according to its unique stationary distribution.

Assumption 3. For each $a \in [0,1]$, $\{s_{at}\}_{t=-\infty}^{\infty}$ has the same probability distribution and state space as does $\{s_t\}$.

Assumption 4. The random variables $\underset{\sim}{s}_a = (\ldots, s_{a,-1}, s_{a0}, s_{a1}, \ldots)$, $0 \leqq a \leqq 1$, are mutually independent.

Assumption 5. (u,ω,ρ): $[0,1] \to \mathscr{U} \times R_+^{L|S|} \times (0,\infty)$ is measurable, and maps into $U \times W \times [\rho_1,\rho_2]$, where U and W are compact and $0 < \rho_1 < \rho_2 < \infty$.

Assumption 6. For all $u \in U$ and s, $u(\cdot, s)$: $R^L_+ \to R$ is strictly concave and monotonic. By monotonic, I mean that $\partial u(x, s)/\partial x_k > 0$, for all x and k.

Assumption 7. $\lim_{x_k \to \infty} \partial u(x, s)/\partial x_k = 0$, uniformly in x, k, s and $u \in U$.

Assumption 8. For all $\omega \in W$, $\omega_k(s) > 0$, for all k and s.

The next assumption says that there is a uniformly worst state $\underline{s}$.

Assumption 9. There is $\underline{s} \in S$ such that for all $u \in U$, $\partial u(x, s)/\partial x_k \leqq \partial u(x, \underline{s})/\partial u_k$, for all x, k and s. Also, for all $\omega \in W$, $\omega_k(\underline{s}) < \omega_k(s)$, for all $s \neq \underline{s}$ and for all k. Also, Prob $[s_1 = \underline{s} | s_0 = s] > 0$, for all $s \in S$.

4. Theorems

I assume that Assumptions 1–9 all apply.

The first theorem generalizes theorem 3.4 in Schechtman and Escudero (1977). It is also similar to proposition 4 in Lucas (1980). These authors assume that the s_t are independent and identically distributed.

Theorem 1. Corresponding to each $p \in \operatorname{int} \Delta^{L-1}$ *and* $(u, \omega, \rho) \in U \times W \times (0, \infty)$, *there exists a unique stationary demand* $\xi(p)$ *and stationary initial balance function* $M_{-1}(p)$.

Theorem 2. There exists a stationary equilibrium.

The next theorem says that if a consumer has a low rate of time preference, then his stationary money balances will be large.

Theorem 3. For every $M > 0$ *and* $\varepsilon > 0$, *there is a* $\underline{\rho} > 0$ *which depends only on U and W and is such that for every a if* $\rho_a \leqq \underline{\rho}$, *then* $\text{Prob}[M_{a,-1}(p, \underline{s}) \leqq M] < \varepsilon$, *where p is the stationary equilibrium price vector.*

The next corollary says that the aggregate or average money supply goes to infinity as the rates of time preference of all consumers go to zero. One could normalize prices so that the equilibrium money supply would equal some constant. Then, Theorem 3 would imply that prices go to zero as consumers' rates of time preference go to zero.

Corollary 1. For every $M > 0$, there exists $\underline{\rho} > 0$ such that $\int_0^1 \mathrm{E}M_a(p)\,\mathrm{d}a \geqq M$ if $\rho_2 \leqq \underline{\rho}$, where ρ_2 is as in Assumption 7.

In Bewley (1977), constancy of the marginal utility of money was referred to as the permanent income hypothesis. The last theorem says that a consumer satisfies the permanent income hypothesis approximately if his pure rate of time preference is small.

Theorem 4. For every $\varepsilon > 0$, there exists $\underline{\rho} > 0$ which depends only on U and W and is such that for every a, if $\rho_a \leqq \underline{\rho}$, then $\mathrm{Prob}[|\lambda_{a0}(p, \underset{\sim}{s}) - \lambda_{a\infty}(p)| > \varepsilon] < \varepsilon$, *where p is the stationary equilibrium price vector.*

5. Preliminary results

For the moment, (u, ω, ρ) denotes a fixed element of $U \times W \times (0, \infty)$, where U and W are as in Assumption 5. Similarly, p is a fixed element of $\operatorname{int} \Delta^{L-1}$. I now study the problem

$$\max \left\{ \mathrm{E} \sum_{t=0}^{\infty} (1+\rho)^{-t} u(x_t(\underset{\sim}{s}), s_t) | \underset{\sim}{x} \in \beta(p, M) \right\}, \tag{3}$$

where $\beta(p, M)$ is as in definition (1), $p \in \operatorname{int} \Delta^{L-1}$ and M is a positive number.

I now state a series of assertions which I do not prove. The proofs are contained in Bewley (1977) or are obtained by slight modifications of proofs contained in Schechtman and Escudero (1977).

> Problem (3) has a solution, call it $\underset{\sim}{x}(M) = (x_0(M), x_1(M), \dots)$. The solution is unique up to sets of probability zero. (4)

Define $V(M, s)$, for $s \in S$, by $V(M, s) = \mathrm{E}\left[\sum_{t=0}^{\infty} (1+\rho)^{-t} u(x_t(M, \underset{\sim}{s}), s_t) | s_0 = s\right]$.

> For each $s \in S$, $V(M, s)$ is a continuously differentiable function of M. (5)

Let $\lambda(M, s) = (\mathrm{d}/\mathrm{d}M)V(M, s)$. $\lambda(M, s)$ is the marginal utility of money.

> For each $s \in S$, $\lambda(M, s)$ is a continuous, positive and strictly decreasing function of M. (6)

In the optimal program $\underset{\sim}{x}(M) = (x_0(M), x_1(M), \ldots)$, the random variable $x_0(M, \underset{\sim}{s})$ depends only on s_0. Define g: $[0, \infty) \times S \to R^L_+$ by $g(M, s_0) = x_0(M, \underset{\sim}{s})$.

$$\partial u(g(M, s), s)/\partial x_k \leqq \lambda(M, s)p_k, \text{ for all } k \text{ and } s, \text{ with equality if } g_k(M, s) > 0. \tag{7}$$

Let $M_0(M, \underset{\sim}{s}) = M + p \cdot (\omega(s_0) - g(M, s_0))$.

$$\lambda(M, s) \geqq (1 + \rho)^{-1}\mathrm{E}[\lambda(M_0(M, \underset{\sim}{s}), s_1)|s_0 = s], \text{ with equality if } M_0(M, \underset{\sim}{s}) > 0. \tag{8}$$

For all $s \in S$, $p \cdot g(M, s)$ is a continuous, non-decreasing function of M. (9)

For all $s \in S$, $M - p \cdot g(M, s)$ is a continuous, non-decreasing function of M. (10)

This implies that $M_0(M, \underset{\sim}{s})$ is a non-decreasing function of M.

