SUPPLEMENT TO "PUTTING THE 'FINANCE' INTO 'PUBLIC FINANCE': A THEORY OF CAPITAL GAINS TAXATION"

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1. APPENDIX FOR SECTION 1

1.1. *Mapping the Neoclassical Growth Model into Our Setup*

As is standard, a representative consumer has preferences $\sum_{t=0}^{\infty} \beta^t U(C_t)$ where C_t is consumption, the consumption good is produced according to a constant-returns technology $\mathcal{Y}_t = f(K_t, A_t L_t)$ where K_t is capital, A_t is productivity and L_t is labor, labor is supplied inelastically $L_t = 1$, and the resource constraint is

$$C_t + I_t = \mathcal{Y}_t, \quad K_{t+1} = I_t + (1-\delta)K_t, \tag{A-1}$$

where I_t is investment. Importantly, the fact that the consumption good $\mathcal{Y}_t$ can be converted into investment one-for-one immediately pins down the unit price of capital (relative to consumption) at one. We discuss this property in more detail momentarily where we also discuss how to break it.

The Price of Capital in Variants of the Growth Model

To understand why the unit price of capital is pinned down at one in the growth model and to see how to break this result, it is useful to consider a more general model in which the result does not necessarily hold: a two-sector growth model with a separate investment goods production sectors. The model is the same as in Section 1.5 except that the resource constraint is

$$C_t + \iota_t = \mathcal{Y}_t, \qquad I_t = G_t(\iota_t), \qquad K_{t+1} = I_t + (1-\delta)K_t. \tag{A-2}$$

Here ι_t units of the consumption produce I_t units of investment according to a production function G_t which is increasing but which may be concave $G_t'' \leq 0$ or may vary over time. Profit maximization of investment goods producer is

$$\max_{\iota_t} \; p_t G_t(\iota_t) - \iota_t.$$

As long as the marginal product $G'(\iota_t)$ is positive, producers choose ι_t to satisfy the optimality condition

$$p_t G_t'(\iota_t) = 1. \tag{A-3}$$

This model has two interesting polar special cases:

1. Neoclassical growth model: $I_t = G(\iota_t) = \iota_t$ so that $G'(\iota_t) = 1$. In this case the resource constraints (A-2) become $C_t + I_t = \mathcal{Y}_t$ and the optimality condition (A-3) implies $p_t = 1$.
2. Capital in fixed supply (Section 6.1): $G(\iota_t) = 0$ for all ι_t and $\delta = 0$. In this case $I_t = \iota_t = 0$ so that the resource constraints (A-2) become $C_t = \mathcal{Y}_t$. The price of capital p_t is pinned down by market clearing $I_t = 0$ rather than the optimality condition (A-3) because there is no optimization problem for investment goods production.

Growth Model with a Stock Market

This appendix spells out in more detail a decentralization of the growth model in which households trade shares in the representative firm which are in unit fixed supply. The budget constraint of the representative household is

$$p_t(S_{t+1} - S_t) + C_t = Y_t + D_t S_t.$$

Here, each share S_t is a claim on the profits of the representative firm. In equilibrium, shares are in unit fixed supply, so $S_t = 1$, and enoting the wage by W_t, the firm's cash flows are $D_t = \mathcal{Y}_t - W_t L_t - I_t$. Using that labor is paid its marginal product $W_t = f_L(K_t, A_t L_t)A_t$, that $f(K_t, A_t L_t) = f_K(K_t, A_t L_t)K_t + f_L(K_t, A_t L_t)A_t L_t$ because of constant returns, and $L_t = 1$:

$$D_t = f_K(K_t, A_t)K_t + (1-\delta)K_t - K_{t+1}. \tag{A-4}$$

The discount rate is still given by equation (12) in the paper and hence $R_{t+1} = f_K(K_{t+1}, A_{t+1}) + 1 - \delta$ in equilibrium.

We next show that the share price equals the value of the capital stock $p_t = K_{t+1}$. First, as usual, the share price equals the present-discounted value of dividends, i.e. it satisfies equation (7) in the paper with $T = \infty$.

LEMMA A-1: *The share price equals the value of the capital stock: $p_t = K_{t+1}$.*

PROOF: Because the share price satisfies equation (7) in the paper with $T = \infty$ it equivalently satisfies

$$p_t = R_{t+1}^{-1}(D_{t+1} + p_{t+1}).$$

Using $R_{t+1} = f_K(K_{t+1}, A_{t+1}) + 1 - \delta$ and (A-4)

$$p_t(f_K(K_{t+1}, A_{t+1}) + 1 - \delta) = (f_K(K_{t+1}, A_{t+1})K_{t+1} + (1-\delta)K_{t+1} - K_{t+2}) + p_{t+1}.$$

The p_t sequence satisfying this equation is $p_t = K_{t+1}$ as claimed. *Q.E.D.*

Remark on Lemma A-1. Note that it is still true that the price per unit of capital (rather than the price of the entire capital stock $p_t = K_{t+1}$) equals one.

Balanced Growth Path (BGP). Assume that productivity A_t grows at a constant rate

$$A_{t+1} = GA_t, \quad G > 1 \quad \Rightarrow \quad A_t = G^t A_0$$

and that households have isoelastic preferences

$$U(C) = \frac{C^{1-1/\sigma}}{1 - 1/\sigma}.$$

Then the economy has a BGP on which the capital stock, output, and consumption all grow at the same rate G. On this BGP, the asset return is constant

$$R_{t+1} = f_K(K_{t+1}, A_{t+1}) + 1 - \delta = f_K(K_0^*, A_0) + 1 - \delta = \overline{R}$$

where we have used that $f_K(K_t, A_t)$ is homogeneous of degree zero in (K_t, A_t). The initial location of the BGP K_0^* is pinned down by the discount rate