Now let the initial holdings of money be a random variable, M_{-1}: $\Sigma \to [0, \infty)$, and consider the problem

$$\max\left\{\mathrm{E}\sum_{t=0}^{\infty}(1 + \rho)^{-t}u(x_t(\underset{\sim}{s}), s_t)|\underset{\sim}{x} \in \beta(p, M_{-1})\right\}. \tag{11}$$

The function g defined above generates a solution to this problem. Define $M_t(M, \underset{\sim}{s})$, for $t = -1, 0, 1, \ldots$, by induction on t as follows.

$$M_{-1}(M_{-1}, \underset{\sim}{s}) = M_{-1}(\underset{\sim}{s}).$$

Given $M_t(M_{-1}, \underset{\sim}{s})$,

$$M_{t+1}(M_{-1}, \underset{\sim}{s}) = M_t(M_{-1}, \underset{\sim}{s}) + p \cdot \omega(s_{t+1}) - g(M_t(M_{-1}, \underset{\sim}{s}), s_t).$$

Let

$$x_t(\underset{\sim}{s}) = g(M_t(M_{-1}, \underset{\sim}{s}), s_t).$$

$\underset{\sim}{x}$ is the program generated by g, and $M_t(M_{-1}, \underset{\sim}{s})$ is the consumer's money balance at the end of period t.

$$x \text{ solves problem (11). The solution is unique up to sets of probability zero.} \tag{12}$$

We now define some useful functions. For $\lambda > 0$ and $s \in S$, let $G(\lambda, s)$ be the unique vector $x \in R_+^L$ such that $\partial u(x, s)/\partial x_k \leqq \lambda p_k$, for all k, with equality if $x_k > 0$. By Assumption 7, $G(\lambda, s)$ is well-defined for all $\lambda > 0$. Given $w > 0$, let $\Lambda(w, s)$ be the Lagrange multiplier associated with the problem $\max\{u(x, s)|p \cdot x \leqq w\}$. Let $\Lambda(0, s) = \lim_{w \to 0}\Lambda(w, s)$. I state the following facts without proof.

$G(\lambda, s)$ is a continuous function of λ, for all s. Also, $p \cdot G(\lambda, s)$ is decreasing in λ, for λ such that $p \cdot G(\lambda, s) > 0$. Finally, $\lim_{\lambda \to 0} p \cdot G(\lambda, s) = \infty$, and $p \cdot G(\lambda, s) = 0$, for λ sufficiently large. (13)

$\lambda(w, s)$ is a continuous, strictly decreasing function of w, for all s. $\Lambda(p \cdot G(\lambda, s), s) = \lambda$. (14)

$$g(M, s) = G(\lambda(M, s), s) \quad \text{and} \quad \lambda(M, s) = \Lambda(p \cdot g(M, s), s).$$

It follows easily from (13) that there is a unique value of λ, call it λ_∞, such that

$$Ep \cdot G(\lambda_\infty, s_0) = Ep \cdot \omega(s_0). \tag{15}$$

Now think of u, ω, ρ and p as variables. $\lambda(u, \omega, \rho, p, M, s)$, $g(u, \omega, \rho, p, M, s)$, $M_t(u, \omega, \rho, p, M, s)$, $G(u, p, \lambda, s)$, $\lambda(u, p, w, s)$, and $\lambda_\infty(u, \omega, p)$ denote the functions defined before with the above variables made explicit.

Let

$$\bar{\lambda} = \max\left\{\frac{\partial u(0, s)}{\partial x_k}\Big|u \in U, s \in S \text{ and } k = 1, \ldots, L\right\}.$$

$\bar{\lambda}$ exists and is finite, since U is compact and each u in U is strictly monotone. It should be clear that

$$\lambda(u, \omega, \rho, p, M, s) \leqq \bar{\lambda} \max_k p_k^{-1}. \tag{16}$$

6. Some lemmas

The following lemmas give control over the money holdings of consumers. The first lemma states that the stationary money holdings of consumers are uniformly bounded. In what follows, $\Delta_\delta^{L-1} = \{ p \in \text{int}\, \Delta^{L-1} | p_k \geqq \delta, \text{ for all } k \}$.

Lemma 1. For each $\delta > 0$, *there exists a continuous, non-increasing function of* ρ, $\overline{M}(\rho, \delta)$, *such that* $p \cdot g(u, \rho, \omega, p, M, s) > p \cdot \omega(s)$, *whenever* $M \geqq \overline{M}(p, \delta)$, *for all* $(u, \omega, p, s) \in U \times W \times \Delta_\delta^{L-1} \times S$.

Since $p \cdot g(u, \omega, \rho, p, M, s)$ is a non-decreasing function of M, statement (9) implies that $M_0(u, \omega, \rho, p, M_2, S) \leqq M_0(u, \omega, \rho, p, M_1, s) + M_2 - M_1$, whenever $M_2 < M_1$. Hence, Lemma 1 implies that

$$M_0(u, \omega, \rho, p, M, s) \leqq \overline{M}(\rho, \delta), \text{ whenever } M \leqq \overline{M}(\rho, \delta), \\ \text{for all } (u, \omega, \rho, s) \in U \times W \times \Delta_\infty^{L-1} \times S. \tag{17}$$

That is, $\overline{M}(\rho, \delta)$ is an effective upper bound on stationary money balances.

Lemma 1 generalizes theorem 3.3 in Schechtman and Escudero (1977).

Proof of Lemma 1. First of all, I introduce some bounds. Each of the bounds (18)–(21) below applies for all $\rho > 0$, $M \geqq 0$ and for all $(u, \omega, p, s) \in U \times W \times \Delta_\delta^{L-1} \times S$. By (16),

$$\lambda(u, \omega, \rho, p, M, s) \leqq \delta^{-1}\bar{\lambda}. \tag{18}$$

The compactness of W implies:

$$\text{There exists } \bar{w} \text{ such that } p \cdot \omega(s) \leqq \bar{w}. \tag{19}$$

The compactness of U implies:

$$\text{There exists } \underline{\lambda} > 0 \text{ such that } \lambda(u, \omega, \rho, p, M, s) \geqq \underline{\lambda}, \\ \text{whenever } p \cdot g(u, \omega, \rho, p, M, s) \leqq \bar{w}. \tag{20}$$

Assumption 7 implies:

$$\text{There exists } \hat{w} \geqq \bar{w} \text{ such that } p \cdot g(u, \omega, \rho, p, M, s) \leqq \hat{w}, \\ \text{whenever } \lambda(u, \omega, \rho, p, M, s) \geqq \underline{\lambda}. \tag{21}$$

Let $N(\rho, \delta)$ be the smallest positive integer n such that $(1 + \rho)^n \underline{\lambda} \leqq \delta^{-1}\bar{\lambda} + 1$. Let $\bar{M}(\rho, \delta)$ be any continuous, non-increasing function of ρ such that $\bar{M}(\rho, \delta) > (N(\rho, \delta) + 1)\hat{w}$, for all $\rho > 0$.

I now prove that $\bar{M}(\rho, \delta)$ is as in the lemma. Suppose that $\bar{M}(\rho, \delta)$ did not satisfy the conditions of the lemma. Then, for some $\rho > 0$, some $M \geqq \bar{M}(\rho, \delta)$ and some $(u, \omega, p, \bar{s}_0) \in U \times W \times \Delta_\delta^{L-1} \times \mathscr{S}$,

$$p \cdot g(u, \omega, \rho, p, M, \bar{s}_0) \leqq p \cdot \omega(\bar{s}_0). \tag{22}$$

Since u, ω, ρ and p are now fixed, I drop them from the variables of the functions g, λ and M_t. I prove the following:

There exist $\bar{s}_1, \ldots, \bar{s}_{N(\rho,\delta)+1}$, such that for $t = 0, 1, \ldots, N(\rho, \delta) + 1$, $\lambda(M_{t-1}(M, \underset{\sim}{s}), \bar{s}_t) \geqq (1 + \rho)^t \underline{\lambda}$ and $M_{t-1}(M, \underset{\sim}{s}) \geqq M - t\hat{w}$, whenever $\underset{\sim}{s}$ is such that $s_n = \bar{s}_n$, for $0 \leqq n \leqq t$. (23)

This statement implies that inequality (22) is impossible. For the statement implies that there exists $\underset{\sim}{s}$ such that $\lambda(M_{N(\rho,\delta)}(M, \underset{\sim}{s}), s_{N(\rho,\delta)+1}) \geqq (1 + \rho)^{N(\rho,\delta)}\underline{\lambda} > \bar{\lambda}$, which contradicts inequality (18). Thus, (23) implies the lemma.

Statement (23) is proved by induction of t. By (19) and (22), $p \cdot g(M, \bar{s}_0) \leqq p \cdot \omega(\bar{s}_0) \leqq \bar{w}$, so that by (20), $\lambda(M, \bar{s}_0) \geqq \underline{\lambda}$. Since $\bar{w} \leqq \hat{w}$ it follows that $M_0(M, \bar{s}_0) \geqq M - \hat{w}$. This proves (23) for $t = 0$. Suppose that $\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_T$ have been defined and satisfy the two inequalities of (23) for $0 \leqq t \leqq T \leqq N(\rho, \delta)$. If $\underset{\sim}{s}$ is such that $s_n = \bar{s}_n$, for $0 \leqq n \leqq T$, then $\lambda(M_{T-1}(M, \underset{\sim}{s}), \bar{s}_T) \geqq (1 + \rho)^T \underline{\lambda} \geqq \underline{\lambda}$, so that by (21), $p \cdot g(M_{T-1}(M, \underset{\sim}{s}), \bar{s}_T) \leqq \hat{w}$ and $M_T(M, \underset{\sim}{s}) \geqq M_{T-1}(M, \underset{\sim}{s}) - \hat{w} \geqq M - (T + 1)\hat{w}$. Since $M - (T + 1)\hat{w} \geqq \bar{M}(\rho, \delta) - (T + 1)\hat{w} > 0$, it follows from inequality (8) that if $\underset{\sim}{s}$ is such that $s_n = \bar{s}_n$, for $0 \leq n \leq T$, then $\lambda_T(M_{T-1}(M, \underset{\sim}{s}), \bar{s}_T) = (1 + \rho)^{-1}\mathrm{E}[\lambda(M_T(M, \underset{\sim}{s}), s_{T+1}) | s_n = \bar{s}_n$, for $0 \leqq n \leqq T]$. Therefore, there exists $\bar{s}_{T+1} \in S$ such that $\lambda(M_T(M, \underset{\sim}{s}), s_{T+1}) \geqq (1 + \rho)\lambda(M_{T-1}(M, \underset{\sim}{s}), \bar{s}_T) \geqq (1 + \rho)^{T+1}\underline{\lambda}$, whenever $\underset{\sim}{s}$ is such that $s_n = \bar{s}_n$, for $0 \leqq n \leqq T + 1$. This completes the induction step in the proof of (23). □

Lemma 2. Let $\delta > 0$ and let $\underset{\sim}{s}$ be as in Assumption 9. There is a continuous, positive and increasing function of ρ, $\underline{M}(\rho, \delta)$, such that $p \cdot g(u, \omega, \rho, p, M, \underset{\sim}{s}) = p \cdot \omega(\underset{\sim}{s}) + M$, whenever $M \leqq \underline{M}(\rho, \delta)$, for all $(u, \omega, p, s) \in U \times W \times \Delta_\delta^{L-1} \times S$.

Proof. Let $\underline{M}(u, \omega, \rho, p)$ be the unique value of M which solves the equation $\Lambda(u, p, p \cdot \omega(\underset{\sim}{s}) + M, \underset{\sim}{s}) = (1 + \rho)^{-1}\Lambda(u, p, p \cdot \omega(\underset{\sim}{s}), s)$, where Λ is as in

14). By (14), $\underline{M}(u, \omega, \rho, p) > 0$. It is easy to see that $M(u, \omega, \rho, p)$ is continuous in u, ω and p and increasing in ρ. Let $\underline{M}(\rho, \delta) = \min\{\underline{M}(u, \omega, \rho, p) | (u, \omega, p) \in U \times W \times \Delta_\delta^{L-1}\}$. Clearly, $\underline{M}(\rho, \delta) > 0$ and $\underline{M}(\rho, \delta)$ is continuous and increasing in ρ.

I claim that $\underline{M}(\rho, \delta)$ is as in the lemma. Let $(u, \omega, p) \in U \times W \times \Delta_\delta^{L-1}$. Assumption 9 implies that $\lambda(u, \omega, \rho, p, 0, s) \leqq \Lambda(u, p, p \cdot \omega(\underset{\sim}{s}), \underset{\sim}{s})$, for all s. It follows that if $M < \underline{M}(\rho, \delta)$, then $\lambda(u, \omega, \rho, p, M, \underset{\sim}{s}) \geqq \Lambda(u, p, p \cdot \omega(\underset{\sim}{s}) + M, \underset{\sim}{s}) > (1 + \rho)^{-1}\Lambda(u, p, p \cdot \omega(\underset{\sim}{s}), \underset{\sim}{s}) \geqq (1 + \rho)^{-1} \times \mathrm{E}[\lambda(u, \omega, \rho, p, 0, s_1) | s_0 = \underset{\sim}{s}]$. Hence, inequality (8) implies that $M_0(u, \omega, \rho, p, M, \underset{\sim}{s}) = 0$. That is, $p \cdot g(u, \omega, \rho, p, M, \underset{\sim}{s}) = p \cdot \omega(\underset{\sim}{s}) + M$, if $M < \underline{M}(\rho, \delta)$. By continuity, the same inequality holds if $M = \underline{M}(\rho, \delta)$. □

7. *Proof of Theorem 1*

The next lemma asserts that there exists a stationary distribution of money.

Lemma 3. Corresponding to each $(u, \omega, \rho) \in U \times W \times (0, \infty)$, *and to each* $p \in \operatorname{int} \Delta^{L-1}$, *there is a stationary distribution of initial balances,* $M_{-1}(u, \omega, \rho, p, \underset{\sim}{s})$, *which is unique up to sets of probability zero.*

Proof. Fix (u, ω, ρ) and p. From now on, I drop these variables from the functions M_t, so that $M_t(M, \underset{\sim}{s}) = M_t(u, \omega, \rho, p, M, \underset{\sim}{s})$.

Choose $\delta > 0$ so small that $p \in \Delta_\delta^{L-1}$ and let $\overline{M}(\rho, \delta)$ and $\underline{M}(\rho, \delta)$ be as in Lemmas 1 and 2, respectively. Let N be the smallest integer such that $N\underline{M}(\rho, \delta) > \overline{M}(\rho, \delta)$. For each $T \geqq N$; let $\Sigma_T = \{\underset{\sim}{s} \in \Sigma |$ there is t such that $-T \leqq t \leqq -N$ and $s_t = s_{t+1} = \cdots = s_{t+N-1} = \underset{\sim}{s}\}$. Observe that $\Sigma_T \subset \Sigma_{T+1}$ and $\Sigma_T \in \mathscr{S}_{-1}$, for all T.

I now show that

$$\lim_{T \to \infty} P(\Sigma_T) = 1. \tag{24}$$

Let $\alpha = \min_{s \in S} \operatorname{Prob}[s_1 = \underset{\sim}{s} | s_0 = s]$. By Assumption 9, $\alpha > 0$. Since $\{s_t\}$ is Markov, $P(\Sigma_t) \geqq 1 - (1 - \alpha^N)^{[TN^{-1}]}$, where $[r]$ is the largest integer less than or equal to r. This proves that $\lim_{T \to \infty} P(\Sigma_T) = 1$.