$$\overline{R} = \frac{1}{\beta} G^{1/\sigma}.$$

On the BGP, the asset price grows at a constant rate resulting in capital gains

$$\frac{p_{t+1}}{p_t} = \frac{K_{t+2}}{K_{t+1}} = G.$$

From (A-4), the dividend yield is given by

$$\frac{D_{t+1}}{p_t} = \frac{f_K(K_{t+1}, A_{t+1})K_{t+1} + (1-\delta)K_{t+1} - K_{t+2}}{K_{t+1}} = f_K(K_{t+1}, A_{t+1}) + 1 - \delta - G = \overline{R} - G, \tag{A-5}$$

so that

$$\frac{D_{t+1}}{p_t} + \frac{p_{t+1}}{p_t} = \overline{R}$$

as expected. Also note that from (A-5) we have

$$p_t = \frac{D_{t+1}}{\overline{R} - G}.$$

Therefore, all capital gains are driven entirely by growing cash flows and the price-dividend ratio is constant as in the Gordon growth model (**?**). Also note that all capital gains are, in fact, unrealized. This is because, in equilibrium, the representative household does not buy or sell any shares (which are in fixed supply).

1.2. *Proof of Lemma 1*

Since preferences are assumed to be homothetic, we can write them as $U(\{c(s^t)\}) = G(H(\{c(s^t)\}))$, where H is homogenous of degree ρ and G is a monotonic function. We assume that G and H are such that $G(H(\cdot))$ is differentiable, strictly increasing, and strictly concave. The first-order condition for person θ in history s^t from problem (13) in the paper is

$$\omega(\theta) G'(H) \frac{\partial H(\{c(s^t, \theta)\})}{\partial c(s^t)} = \lambda(s^t),$$

where $\lambda(s^t)$ is the multiplier on the resource constraint in history s^t. The solution to the first-order conditions and resource constraints is the unique optimal allocation.

Given the unique optimum, we guess and verify that $c(s^t, \theta) = \Omega(\theta) C(s^t)$ satisfies the first-order conditions and resource constraints for some $\Omega(\theta)$. Let $\bar{H} \equiv H(\{C(s^t)\})$, which is pinned down by the aggregate resource conditions in equation (13) in the paper. This implies that $H(\{c(s^t,\theta)\}) = H(\{\Omega(\theta)C(s^t)\}) = \Omega(\theta)^\rho \bar{H}$, where the last equality uses the homogeneity of H. The first-order condition for θ in state s^t becomes

$$\begin{aligned}\lambda(s^t) &= \omega(\theta)G'(H)\frac{\partial H(\{c(s^t,\theta)\})}{\partial c(s^t)} \\ &= \omega(\theta)G'(\Omega(\theta)^\rho \bar{H})\frac{\partial H(\Omega(\theta)\{C(s^t)\})}{\partial c(s^t)} \\ &= \omega(\theta)G'(\Omega(\theta)^\rho \bar{H})\Omega(\theta)^{\rho-1}\frac{\partial H(\{C(s^t)\})}{\partial c(s^t)} \\ &= \omega(\theta)G'(\Omega(\theta)^\rho \bar{H})\Omega(\theta)^{\rho-1}\frac{\partial \bar{H}}{\partial c(s^t)},\end{aligned}$$

where the second equality uses our conjectured consumption rule, the third uses the homogeneity of H, and the last uses the definition of $\bar{H}$. This condition implies that $\omega(\theta)G'(\Omega(\theta)^\rho \bar{H})\Omega(\theta)^{\rho-1}$ is constant across θ and s^t. That is, there is an $\bar{\Omega}$ such that

$$\omega(\theta)G'(\Omega(\theta)^\rho \bar{H})\Omega(\theta)^{\rho-1} = \bar{\Omega} \quad \forall\theta.$$

Since $G(H(\cdot))$ is strictly concave, the left-hand side is strictly decreasing in $\Omega(\theta)$ and we can invert it to solve for $\Omega(\theta)$ given $\bar{\Omega}$. That is, $\Omega(\theta) = f(\omega(\theta), \bar{\Omega})$, where f is strictly increasing in its first argument. We can solve for $\bar{\Omega}$ as the solution to

$$\int f(\omega(\theta), \bar{\Omega})dF(\theta) = 1,$$

which ensures that all resource conditions are satisfied.

B. APPENDIX FOR SECTION 3

B.1. *Proof of Proposition 1*

We begin with proving the first equation in Proposition 1. From the budget constraint (22) in the paper we have

$$\begin{aligned}\overline{T}_0(\theta) &= y_0(\theta) - \overline{c}_0(\theta) + \overline{p}\,\overline{x}(\theta) \\ T_0(\theta) &= y_0(\theta) - c_0(\theta) + px(\theta),\end{aligned}$$

where we denote by $\overline{c}_t(\theta)$, $t = 0,1$, consumption at the old prices and dividends. Subtracting the former from the latter, we obtain

$$\Delta T_0(\theta) = T_0(\theta) - \overline{T}_0(\theta) = \overline{x}(\theta)\Delta p + p(x(\theta) - \overline{x}(\theta)) - (c_0(\theta) - \overline{c}_0(\theta)). \tag{A-6}$$

By the second-period budget constraint (23) in the paper and the normalization that $T_1(\theta)$ is held fixed, we have

$$c_1(\theta) - \overline{c}_1(\theta) = \overline{k}_1(\theta)\Delta D - D(x(\theta) - \overline{x}(\theta)) \tag{A-7}$$

and thus

$$p(x(\theta) - \overline{x}(\theta)) = \frac{p}{D}\left[\overline{k}_1(\theta)\Delta D - (c_1(\theta) - \overline{c}_1(\theta))\right].$$

Substituting in (A-6), we obtain

$$\Delta T_0(\theta) = \overline{x}(\theta)\Delta p + \frac{p}{D}\overline{k}_1(\theta)\Delta D - \left[(c_0(\theta) - \overline{c}_0(\theta)) + \frac{p}{D}(c_1(\theta) - \overline{c}_1(\theta))\right].$$