Lemma 2 and the choice of N imply that $M_{N-1}(M, \underset{\sim}{s}) = 0$ if $s_0 = s_1 = \cdots = s_{N-1} = \underset{\sim}{s}$, provided that $0 \leqq M \leqq \overline{M}$. It follows that if $\underset{\sim}{s} \in \Sigma_T$, then $M_T(M, \sigma^{-T-1} s) = M_T(0, \sigma^{-T-1} \underset{\sim}{s})$, provided that $M \leqq \overline{M}(\rho, \delta)$. [$M_T(M, \sigma^{-T-1}s)$ is the money balance of the consumer at the end of period minus one if he has M units at the end of period $-T-2$.] Statement (17) implies that $M_T(0, \sigma^{-T-1}\underset{\sim}{s}) \leqq \overline{M}(\rho, \delta)$, for all T. Hence,

$$M_T\left(0, \sigma^{-T-1}\underset{\sim}{s}\right) = M_{T'}\left(0, \sigma^{-T'-1}\underset{\sim}{s}\right) \quad \text{if} \quad T' \geqq T \quad \text{and} \quad \underset{\sim}{s} \in \Sigma_T. \quad (25)$$

Assertions (24) and (25) imply that $\lim_{T \to \infty} M_T(0, \sigma^{-T-1}\underset{\sim}{s})$ exists, for almost every $\underset{\sim}{s}$. Let $M_{-1}(\underset{\sim}{s})$ be this limit. $M_{-1}(\underset{\sim}{s})$ is the stationary money balance. In order to see that it is stationary, observe that

$$\begin{aligned} M_0(M_{-1}, \underset{\sim}{s}) &= M_0\left(\lim_{T \to \infty} M_T\left(0, \sigma^{-T-1}\underset{\sim}{s}\right), \underset{\sim}{s}\right) \\ &= \lim_{T \to \infty} M_0\left(M_T\left(0, \sigma^{-T-1}\underset{\sim}{s}\right), \underset{\sim}{s}\right) \\ &= \lim_{T \to \infty} M_{T+1}\left(0, \sigma^{-T-2}\sigma\underset{\sim}{s}\right) = M_{-1}(\sigma\underset{\sim}{s}). \end{aligned}$$

It should be clear that M_{-1} is unique up to sets of probability zero. □

It is now possible to prove Theorem 1. Let $(u, \omega, \rho) \in U \times W \times (0, \infty)$ and let $p \in \operatorname{int} \delta^{L-1}$. Let $x_t(\underset{\sim}{s}) = g(u, \omega, \rho, p, M_{t-1}(u, \omega, \rho, p, \underset{\sim}{s}), s_t)$. $\underset{\sim}{x} = (x_0, x_1, \dots)$ is a stationary program. By (12), $\underset{\sim}{x}$ solves problem (2) with initial balances $M_{-1}(u, \omega, \rho, p, s)$. Thus, I may let $\xi(p) = \underset{\sim}{x}$ and $M_{-1}(p) = M_{-1}(u, \omega, \rho, p, \underset{\sim}{s})$.

8. Continuity of the stationary distribution

In this section, I demonstrate that the stationary distribution of money is continuous with respect to the parameters u, ω, ρ and p. Let $\overline{M}(\rho, \delta)$ be as in Lemma 1 and let ρ_1 and ρ_2 be as in Assumption 7.

Lemma 4. For each $\delta < 0$ and $s \in S$, $g(u, \omega, \rho, p, M, s)$ and $\lambda(u, \omega, \rho, p, M, s)$ are uniformly continuous with respect to (u, ω, ρ, p, M) on $U \times [\rho_1, \rho_2] \times W \times \Delta_\delta^{L-1} \times [0, \overline{M}(\rho_1, \delta)]$.

Proof. It is easy to see that these functions are continuous. Uniform continuity follows because the set $U \times [\rho_1, \rho_2] \times W \times \Delta_\delta^{L-1} \times [0, \overline{M}(\rho, \delta)]$ is compact. □

Corollary 2. For each $\delta > 0$, $t = 0, 1, \dots,$ and each $\underset{\sim}{s} \in \Sigma$, $M_t(u, \omega, \rho, p, M, \underset{\sim}{s})$ is uniformly continuous with respect to (u, ω, ρ, p, M) on $U \times W \times [\rho_1, \rho_2] \times \Delta_\delta^{L-1} \times [0, \overline{M}(\rho_1, \delta)]$.

Given $\delta > 0$, let $N(\delta)$ be the smallest integer such that $N(\delta)\underline{M}(\rho_1, \delta) > \overline{M}(\rho_1, \delta)$. For each $T \geqq N(\delta)$, define $\Sigma_T(\delta)$ as in the proof of Lemma 3. That is $\Sigma_T(\delta) = \{\underset{\sim}{s} \in \Sigma |$ there is t such that $-T \leqq t \leqq -N(\delta)$ and $s_t = s_{t-1} = \cdots = s_{T+N(\delta)-1} = \underset{\sim}{s}\}$.

Lemma 5. Let $\delta > 0$. For any $T \geqq N(\delta)$ and any $t \geqq T$, $M_t(u, \omega, \rho, p, 0, \sigma^{-t-1}\underset{\sim}{s})$ is uniformly continuous with respect to (u, ω, ρ, p) on $U \times W \times [\rho_1, \rho_2] \times \Delta_\delta^{L-1}$, uniformly for $\underset{\sim}{s} \in \Sigma_T$.

Proof. Note that it is appropriate to use ρ_1 to define $N(\delta)$, since $\underline{M}(\rho, \delta)$ is increasing in ρ and $\overline{M}(\rho, \delta)$ is non-increasing in ρ. If $\underset{\sim}{s} \in \Sigma_T$, then for some n such that $t - T < n \leqq t$, $M_n(u, \omega, \rho, p, 0, \sigma^{-t-1}\underset{\sim}{s}) = 0$ and so $M_t(u, \omega, \rho, p, 0, \sigma^{-t-1}\underset{\sim}{s}) = M_{t-n}(u, \omega, \rho, p, 0, \sigma^{-t+n-1}\underset{\sim}{s})$. Since $0 \leqq t - n < T$, the lemma follows from the previous corollary. □

Let $M_{-1}(u, \omega, \rho, p, s)$ be the stationary initial balances defined in the previous section and suppose that $p \in \Delta_\delta^{L-1}$ and $\rho \geqq \rho_1$. $M_{-1}(u, \omega, \rho, p, \underset{\sim}{s})$ is defined only up to sets in Σ of probability zero. However, I can choose M_{-1} so that

$$M_{-1}(u, \omega, \rho, p, \underset{\sim}{s}) = M_t\big(u, \omega, \rho, p, 0, \sigma^{-t-1}\underset{\sim}{s}\big), \qquad \text{for } \underset{\sim}{s} \in \Sigma_T(\delta), \quad \text{for all } T. \tag{26}$$

For the rest of this paper, I will assume that $M_{-1}(u, \omega, \rho, p, \underset{\sim}{s})$ has been chosen in this way.

Corollary 3. Let $\delta > 0$. For each T, $M_{-1}(u, \omega, \rho, p, \underset{\sim}{s})$ is uniformly continuous with respect to (u, ω, ρ, p) on $U \times W \times [\rho_1, \rho_1] \times \Delta_\delta^{L-1}$, uniformly for $\underset{\sim}{s} \in \Sigma_T$.

Let $x_0(u, \omega, \rho, p, \underset{\sim}{s}) = g(u, \omega, \rho, p, M_{-1}(u, \omega, \rho, p, \underset{\sim}{s}), s_0)$, which is the stationary demand of a consumer of type (u, ω, ρ) when prices are p.

Corollary 4. Let $\delta > 0$. For each T, $x_0(u, \omega, \rho, p, \underset{\sim}{s})$ is uniformly continuous with respect to (u, ω, ρ, p) on $U \times W \times [\rho_1, \rho_2] \times \Delta_\delta^{L-1}$, uniformly for $\underset{\sim}{s} \in \Sigma_T$.

The last two corollaries are used to prove Theorem 2.

9. Proof of Theorem 2

The next lemma defines a lower bound on equilibrium prices.

Lemma 6. There exists $\delta > 0$ such that if $p \in \operatorname{int} \Delta^{L-1}$ and $p_k \leqq \delta$, for any k, then $\mathrm{E}\prod_{k=1}^{L} g_k(u, \omega, \rho, p, M, s_0) \geqq 2\mathrm{E}\prod_{k=1}^{L} \omega_k(s_0)$, *for all* $(u, \omega, \rho, M) \in U \times W \times (0, \infty) \times [0, \infty)$.

Proof. By Assumption 8, $\omega_k(s) > 0$, for all $\omega \in W$ and all k and s. Since W is compact, there is $\varepsilon > 0$ such that $\varepsilon^{-1} \geqq \omega_k(s) \geqq \varepsilon$, for all $\omega \in W$ and all k and s.

Lemma 2 implies that $p \cdot g(u, \omega, \rho, p, M, \underline{s}) \geqq p \cdot \omega(\underline{s}) \geqq \varepsilon$, for all $(u, \omega, \rho, p, M) \in U \times W \times (0, \infty) \times \operatorname{int} \Delta^{L-1} \times [0, \infty)$, where $\underline{s}$ is as in Assumption 9.

The compactness of U and the monotonicity of utility functions in U (Assumption 6) imply that there exists $\delta > 0$ such that

$$\sum_{k=1}^{L} \bar{x}_k \geqq 2(\operatorname{Prob}[s_0 = \underline{s}])^{-1} L \varepsilon^{-1},$$

whenever $\bar{x}$ solves the problem $\max\{u(x, \underline{s}) | p \cdot x \leqq w\}$ for some $u \in U$, some $w \geqq \varepsilon$, and some $p \in \operatorname{int} \Delta^{L-1}$ such that $p_k \leqq \delta$, for some k.

If $p_k \leqq \delta$, for some k, then

$$\begin{aligned} \mathrm{E} \sum_{k=1}^{L} g_k(u, \omega, \rho, p, M, s_0) &\geqq \operatorname{Prob}[s_0 = \underline{s}] \sum_{k=1}^{L} g_k(u, \omega, \rho, p, M, \underline{s}) \\ &\geqq 2L\varepsilon^{-1} \geqq 2\mathrm{E} \sum_{k=1}^{L} \omega_k(s_0). \quad \square \end{aligned}$$

Now let $\xi_a(p) = (x_{a0}(p), x_{a1}(p), \ldots)$ and $M_{a,-1}(p)$ be the stationary demand and initial balance distribution, respectively, for agent $a \in [0, 1]$ of characteristics (u_a, ω_a, ρ_a). Let

$$z(p) = \int_0^1 \mathrm{E}x_{a0}(p)\, \mathrm{d}a - \int_0^1 \mathrm{E}\omega_a(s_0)\, \mathrm{d}a,$$

where

$$\mathrm{E}x_{a0}(p) = \int x_{a0}(p, \underline{s}) P(\mathrm{d}\underline{s}).$$

Since $x_{a0}(p, \underset{\sim}{s}) = g(u_a, \omega_a, \rho_a, p, M_{a,-1}(\underset{\sim}{s}), s_0)$, it follows from the previous lemma that

$$\sum_{k=1}^{L} z_k(p) > 0 \quad \text{if} \quad p \in \operatorname{int} \Delta^{L-1} \quad \text{and} \quad p_k \leqq \delta, \quad \text{for any } k. \tag{27}$$

Since the program $\zeta_a(p)$ is associated with a stationary distribution of money balances, it follows that $\mathrm{E}p \cdot x_{a0}(p) - \mathrm{E}p \cdot \omega_a(s_0) = \mathrm{E}M_{a0} - \mathrm{E}M_{a,-1} = 0$, where $M_{a0}(\underset{\sim}{s}) = M_{a,-1}(\sigma \underset{\sim}{s})$. Hence

$$p \cdot z(p) = 0 \quad \text{for all} \quad p \in \operatorname{int} \Delta^{L-1}. \tag{28}$$

Since $(u_a, \omega_a, \rho_a) \in U \times W \times [\rho_1, \rho_2]$, for all a, it follows from Corollary 4 that

$$z(p) \text{ is continuous on } \operatorname{int} \Delta^{L-1}. \tag{29}$$

Statements (27)–(29) are the conditions of lemma 1 of Hildenbrand (1974, p. 150). This lemma asserts that there exists $\bar{p} \in \operatorname{int} \Delta^{L-1}$ such that $z(\bar{p}) = 0$. $\bar{p}$ is the stationary equilibrium price vector. This proves Theorem 2. For future reference, I record the following:

$$\text{Any stationary equilibrium price vector, } p, \text{ belongs to } \Delta_\delta^{L-1}, \text{ where } \delta \text{ is as in Lemma 6. } \delta \text{ does not depend on } \rho_1 \text{ and } \rho_2. \tag{30}$$

10. Proof of Theorem 3

I prove the following lemma, where δ is as in Lemma 6:

Lemma 7. For every $M > 0$ and $\varepsilon > 0$, there exists $\underline{\rho} > 0$ and a positive integer T such that Prob$[M_T(u, \omega, \rho, p, 0, \underset{\sim}{s}) \geqq \bar{M}] \geqq 1 - \varepsilon$, *for all* $(u, \omega, \rho, p) \in U \times W \times (0, \underline{\rho}] \times \Delta_\delta^{L-1}$.

This lemma implies the theorem, for as has just been pointed out, any equilibrium price vector belongs to Δ_δ^{L-1}. Also, $(u_a, \omega_a) \in U \times W$, so that the lemma applies to every consumer a. Finally, $M_{a,-1}(p, \sigma^{T+1}\underset{\sim}{s}) = M_T(u_a, \omega_a, \rho_a, p, M_{a,-1}, \underset{\sim}{s}) \geqq M_T(u_a, \omega_a, \rho_a, p, 0, \underset{\sim}{s})$. Hence if $\rho_a \leqq \underline{\rho}$ and p is an equilibrium price vector, then Prob$[M_{a,-1}(p, \underset{\sim}{s}) \geqq \bar{M}] =$ Prob$[M_{a,-1}(p, \sigma^{T+1}\underset{\sim}{s}) \geqq M] \geqq$ Prob$[M_T(u_a, \omega_a, \rho_a, p, 0, \underset{\sim}{s}) \geqq \bar{M}] \geqq 1 - \varepsilon$.