Next, by Lemma 1 we have

$$\Delta T_0(\theta) = \overline{x}(\theta)\Delta p + \frac{p}{D}\overline{k}_1(\theta)\Delta D - \Omega(\theta)\left[C_0 - \overline{C}_0 + \frac{p}{D}(C_1 - \overline{C}_1)\right]. \tag{A-8}$$

To rewrite the expression in square brackets, we work with the aggregate resource constraints. Since $\int T_0(\theta)dF(\theta) = \int T_1(\theta)dF(\theta) = 0$ we have $C_0 = pX + Y_0$ and $C_1 = D(K_0 - X) + Y_1$. Therefore

$$C_0 - \overline{C}_0 = pX - \overline{p}\overline{X} = \overline{X}\Delta p + p\Delta X,$$
$$C_1 - \overline{C}_1 = D(K_0 - X) - \overline{D}(K_0 - \overline{X}) = K_0\Delta D - D\Delta X - \overline{X}\Delta D = \overline{K}_1\Delta D - D\Delta X$$

where $\Delta X = X - \overline{X}$. Combining yields

$$C_0 - \overline{C}_0 + \frac{p}{D}(C_1 - \overline{C}_1) = \overline{X}\Delta p + \frac{p}{D}\overline{K}_1\Delta D.$$

Substituting in (A-8) delivers the final result. The proof of the second equation in Proposition 1 follows the same steps, except that we write equation (A-6) equivalently as

$$\Delta T_0(\theta) = x(\theta)\Delta p + \overline{p}(x(\theta) - \overline{x}(\theta)) - (c_0(\theta) - \overline{c}_0(\theta)) \tag{A-9}$$

and equation (A-7) as

$$c_1(\theta) - \overline{c}_1(\theta) = k_1(\theta)\Delta D - \overline{D}(x(\theta) - \overline{x}(\theta))$$

and analogously for the aggregate resource constraints.

B.2. *Proof of Lemma 2*

As in the Lemma, denote the old price by $\overline{p}$ and the new price by $p = \overline{p} + \Delta p$. Similarly, denote the old dividend by $\overline{D}$ and the new dividend by $D = \overline{D} + \Delta D$. Denote the original consumption bundle by $(\overline{c}_0(\theta), \overline{c}_1(\theta))$. Slutsky compensation is defined as the change in the investor's total budget $y_0(\theta)$ that keeps the original consumption bundle $(\overline{c}_0(\theta), \overline{c}_1(\theta))$ affordable at the new asset price p and dividend D. In the remainder of the proof, we suppress the dependence of variables on θ for notational simplicity.

The lifetime budget line at the original price is the set of points (c_0, c_1) such that

$$c_0 + \frac{\overline{p}}{\overline{D}}c_1 = y_0 + \frac{\overline{p}}{\overline{D}}y_1 + \overline{p}k_0 \tag{A-10}$$

The Slutsky-compensated budget line at the new price is the set of points (c_0, c_1) such that

$$c_0 + \frac{p}{D}c_1 = y_0 + \frac{p}{D}y_1 + pk_0 + \Delta y_0, \tag{A-11}$$

where Δy_0 is the Slutsky compensation term. The aim is to solve for Δy_0 such that the two budget lines intersect at the point $(c_0, c_1) = (\overline{c}_0, \overline{c}_1)$, i.e. so that the original consumption bundle remains affordable at the new prices. To this end, evaluate (A-10) and (A-11) at $(\bar{c}_0, \bar{c}_1)$ and subtract the old budget constraint (A-10) from the new budget constraint (A-11)

$$\left(\frac{p}{D} - \frac{\overline{p}}{\overline{D}}\right)\bar{c}_1 = \left(\frac{p}{D} - \frac{\overline{p}}{\overline{D}}\right) y_1 + k_0 \Delta p + \Delta y_0$$

Rearranging, we have

$$\Delta y_0 = \left(\frac{p}{D} - \frac{\overline{p}}{\overline{D}}\right)(\bar{c}_1 - y_1) - k_0 \Delta p = \left(\frac{p}{D} - \frac{p}{\overline{D}} + \frac{\Delta p}{\overline{D}}\right)(\bar{c}_1 - y_1) - k_0 \Delta p$$

where the second equality used $\overline{p} = p - \Delta p$. Using the second-period budget constraint (15), which implies $\bar{c}_1 - y_1 = \overline{D}\overline{k}_1$, yields

$$\Delta y_0 = \left(\frac{p}{D} - \frac{p}{\overline{D}}\right)\overline{D}\overline{k}_1 + (\overline{k}_1 - k_0)\Delta p = p\left(\frac{\overline{D}}{D} - 1\right)\overline{k}_1 + (\overline{k}_1 - k_0)\Delta p = -\frac{p}{D}\overline{k}_1 \Delta D - \overline{x}\Delta p$$

where the last equality uses $\overline{x} = k_0 - \overline{k}_1$. Reintroducing the explicit dependence on θ yields the in the lemma.

B.3. *Endogenous payout policy and share repurchases*

The capital-structure neutral reformulation of our setup is easiest to explain in the multi-period model of Section 1. Consider a firm that produces an income stream (i.e. earnings minus investment) $\{\Pi_t\}_{t=0}^T$ from its fundamental (e.g., non-financial) operations. Investors have budget constraint (2) and we assume for simplicity that the only asset at their disposal is firm shares so that $k_t(\theta)$ denotes share holdings, p_t denotes the share price, and D_t denotes the business dividends per share. The firm's cash flows are distributed to shareholders through both dividends and share repurchases:

$$\Pi_t = \mathcal{K}_t D_t + (\mathcal{K}_t - \mathcal{K}_{t+1}) p_t \tag{A-12}$$

where and $\mathcal{K}_t = \int k_t(\theta) dF(\theta)$ denotes the total amount of outstanding shares. When $\mathcal{K}_{t+1} < \mathcal{K}_t$ the business is repurchasing its own shares. From this equation it is already apparent that share repurchases and dividend payments are equivalent means of distributing cash flows $\{\Pi_t\}_{t=0}^T$ to shareholders as a whole. When the business repurchases its shares (i.e., $\mathcal{K}_{t+1} < \mathcal{K}_t$) this results in an income stream $(k_t(\theta) - k_{t+1}(\theta))p_t$ for those individual selling their shares to the business.