In order to prove Lemma 7, I make use of a program which is optimal for $\rho = 0$. Lemmas 5.6 and 5.10 in Bewley (1977) assert that $\lambda(u, \omega, \rho, p, M, s)$ and $p \cdot g(u, \omega, \rho, p, M, s)$ are, respectively, non-decreasing and non-increasing functions of ρ. Since $\lambda(u, \omega, \rho, p, M, s) \leqq \bar{\lambda} \max_k p_k^{-1}$ [by (16)] and since $p \cdot g(u, \omega, \rho, p, M, s) \geqq 0$, the limits exist as ρ goes to zero. The existence of these limits implies that $g(u, \omega, \rho, p, M, s)$ converges as ρ goes to zero. Let $\lambda(u, \omega, 0, p, M, s) = \lim_{\rho \to 0} \lambda(u, \omega, \rho, p, M, s)$ and $g(u, \omega, 0, p, M, s) = \lim_{\rho \to 0} g(u, \omega, \rho, p, M, s)$. It is easy to see that

$$\frac{\partial u}{\partial x_k}(g(u, \omega, 0, p, M, s), s) \leqq \lambda(u, \omega, 0, p, M, s) p_k,$$
$$\text{for all } k, \text{ with equality if } g_k(u, \omega, 0, p, M, s) > 0. \tag{31}$$

Let $M_t(u, \omega, 0, p, M, \underset{\sim}{s})$ be the money balance at the end of period t determined by the $g(u, \omega, 0, p, \cdot, s)$ and by initial balances M,

Lemma 8. If $\rho' > \rho'' \geqq 0$, then $M_t(u, \omega, \rho', p, M, \underset{\sim}{s}) \leqq M_t(u, \omega, \rho'', p, M, \underset{\sim}{s})$, for all t, u, ω, p, M and $\underset{\sim}{s}$.

Proof. For brevity, I drop u, ω and p from the variables of M_t.

The proof is by induction on t. The statement is true for $t = -1$, since $M_{-1}(\rho', M, s) = M = M_{-1}(\rho'', M, s)$. Suppose by induction that it is true for t. Then,

$$\begin{aligned} M_{t+1}(\rho', M, \underset{\sim}{s}) &= M_t(\rho', M, \underset{\sim}{s}) + p \cdot \omega(s_{t+1}) \\ &\quad - p \cdot g(\rho', M_t(\rho', M, \underset{\sim}{s}), s_{t+1}) \\ &\leqq M_t(\rho'', M, \underset{\sim}{s}) + p \cdot \omega(s_{t+1}) \\ &\quad - p \cdot g(\rho', M_t(\rho'', M, \underset{\sim}{s}), s_{t+1}) \\ &\leqq M_t(\rho'', M, \underset{\sim}{s}) + p \cdot \omega(s_{t+1}) \\ &\quad - p \cdot g(\rho'', M_t(\rho'', M, \underset{\sim}{s}), s_{t+1}) \\ &= M_{t+1}(\rho'', M, \underset{\sim}{s}). \end{aligned}$$

The first inequality follows from inequality (10) and from the induction hypothesis. The second inequality follows from the fact that $p \cdot g(\rho, M, s)$ is a non-increasing function of ρ. This completes the induction step. □

Lemma 9. $\lim_{\rho \to 0} M_t(u, \omega, \rho, p, M, \underset{\sim}{s}) = M_t(u, \omega, 0, p, M, \underset{\sim}{s})$, *for all* t, u, ω, p, M *and* $\underset{\sim}{s}$.

Proof. I again drop u, ω and p from the variables of M_t and proceed by induction on t. The statement is clearly true for $t = -1$, so suppose that it is true for t. Then,

$$\begin{aligned} & M_{t+1}(\rho, M, s) - p \cdot \omega(s_{t+1}) \\ &= M_t(\rho, M, s) - p \cdot g(\rho, M_t(\rho, M, \underset{\sim}{s}), s_{t+1}) \\ &\geqq M_t(\rho, M, \underset{\sim}{s}) - p \cdot g(\rho, M_t(0, M, \underset{\sim}{s}), s_{t+1}). \end{aligned} \tag{32}$$

The above inequality follows from the previous lemma and the fact that $p \cdot g(\rho, M, \underset{\sim}{s})$ is a non-decreasing function of M [statement (9)]. By the induction assumption and the definition of $g(0, M, s)$,

$$\begin{aligned} & \lim_{\rho \to 0} \left[M_t(\rho, M, \underset{\sim}{s}) - p \cdot g(\rho, M_t(0, M, \underset{\sim}{s}), s_{t+1}) \right] \\ &= M_t(0, M, \underset{\sim}{s}) - p \cdot g(0, M_t(0, M, \underset{\sim}{s}), s_{t+1}) \\ &= M_{t+1}(0, M, \underset{\sim}{s}) - p \cdot \omega(s_{t+1}). \end{aligned} \tag{33}$$

Inequality (32) and (33) imply that $\lim_{\rho \to 0} M_{t+1}(\rho, M, \underset{\sim}{s}) \geqq M_{t+1}(0, M, \underset{\sim}{s})$. Since by the previous lemma the opposite inequality is valid, the lemma is proved. □

Let $\lambda_\infty(u, \omega, p)$ be defined by (15). In Bewley (1977) it is proved that

$$\lambda(u, \omega, 0, p, M, s) \geqq \lambda_\infty(u, \omega, p), \tag{34}$$

for all values of the variables.

I now prove that the above inequality is strict in the case of this paper.

Lemma 10. $\lambda(u, \omega, 0, p, M, s) > \lambda_\infty(u, \omega, p)$, *for all* $(u, \omega, p, M, s) \in U \times W \times \operatorname{int} \Delta^{L-1} \times [0, \infty) \times S$.

Proof. Fix $(u, \omega, p) \in U \times W \times \operatorname{int} \Delta^{L-1}$ and let $\lambda_\infty = \lambda_\infty(u, \omega, p)$ and $x_\infty(s) = G(u, p, \lambda_\infty, s)$, where G is as in (13). By the definition of λ_∞, $\mathrm{E}p \cdot x_\infty(s_0) = \mathrm{E}p \cdot \omega(s_0)$. Assumption 9 implies that $\mathrm{E}p \cdot \omega(s_0) > p \cdot \omega(\underset{\sim}{s})$ and that $p \cdot x_\infty(\underset{\sim}{s}) \geqq \mathrm{E}p \cdot x_\infty(s_0)$. Therefore, $p \cdot x_\infty(\underset{\sim}{s}) > p \cdot \omega(\underset{\sim}{s})$.

Let $\lambda(M, s) = \lambda(u, \omega, 0, p, M, s)$ and let $M_t(M, \underline{s}) = M_t(u, \omega, 0, p, M, \underline{s})$. Lemma 8.1 of Bewley (1977) asserts that

$$\lambda(M, s) \geqq \mathrm{E}[\lambda(M_0(M, \underline{s}), s_1)|s_0 = s], \quad \text{for all } s. \tag{35}$$

This inequality and Assumption 2 imply that in order to prove the lemma it is enough to prove that

$$\lambda(M, \underline{s}) > \lambda_\infty, \quad \text{for all} \quad M \geqq 0. \tag{36}$$

Let $M' = \inf\{M \geqq 0 | \lambda(M, \underline{s}) = \lambda_\infty\}$ and suppose that $M' < \infty$. Let $\hat{M} = M' + \frac{1}{2}(p \cdot x_\infty(\underline{s}) - p \cdot \omega(\underline{s}))$. Since $\hat{M} > M'$ and $\lambda(M, \underline{s})$ is a non-increasing function of M, it follows from (34) that $\lambda(\hat{M}, \underline{s}) = \lambda_\infty$. Hence, $M_0(\hat{M}, \underline{s}) = M + p \cdot \omega(\underline{s}) - p \cdot x_\infty(\underline{s}) < M'$, so that $\lambda(M_0(\hat{M}, \underline{s}), \underline{s}) > \lambda_\infty$. This inequality, together with inequalities (34) and (35), implies that $\lambda(\hat{M}, \underline{s}) > \lambda_\infty$. This contradiction implies that $M' = \infty$. This proves (36) and hence the lemma. □

Theorems 3.1 and 3.2 of Bewley (1977) assert that

$$\lim_{M \to \infty} \lambda(u, \omega, 0, p, M, s) = \lambda_\infty(u, \omega, p),$$

and

$$\lim_{t \to \infty} \lambda(u, \omega, 0, p, M_t(u, \omega, 0, p, 0, \underline{s}), s_t) = \lambda_\infty(u, \omega, p),$$

almost surely, for all values of the variables. These facts together with the previous lemma imply that

$$\lim_{t \to \infty} M_t(u, \omega, 0, p, 0, \underline{s}) = \infty \quad \text{almost surely,}$$
$$\text{for all} \quad (u, \omega, p) \in U \times W \times \operatorname{int} \Delta^{L-1}. \tag{37}$$

I may now prove Lemma 7.

Proof of Lemma 7. Since $U \times W \times \Delta_\delta^{L-1}$ is compact, it is enough to show that for each $(\bar{u}, \bar{\omega}, \bar{p})$ in this set, there exist a postive integer T, a $\rho > 0$ and a neighborhood $\mathcal{N}$ of $(\bar{u}, \bar{\omega}, \bar{p})$ such that $\operatorname{Prob}[M_T(u, \omega, \rho, p, 0, \underline{s}) \geqq M] \geqq 1 - \varepsilon$, for all $(u, \omega, p) \in \mathcal{N}$ and for all $\rho \leqq \rho$.

So, let $(\bar{u}, \bar{\omega}, \bar{p})$ be fixed. By (37), there exists T such that $\operatorname{Prob}[M_T(\bar{u}, \bar{\omega}, 0, \bar{p}, 0, \underline{s}) \geqq M + 2] \geqq 1 - \varepsilon$. M_T depends on s only through

$s_0, \ldots, s_T$. Since there are only finitely many of these sequences, Lemma 9 implies that there is $\underline{\rho} > 0$ so small that $\text{Prob}[M_T(\bar{u}, \bar{\omega}, \underline{\rho}, \bar{p}, 0, \underset{\sim}{s}) \geqq M + 1] \geqq 1 - \varepsilon$. It follows from the uniform continuity of M_T with respect to u, ω and p (Corollary 2) that there exists a neighborhood $\mathcal{N}$ of $(\bar{u}, \bar{\omega}, \bar{p})$ such that $\text{Prob}[M_T(u, \omega, \underline{\rho}, p, 0, \underset{\sim}{s}) \geqq M] \geqq 1 - \varepsilon$, for all $(u, \omega, p) \in \mathcal{N}$. Since M_T is non-increasing in ρ (Lemma 8), $\text{Prob}[M_T(u, \omega, \rho, p, 0, \underset{\sim}{s}) \geqq M] \geqq 1 - \varepsilon$, if $0 \leqq \rho \leqq \underline{\rho}$, for $(u, \omega, p) \in \mathcal{N}$. □

11. *Proof of Theorem 4*

The proof depends on the following lemma.

Lemma 11. Let $\delta > 0$. $\lim_{M \to \infty} \lambda(u, \omega, 0, p, M, s) = \lambda_\infty(u, \omega, p)$ *uniformly in* $(u, \omega, p) \in U \times W \times \Delta_\delta^{L-1}$.

Proof. We here follow the proof of lemma 7.9 in Bewley (1979). It is proved there that $\lambda(u, \omega, 0, p, s) \leqq \lambda_\infty(u, \omega, p) + \varepsilon + \eta(u, \omega, p, M, s)$, for all s and all $M \geqq 1$, where $\lim_{M \to \infty} \eta(u, \omega, p, M, s) = 0$, for all s.

I now prove:

$$\text{If } M \text{ is sufficiently large, then } \eta(u, \omega, p, M, s) \leqq \varepsilon, \text{ for all } (u, \omega, p, s) \in U \times W \times \Delta_\delta^{L-1} \times S. \tag{38}$$

Let

$$N_T(u, \omega, p, \underset{\sim}{s}) = \sum_{t=0}^{T} \left[p \cdot \omega(s_t) - p \cdot G(u, p, \lambda_\infty(u, \omega, p) + \varepsilon, s_t) \right],$$

where G is as in (13). $N_T(u, \omega, p, \underset{\sim}{s})$ is the money balance at the end of period T if the consumer has no money at the beginning of period zero and if he keeps his marginal utility of money always equal to $\lambda_\infty(u, \omega, p) + \varepsilon$.

In the notation of this paper,

$$\eta(u, \omega, p, M, s) = \text{Prob}\left[\inf_{0 \leq t < \infty} N_t(u, \omega, p, \underset{\sim}{s}) \leqq 1 - M | s_0 = s \right]. \tag{39}$$

The strong law of large numbers implies that

$$\lim_{t \to \infty} \left[N_T(u, \omega, p, \underset{\sim}{s}) - \frac{t+1}{2} \times \mathrm{E}(p \cdot \omega(s_0) - p \cdot G(u, p, \lambda_\infty(u, \omega, p) + \varepsilon, s_t)) \right] = \infty,$$

almost surely, for each u, ω, and p.

Fix $(\bar{u}, \bar{\omega}, \bar{p}) \in U \times W \times \Delta_\delta^{L-1}$. By the previous equation, there exists T such that

$$\text{Prob}\left[N_t(\bar{u}, \bar{\omega}, \bar{p}, \underline{s}) - \frac{t+1}{2}\text{E}(p \cdot \bar{\omega}(s_0) - p \cdot G(\bar{u}, \bar{p}, \lambda_\infty(\bar{u}, \bar{\omega}, \bar{p}) + \varepsilon, s_0)) \geqq 0, \text{ for all } t > T | s_0 = s\right] > 1 - \varepsilon, \quad \text{for all } s. \tag{40}$$

Clearly, there exists M such that

$$\text{Prob}\left[N_t(\bar{u}, \bar{\omega}, \bar{p}, \underline{s}) \leqq -M, \text{for some } t = 0, 1, \ldots, T | s_0 = s\right] = 0, \quad \text{for all } s. \tag{41}$$

Since $p \cdot \omega(s) - p \cdot G(u, p, \lambda_\infty(u, \gamma, p) + \varepsilon, s)$ is continuous in u, ω and p, there exists a neighborhood $\mathscr{N}$ of $(\bar{u}, \bar{\omega}, \bar{p})$ such that for all $(u, \omega, p) \in \mathscr{N}$,

$$\text{Prob}\left[N_t(u, \omega, p, \underline{s}) \leqq -M + 1, \text{ for some } t = 0, 1, \ldots, T | s_0 = s\right] = 0, \quad \text{for all } s, \tag{42}$$

and

$$|p \cdot \omega(s) - p \cdot G(u, p, \lambda_\infty(u, \omega, p) + \varepsilon, s) - p \cdot \bar{\omega}(s) + p \cdot G(\bar{u}, \bar{p}, \lambda_\infty(\bar{u}, \bar{\omega}, \bar{p}) + \varepsilon, s)| < \tfrac{1}{2}\text{E}\left[p \cdot \bar{\omega}(s_0) - p \cdot G(\bar{u}, \bar{p}, \lambda_\infty(\bar{u}, \bar{\omega}, \bar{p}) + \varepsilon, s_0)\right], \quad \text{for all } s. \tag{43}$$

[The definition of λ_∞ implies that the right-hand side of (43) is positive.]