Denoting by $s_t(\theta) \equiv k_t(\theta)/\mathcal{K}_t$ the individual's ownership share of the business and by $V_t \equiv \mathcal{K}_t p_t$ the market value of the business, we can combine the individual and business budget constraints, equation (2) in the paper and (A-12), to obtain:

$$c_t(\theta) + V_t(s_{t+1}(\theta) - s_t(\theta)) = y_t(\theta) + \Pi_t s_t(\theta) \tag{A-13}$$

This budget constraint has the same form as equation (2) in the paper, except that (i) the dividend per share D_t is replaced by the income stream from operations Π_t, (ii) the price per share p_t is replaced by the market value of the firm V_t, and (iii) the number of shares held by the individuals $k_t(\theta)$ is replaced by the ownership share in the business $s_t(\theta)$. An alternative viewpoint on this consolidated budget constraint is to consider the return to investing in the business. See **?** for more discussion on this capital-structure neutral reformulation.

B.4. *Alternative implementations: taxes on expenditure or total capital income*

This appendix contains the details for Section 3.4.

An expenditure tax

Denote by $\{\overline{c}_t(\theta)\}$, $t=0,1$, the optimal consumption allocation at the old prices and dividends $\overline{p}$ and $\overline{D}$ (i.e., the solution to the Pareto problem (21) in the paper), and by $\{c_t(\theta)\}$ at the new prices and dividends $p=\overline{p}+\Delta p$ and $D=\overline{D}+\Delta D$. Let $\{\overline{T}_t(\theta)\}$, $t=0,1$, be some lump-sum taxes that implement the optimum at the old prices and dividends. Finally, let $\widehat{c}_t(\theta)$, $t=0,1$, denote investor θ's *individually* optimal consumption allocation under the new prices and dividends but when taxes are held fixed at the old level. Formally, $\widehat{c}_t(\theta)$ solves

$$\max_{c_0(\theta),c_1(\theta),x(\theta)} U(c_0(\theta),c_1(\theta)) \quad \text{s.t.} \quad \text{equations (22) and (23) in the paper}$$

when taxes are given by $\{\overline{T}_t(\theta)\}$. Then we have the following result:

PROPOSITION A-1: *Suppose asset prices increase from $\overline{p}$ to $p=\overline{p}+\Delta p$ and dividends from $\overline{D}$ to $D=\overline{D}+\Delta D$. The optimal consumption allocation at the new prices and dividends $\{c_t(\theta)\}$, $t=0,1$, can be implemented with taxes given by*

$$T_t(\theta)=\overline{T}_t(\theta)+\Delta\widehat{c}_t(\theta)-\Omega(\theta)\Delta C_t, \quad t=0,1$$

where $\Delta\widehat{c}_t(\theta)\equiv\widehat{c}_t(\theta)-\overline{c}_t(\theta)$ and $\Delta C_t=\int c_t(\theta)dF(\theta)-\int\overline{c}_t(\theta)dF(\theta)$.

PROOF: An investor's present-value budget constraint at the new prices and dividends (p,D) and new taxes $T_t(\theta)$ is

$$c_0(\theta)+\frac{p}{D}c_1(\theta)+T_0(\theta)+\frac{p}{D}T_1(\theta)=y_0(\theta)+\frac{p}{D}y_1(\theta)+pk_0(\theta) \tag{A-14}$$

The present-value budget constraint at the new prices and dividends (p,D) but *old* taxes $\overline{T}_t(\theta)$ is

$$\widehat{c}_0(\theta)+\frac{p}{D}\widehat{c}_1(\theta)+\overline{T}_0(\theta)+\frac{p}{D}\overline{T}_1(\theta)=y_0(\theta)+\frac{p}{D}y_1(\theta)+pk_0(\theta) \tag{A-15}$$

Subtracting (A-15) from (A-14) yields

$$c_0(\theta)-\widehat{c}_0(\theta)+\frac{p}{D}(c_1(\theta)-\widehat{c}_1(\theta))+T_0(\theta)-\overline{T}_0(\theta)+\frac{p}{D}(T_1(\theta)-\overline{T}_1(\theta))=0$$

which we can rewrite as

$$\Delta T_0(\theta)+\frac{p}{D}\Delta T_1(\theta)=\widehat{c}_0(\theta)-c_0(\theta)+\frac{p}{D}(\widehat{c}_1(\theta)-c_1(\theta)) \tag{A-16}$$

Next, observe that, for $t=0,1$,

$$\widehat{c}_t(\theta)-c_t(\theta)=\widehat{c}_t(\theta)-\overline{c}_t(\theta)-(c_t(\theta)-\overline{c}_t(\theta))=\Delta\widehat{c}_t(\theta)-\Delta c_t(\theta)=\Delta\widehat{c}_t(\theta)-\Omega(\theta)\Delta C_t,$$

where the last step uses Lemma 1. Substituting back in equation (A-16) yields

$$\Delta T_0(\theta)+\frac{p}{D}\Delta T_1(\theta)=\Delta\widehat{c}_0(\theta)-\Omega(\theta)\Delta C_0+\frac{p}{D}(\Delta\widehat{c}_1(\theta)-\Omega(\theta)\Delta C_1).$$