Inequalities (40) and (43) imply that

$$\text{Prob}\left[N_t(u, \omega, p, \underline{s}) \geqq 0, \text{for all } t > T | s_0 = s\right] > 1 - \varepsilon, \quad \text{for all } (u, \omega, p) \in \mathscr{N}. \tag{44}$$

Equation (42) and inequality (44) imply that $\eta(u, \omega, p, M, s) < \varepsilon$, for all $(u, \omega, p) \in \mathscr{N}$ and for all s. Since $\eta(u, \omega, p, M, s)$ is non-increasing in M and since $U \times W \times \Delta_\delta^{L-1}$ is compact, an obvious compactness argument completes the proof. □

Recall from (30) that any equilibrium price vector belongs to Δ_δ^{L-1}, where δ is as in Lemma 6. It follows that the next lemma proves Theorem 4.

Lemma 12. For every $\varepsilon > 0$, *there exists* $\underline{\rho} > 0$ *such that if* $0 < \rho \leqq \underline{\rho}$, *then* $\text{Prob}[|\lambda(u, \omega, \rho, p, M_{-1}(u, \omega, \rho, p, \underline{s}), s_0) - \lambda_\infty(u, \omega, p)| > \varepsilon] < \varepsilon$, *for all* $(u, \omega, p) \in U \times W \times \Delta_\delta^{L-1}$

Proof. For the moment, assume that

$$p \cdot G(u, p, \lambda_\infty(u, \omega, p) - \varepsilon, s) > 0,$$
$$\text{for all} \quad (u, \omega, p, s) \in U \times W \times \Delta_\delta^{L-1} \times S. \tag{45}$$

Recall that $p \cdot G(u, p, \lambda, s)$ is continuous in all its variables and is strictly decreasing in λ as long as $p \cdot G(u, p, \lambda, s) > 0$ [see (13)]. Therefore, inequality (45) implies that there exists γ, such that $0 < \gamma < 1$, and

$$p \cdot G(u, p, \lambda, s) > p \cdot G(u, p, \lambda_\infty(u, \omega, p), s) + \gamma,$$
$$\text{whenever} \quad \lambda < \lambda_\infty(u, \omega, p) - \varepsilon,$$
$$\text{for all} \quad (u, \omega, p, s) \in U \times W \times \Delta_\delta^{L-1} \times S. \tag{46}$$

Let $\zeta_1 = \varepsilon\gamma(2y + 2)^{-1}$, where $y = \max\{p \cdot G(u, p, \lambda_\infty(u, \omega, p), s)| (u, \omega, p, s) \in U \times W \times \Delta_\delta^{L-1} \times S\}$. Let $\zeta_2 > 0$ be such that $\zeta_2 < \varepsilon$ and $|p \cdot G(u, p, \lambda, s) - p \cdot G(u, p, \lambda_\infty(u, \omega, p), s)| \leqq \zeta_1$, whenever $|\lambda - \lambda_\infty(u, \omega, p)| \leqq \zeta_2$, for all $(u, \omega, p, s) \in U \times W \times \Delta_\delta^{L-1} \times S$.

By Lemma 11, there exists M so large that

$$\lambda(u, \omega, 0, p, M, s) \leqq \lambda_\infty(u, \omega, p) + \zeta_2,$$
$$\text{for all} \quad (u, \omega, p, s) \in U \times W \times \Delta_\delta^{L-1} \times S. \tag{47}$$

By Lemma 7, there exists $\underline{\rho} > 0$ so small that

$$\text{Prob}\left[M_{-1}(u, \omega, \rho, p, \underline{s}) < M\right] < \zeta_1,$$
$$\text{for all} \quad (u, \omega, \rho, p) \in U \times W \times (0, \underline{\rho}] \times \Delta_\delta^{L-1}. \tag{48}$$

I now fix $(u, \omega, \rho, p) \in U \times W \times (0, \rho] \times \Delta_\delta^{L-1}$ and drop these variables from the functions G, λ, λ_∞ and M_{-1}. I prove that

$$\text{Prob}\left[(\lambda(M_{-1}(\underline{s}), s_0) - \lambda_\infty| > \varepsilon\right] < \varepsilon. \tag{49}$$

It follows from the choices of M and $\underline{\rho}$ [(47) and (48)] that

$$\text{Prob}\left[\lambda(M_{-1}(\underset{\sim}{s}), s_0) > \lambda_\infty + \zeta_2\right] < \zeta_1. \tag{50}$$

Since $\zeta_1 < \varepsilon/2$ and $\zeta_2 < \varepsilon$, it follows that

$$\text{Prob}\left[\lambda(M_{-1}(\underset{\sim}{s}), s_0) > \lambda_\infty + \varepsilon\right] < \varepsilon/2. \tag{51}$$

If $M_{-1}(\underset{\sim}{s}) \leqq M$, then $\lambda(M_{-1}(\underset{\sim}{s}), s_0) \leqq \lambda_\infty + \zeta_2$, so that $p \cdot G(\lambda(M_{-1}(\underset{\sim}{s}), s_0), s_0) - p \cdot G(\lambda_\infty, s_0) \geqq -\zeta_1$. If $M_{-1}(\underset{\sim}{s}) < M$, then certainly $p \cdot G(\lambda(M_{-1}(\underset{\sim}{s}), s_0), s_0) \geqq 0$, so that $p \cdot G(\lambda(M_{-1}(s_0), s_0), s_0) - p \cdot G(\lambda_\infty, s_0) \geqq -y$. [$y$ is defined just after (46).] Since $\mathrm{E}p \cdot G(\lambda(M_{-1}(\underset{\sim}{s}), s_0), s_0) = \mathrm{E}p \cdot \omega(s_0) = \mathrm{E}p \cdot G(\lambda_\infty, s_0)$, it follows [using (48)] that

$$\text{Prob}\left[p \cdot G(\lambda(M_{-1}(\underset{\sim}{s}), s_0) > p \cdot G(\lambda_\infty, s_0) + \gamma\right]$$

$$< \gamma^{-1}\zeta_1(y+1) = \varepsilon/2. \tag{52}$$

It now follows from the choice of γ [see (46)] that

$$\text{Prob}\left[\lambda(M_{-1}(s), s_0) < \lambda_\infty - \varepsilon\right] < \varepsilon/2. \tag{53}$$

Inequalities (51) and (53) together imply inequality (49). This proves the lemma, given assumption (45).

If assumption (45) is dropped, then inequalities (51) and (52) remain valid and inequality (53) is replaced by

$$\text{Prob}\left[\lambda(M_{-1}(\underset{\sim}{s}), s_0) < \lambda_\infty - \varepsilon \text{ and } pG(\lambda_\infty - \varepsilon, s_0) > 0\right] < \varepsilon/2. \tag{54}$$

I now sketch verbally how the argument may be completed. There must be some state s such that $p \cdot G(\lambda_\infty - \varepsilon, s) > 0$. If a consumer receives an additional small amount of money, he can choose to wait and spend it only when he arrives in s. He will with high probability arrive in s before time T, if T is large. Therefore, if ρ is small, his utility gain per additional unit of money will with high probability exceed $\lambda_\infty - 2\varepsilon$. It is easy to fill in the details of this argument. □

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