One way of implementing this is to set, in each period $t = 0, 1$,

$$\Delta T_t(\theta) = \Delta \widehat{c}_t(\theta) - \Omega(\theta)\Delta C_t$$

as in Proposition A-1. *Q.E.D.*

Hence, the new optimum can be implemented with a combination of a lump-sum tax equal to $\Delta \widehat{c}_t(\theta)$, which is the amount by which individuals would have changed their consumption after the price and dividend change if taxes had stayed at their old level $\overline{T}_t(\theta)$, and a transfer equal to the difference between the old and new desired consumption $\Delta c_t(\theta) = \Omega(\theta)\Delta C_t$. Notably, if the parameter changes Δp and ΔD are "zero-sum," so that optimal aggregate consumption C_t does not change, then the tax equals $\Delta \widehat{c}_t(\theta)$ meaning that optimal redistributive taxation simply taxes away any increase in consumption from the asset-price and dividend changes (or compensates the corresponding reduction in consumption), i.e. this is a "pure" expenditure tax without an additional transfer. In line with Kaldor's logic, just like Proposition 1, this works for any combination of asset price and dividend changes, i.e. regardless of the source of capital gains. Indeed, we show below that

$$\Delta \widehat{c}_0(\theta) + \frac{p}{D}\Delta \widehat{c}_1(\theta) = \overline{x}(\theta)\Delta p + \frac{p}{D}\overline{k}_1(\theta)\Delta D, \tag{A-17}$$

so the present value of the consumption change (holding taxes fixed) is directly linked to the capital gains and change in dividend income when p and D change. For instance, the investors who would increase their consumption in response to a pure asset price increase (in the absence of a further tax change) are precisely those who sell the asset, and vice versa.

Proof of Equation (A-17). Start with the first-period budget constraint (22) holding fixed the old taxes when prices and dividends change

$$\overline{T}_0(\theta) = y_0(\theta) - \overline{c}_0(\theta) + \overline{p}\,\overline{x}(\theta)$$
$$\overline{T}_0(\theta) = y_0(\theta) - \widehat{c}_0(\theta) + p\widehat{x}(\theta)$$

where we denote by $\widehat{x}(\theta)$ the investor's optimal asset sales at the old taxes but new prices and dividends. Subtracting the former from the latter, we obtain

$$0 = \overline{x}(\theta)\Delta p + p(\widehat{x}(\theta) - \overline{x}(\theta)) - (\widehat{c}_0(\theta) - \overline{c}_0(\theta)). \tag{A-18}$$

By the second-period budget constraint (23) and holding the old taxes $\overline{T}_1(\theta)$ fixed, we have

$$\widehat{c}_1(\theta) - \overline{c}_1(\theta) = \overline{k}_1(\theta)\Delta D - D(\widehat{x}(\theta) - \overline{x}(\theta))$$

and thus

$$p(\widehat{x}(\theta) - \overline{x}(\theta)) = \frac{p}{D}\left[\overline{k}_1(\theta)\Delta D - (\widehat{c}_1(\theta) - \overline{c}_1(\theta))\right].$$

Substituting in (A-18), we obtain

$$0 = \overline{x}(\theta)\Delta p + \frac{p}{D}\overline{k}_1(\theta)\Delta D - \left[\Delta \widehat{c}_0(\theta) + \frac{p}{D}\Delta \widehat{c}_1(\theta)\right].$$

A tax on total capital income

In Section 1, we demonstrated that equation (11) is an equivalent way of writing the budget constraint (2) using an investor’s market value of wealth $a_t(\theta) \equiv p_{t-1}k_t(\theta)$. Thus, as the following proposition shows, an alternative way of writing the first-best tax response in Proposition 1 is in terms of these wealth holdings and the changes in the total returns R_0 and R_1.

PROPOSITION A-2: *Let the asset price change from $\overline{p}$ to $p = \overline{p} + \Delta p$ and dividends from $\overline{D}$ to $D = \overline{D} + \Delta D$ resulting in return changes $\Delta R_0 = R_0 - \overline{R}_0$ and $\Delta R_1 = R_1 - \overline{R}_1$. Then the optimal tax $T_0(\theta)$ (when $T_1(\theta) = 0$) is given by*

$$T_0(\theta) = \overline{T}_0(\theta) + a_0(\theta)\Delta R_0 + \frac{1}{R_1}\overline{a}_1(\theta)\Delta R_1 - \Omega(\theta)\left[A_0\Delta R_0 + \frac{1}{R_1}\overline{A}_1\Delta R_1\right]$$
$$= \overline{T}_0(\theta) + a_0(\theta)\Delta R_0 + \frac{1}{\overline{R}_1}a_1(\theta)\Delta R_1 - \Omega(\theta)\left[A_0\Delta R_0 + \frac{1}{\overline{R}_1}A_1\Delta R_1\right]$$

where $a_1(\theta)$ is investor θ’s period-1 wealth at the new returns, $\overline{a}_1(\theta)$ at the baseline returns, $a_0(\theta) = \overline{a}_0(\theta)$ since p_{-1} is fixed, and A_0, $A_1, \overline{A}_1$ the corresponding aggregate wealth holdings.

PROOF: In a similar manner to Proposition 1, we use the budget constraint in the first period to get

$$\overline{T}_0(\theta) = y_0(\theta) - \overline{c}_0(\theta) - \overline{a}_1(\theta) + \overline{R}_0 a_0(\theta)$$
$$T_0(\theta) = y_0(\theta) - c_0(\theta) - a_1(\theta) + R_0 a_0(\theta).$$

Subtracting the former from the latter, we obtain

$$\Delta T_0(\theta) = T_0(\theta) - \overline{T}_0(\theta) = (\overline{c}_0(\theta) - c_0(\theta)) + (\overline{a}_1(\theta) - a_1(\theta)) + a_0(\theta)\Delta R_0. \tag{A-19}$$

From the second-period budget constraint we have

$$c_1(\theta) - \overline{c}_1(\theta) = R_1 a_1(\theta) - \overline{R}_1\overline{a}_1(\theta). \tag{A-20}$$

Note that we can write it as

$$c_1(\theta) - \overline{c}_1(\theta) = \overline{a}_1(\theta)(R_1 - \overline{R}_1) + R_1(a_1(\theta) - \overline{a}_1(\theta)) \tag{A-21}$$

Thus we have

$$\overline{a}_1(\theta) - a_1(\theta) = \frac{1}{R_1}\left(\overline{c}_1(\theta) - c_1(\theta)\right) + \frac{1}{R_1}\overline{a}_1(\theta)\Delta R_1$$

Replacing in (A-19) we get

$$\Delta T_0(\theta) = a_0(\theta)\Delta R_0 + \frac{1}{R_1}\overline{a}_1(\theta)\Delta R_1 - \left[(c_0(\theta) - \overline{c}_0(\theta)) + \frac{1}{R_1}\left(c_1(\theta) - \overline{c}_1(\theta)\right)\right].$$

Again, by Lemma 1, we know $c_t(\theta) = \Omega(\theta)C_t$. So we have

$$\Delta T_0(\theta) = a_0(\theta)\Delta R_0 + \frac{1}{R_1}\overline{a}_1(\theta)\Delta R_1 - \Omega(\theta)\left[C_0 - \overline{C}_0 + \frac{1}{R_1}(C_1 - \overline{C}_1)\right]. \tag{A-22}$$

Similar to Proposition 1, working with the aggregate resource constraints we have

$$C_0 + A_1 = Y_0 + R_0 A_0,$$

$$C_1 = Y_1 + R_1 A_1.$$

Therefore

$$C_0 - \overline{C}_0 = -\Delta A_1 + A_0 \Delta R_0,$$

$$C_1 - \overline{C}_1 = R_1 A_1 - \overline{R}_1 \overline{A}_1 = R_1 \Delta A_1 + \overline{A}_1 \Delta R_1.$$

Replacing in equation (A-22) we get the final result

$$T_0(\theta) = \overline{T}_0(\theta) + a_0(\theta)\Delta R_0 + \frac{1}{R_1}\overline{a}_1(\theta)\Delta R_1 - \Omega(\theta)\left[A_0 \Delta R_0 + \frac{1}{R_1}\overline{A}_1 \Delta R_1\right]. \tag{A-23}$$

The proof of the second equation in Proposition A-2 follows the same steps. *Q.E.D.*

At first glance, the tax change in Proposition A-2 appears related to a Haig-Simons notion of income: in each period, the additional total capital income $a_t(\theta)\Delta R_t$, including unrealized gains, is taxed. However, there is an important difference. This is easiest to see by considering an increase in the asset price p holding dividends fixed (Special Case 1). In this case, we have

$$\Delta R_0 = \frac{\Delta p}{p_{-1}} > 0 \quad \text{and} \quad \Delta R_1 = \frac{D}{p} - \frac{D}{\overline{p}} < 0$$

since $p = \overline{p} + \Delta p > \overline{p}$. Hence, the investor faces a tax in period 0 (due to the higher return from the increased asset price) but a subsidy in period 1. The reason for the latter is that, whereas the asset price has increased, dividends have not, so the dividend-price ratio and thus the return in period 1 has been reduced, which needs to be compensated.

While the former tax increase (due to the unrealized capital gains in the initial period) indeed corresponds to a Haig-Simons income tax, the latter subsidy (due to the lower dividend-price ratio subsequently) does not. But Proposition A-2 shows that they belong together. Therefore, due to these opposing effects, the total change in taxes is generally ambiguous. In fact, we know from Proposition 1 that it depends solely on whether the investor is a buyer or seller. For instance, when $x(\theta) = 0$ so the individual is not trading, the additional tax in period 0 and the subsidy in period 1 precisely cancel out.

Proposition A-2 naturally extends to the multi-period case, as we show next:

PROPOSITION A-3: *Suppose asset prices change by $\{\Delta p_t\}_{t=0}^{T}$ and dividends by $\{\Delta D_t\}_{t=0}^{T}$ resulting in return changes $\{\Delta R_t\}_{t=0}^{T}$. Then optimal taxes $\{T_t(\theta)\}_{t=0}^{T}$ are such that*

$$\sum_{t=0}^{T} \overline{R}_{0\to t}^{-1} T_t(\theta) = \sum_{t=0}^{T} \overline{R}_{0\to t}^{-1} [\overline{T}_t(\theta) + a_t(\theta)\Delta R_t - \Omega(\theta) A_t \Delta R_t]$$

PROOF: The proof follows exactly analogous steps to the proof of Proposition 5. *Q.E.D.*

Thus, in the multi-period model, a one-off, permanent increase in the asset price would trigger a one-off tax followed by a subsidy forever, in a way that their present value sum is zero for an investor who is not trading. The alternative implementation in Propositions A-2 and A-3 can therefore lead to very volatile taxes over time compared to Proposition 1.

C. APPENDIX FOR SECTION 4

C.1. *Second-best problem for alternative tax instruments*

A tax on c_0. Consider first a tax on period-0 consumption. This means that pre-tax consumption in period 0, given by $z_0(\theta) \equiv px(\theta) + y_0(\theta)$, is observable, so after-tax consumption is $c_0(\theta) = z_0(\theta) - T_0(z_0(\theta))$, where $T_0(z_0)$ is the nonlinear consumption tax in $t = 0$. Hence,

$$x(\theta) = \frac{z_0(\theta) - y_0(\theta)}{p}$$

and we can write the global incentive constraints as

$$\begin{aligned}\mathcal{U}(\theta) &\equiv U\left(c_0(\theta), D\left(k_0(\theta) - \frac{z_0(\theta) - y_0(\theta)}{p}\right) + y_1(\theta)\right)\\ &\geq U\left(c_0(\hat{\theta}), D\left(k_0(\theta) - \frac{z_0(\hat{\theta}) - y_0(\theta)}{p}\right) + y_1(\theta)\right) \ \forall \theta, \hat{\theta}.\end{aligned}$$

The local incentive constraints are therefore given by equation (26) in the paper with $A(\theta) = 0$ and

$$B(\theta) = Dk_0'(\theta) + \frac{D}{p}y_0'(\theta) + y_1'(\theta).$$

A tax on c_1. Consider next a tax on period-1 consumption. Pre-tax consumption in period 1 is $z_1(\theta) \equiv D(k_0(\theta) - x(\theta)) + y_1(\theta)$ and after-tax consumption is $c_1(\theta) = z_1(\theta) - T_1(z_1(\theta))$, where $T_1(z_1)$ is the nonlinear consumption tax in $t = 1$. Hence,

$$x(\theta) = \frac{y_1(\theta) - z_1(\theta)}{D} + k_0(\theta)$$

and we can write the global incentive constraints as

$$\begin{aligned}\mathcal{U}(\theta) &\equiv U\left(\frac{p}{D}(y_1(\theta) - z_1(\theta)) + pk_0(\theta) + y_0(\theta), c_1(\theta)\right)\\ &\geq U\left(\frac{p}{D}(y_1(\theta) - z_1(\hat{\theta})) + pk_0(\theta) + y_0(\theta), c_1(\hat{\theta})\right) \ \forall \theta, \hat{\theta}.\end{aligned}$$

The local incentive constraints are therefore again given by equation (26) in the paper but with

$$A(\theta) = pk_0'(\theta) + y_0'(\theta) + \frac{p}{D}y_1'(\theta)$$

and $B(\theta) = 0$.

Further tax instruments. More generally, for any tax instrument conditioning on some observable choices, we can decompose consumption in each period $t = 0, 1$ into its observable and its unobservable components: $c_t(\theta) = c_t^o(\theta) + c_t^u(\theta)$. For instance, with an assets sales tax, the observable components are $c_0^o(\theta) = z_x(\theta)$ in period 0 and $c_1^o(\theta) = -Dx(\theta)$ in period 1, whereas the unobservable components are $c_0^u(\theta) = y_0(\theta)$ and $c_1^u(\theta) = Dk_0(\theta) + y_1(\theta)$. Hence, the general incentive constraint can be written as

$$\begin{aligned}\mathcal{U}(\theta) &\equiv U\left(c_0^o(\theta) + c_0^u(\theta), c_1^o(\theta) + c_1^u(\theta)\right)\\ &\geq U\left(c_0^o(\hat{\theta}) + c_0^u(\theta), c_1^o(\hat{\theta}) + c_1^u(\theta)\right) \ \forall \theta, \hat{\theta}.\end{aligned}$$

The local incentive constraints are therefore always given by equation (26) in the paper with $A(\theta) = c_0^{u\prime}(\theta)$ and $B(\theta) = c_1^{u\prime}(\theta)$. Note that this general approach also allows for combinations of the tax instruments considered so far. For example, suppose there is both an asset sales tax in period 0 and a wealth tax in period 1. Then $c_0^o(\theta) = z_x(\theta)$ and $c_1^o(\theta) = D(k_0(\theta) - x(\theta))$ while $c_0^u(\theta) = y_0(\theta)$ and $c_1^u(\theta) = y_1(\theta)$, so we obtain $A(\theta) = y_0'(\theta)$ and $B(\theta) = y_1'(\theta)$.[1]

C.2. *Solving the general second-best problem*

For any preferences $U(c_0, c_1) = G(\mathcal{C}(c_0, c_1))$ and any of the tax instruments considered, we can write the second-best Pareto problem as

$$\max_{\{c_0(\theta), c_1(\theta), V(\theta)\}} \int \omega(\theta) G(V(\theta)) dF(\theta)$$

subject to the incentive constraints

$$V'(\theta) = \mathcal{C}_{c_0}(c_0(\theta), c_1(\theta)) A(\theta) + \mathcal{C}_{c_1}(c_0(\theta), c_1(\theta)) B(\theta) \;\; \forall \theta \tag{A-24}$$

where $\mathcal{C}_{c_t} \equiv \partial C / \partial c_t$, the resource constraint

$$Y \geq \int \left(c_0(\theta) + \frac{p}{D} c_1(\theta) \right) dF(\theta) \tag{A-25}$$

with

$$Y \equiv pK_0 + Y_0 + \frac{p}{D} Y_1,$$

and

$$V(\theta) = \mathcal{C}(c_0(\theta), c_1(\theta)) \;\; \forall \theta.$$

It is useful to substitute out $c_0(\theta) = \Phi(V(\theta), c_1(\theta))$ where $\Phi(., c_1)$ is the inverse function of $\mathcal{C}(., c_1)$ with respect to its first argument. This allows us to write the maximization problem in terms of $V(\theta)$ and $c_1(\theta)$ only. Attaching multipliers $\mu(\theta)$ to the incentive constraint for type θ and η to the resource constraint, the corresponding Lagrangian becomes, after integrating by parts,

$$\begin{aligned} \mathcal{L} = & \int \omega(\theta) G(V(\theta)) dF(\theta) - \int \mu'(\theta) V(\theta) d\theta \\ & - \int \mu(\theta) \left[\mathcal{C}_0(\Phi(V(\theta), c_1(\theta)), c_1(\theta)) A(\theta) + \mathcal{C}_1(\Phi(V(\theta), c_1(\theta)), c_1(\theta)) B(\theta) \right] d\theta \\ & - \eta \int \left[\Phi(V(\theta), c_1(\theta)) + \frac{p}{D} c_1(\theta) \right] dF(\theta). \end{aligned}$$

Using the fact that

$$\frac{\partial \Phi}{\partial V} = \frac{1}{\mathcal{C}_0} \quad \text{and} \quad \frac{\partial \Phi}{\partial c_1} = -\frac{\mathcal{C}_1}{\mathcal{C}_0}$$

[1] In this case, the asset sales tax and the wealth tax are not separately determined, but the optimal consumption allocation is.

and dropping arguments to simplify notation, the first-order condition for $c_1(\theta)$ is

$$\mu\left[\left(\frac{\mathcal{C}_{c_0c_0}\mathcal{C}_{c_1}}{\mathcal{C}_{c_0}}-\mathcal{C}_{c_0c_1}\right)A+\left(\frac{\mathcal{C}_{c_0c_1}\mathcal{C}_{c_1}}{\mathcal{C}_{c_0}}-\mathcal{C}_{c_1c_1}\right)B\right]=\eta f\left[\frac{p}{D}-\frac{\mathcal{C}_{c_1}}{\mathcal{C}_{c_0}}\right] \tag{A-26}$$

and for $V(\theta)$

$$\omega f G'(V)=\mu'+\mu\left[\frac{\mathcal{C}_{c_0c_0}}{\mathcal{C}_{c_0}}A+\frac{\mathcal{C}_{c_0c_1}}{\mathcal{C}_{c_0}}B\right]+\frac{\eta f}{\mathcal{C}_{c_0}} \tag{A-27}$$

where $\mathcal{C}_{c_sc_t}$ denotes the second derivates $\partial^2\mathcal{C}/(\partial c_s\partial c_t)$. Together with the incentive constraints (A-24), the resource constraint (A-25) and the boundary conditions $\mu(\underline{\theta})=\mu(\overline{\theta})=0$, equations (A-26) and (A-27) determine the optimal solution $\{V(\theta),c_1(\theta),\mu(\theta),\eta\}$.

C.3. *CES utility and numerical algorithm*

Under the CES preferences given in equation (27) in the paper, it turns out to be convenient to work with

$$\xi(\theta)\equiv\frac{c_0(\theta)}{c_1(\theta)}.$$

Then the first-order conditions (A-26) and (A-27) can be written as

$$\frac{\mu(\theta)}{\sigma c_1(\theta)}\left(\xi(\theta)^{\frac{\sigma-1}{\sigma}}+\beta\right)^{\frac{1}{\sigma-1}}\left(B(\theta)-A(\theta)/\xi(\theta)\right)=\eta f(\theta)\left(\frac{p}{\beta D}-\xi(\theta)^{\frac{1}{\sigma}}\right) \tag{A-28}$$

$$\omega(\theta)f(\theta)G'(V(\theta))=\mu'(\theta)+\frac{\beta\mu(\theta)}{\sigma c_1(\theta)}\frac{B(\theta)-A(\theta)/\xi(\theta)}{\xi(\theta)^{\frac{\sigma-1}{\sigma}}+\beta}+\eta f(\theta)\left(1+\beta\xi(\theta)^{\frac{1-\sigma}{\sigma}}\right)^{\frac{1}{1-\sigma}}. \tag{A-29}$$

Moreover, the incentive constraints (A-24) become

$$V'(\theta)=\left(\xi(\theta)^{\frac{\sigma-1}{\sigma}}+\beta\right)^{\frac{1}{\sigma-1}}\left(\xi(\theta)^{-\frac{1}{\sigma}}A(\theta)+\beta B(\theta)\right) \tag{A-30}$$

and, by definition of CES utility,

$$V(\theta)=c_1(\theta)\left(\xi(\theta)^{\frac{\sigma-1}{\sigma}}+\beta\right)^{\frac{\sigma}{\sigma-1}}. \tag{A-31}$$

We first use (A-28) together with (A-31) to numerically solve for $\xi(\theta)$ as a function of $\mu(\theta)$, $V(\theta)$ and η. Substituting this in (A-29) and (A-30) delivers a system of two ordinary differential equations in $\mu(\theta)$ and $V(\theta)$ that we can solve, given any η, using the boundary conditions $\mu(\underline{\theta})=\mu(\overline{\theta})=0$. Finally, we find η such that the resource constraint (A-25) is satisfied, noting that $c_0(\theta)=\xi(\theta)c_1(\theta)$.

C.4. *Proof of Proposition 2*

We establish the result in a series of steps. We first characterize the first-best allocation $\Gamma^*(\sigma)$ under preferences (27) in the paper.

LEMMA A-2: *With preferences (27), we have*

$$\Omega(\theta) = \frac{\omega(\theta)^{\frac{1}{\gamma}}}{\int \omega(\theta')^{\frac{1}{\gamma}} dF(\theta')}.$$

PROOF: This is a special case of Lemma 1 with $G'(x) = x^{-\gamma}$ and $\rho = 1$. We then have

$$\omega(\theta)\Omega(\theta)^{-\gamma}\bar{H}^{-\gamma} = \bar{\Omega}.$$

Solving this for $\Omega(\theta)$ yields

$$\Omega(\theta) = f(\omega(\theta), \bar{\Omega}) = \left(\frac{\omega(\theta)}{\bar{\Omega}}\right)^{\frac{1}{\gamma}} \bar{H}^{-1}.$$

Integrating, we have

$$\bar{\Omega} = \left(\int \omega(\theta')^{\frac{1}{\gamma}} dF(\theta')\right)^{\gamma} \bar{H}^{-\gamma},$$

which delivers the desired expression. *Q.E.D.*

The next lemma characterizes the first-best allocation for any $\sigma \geq 0$:

LEMMA A-3: *The first-best allocation $\Gamma^*(\sigma)$ for $\sigma \geq 0$ is*

$$c_0^*(\theta,\sigma) = \Omega(\theta)\left(1 + R^{-1}(\beta R)^{\sigma}\right)^{-1}\left(Y_0 + R^{-1}Y_1 + pK_0\right)$$
$$c_1^*(\theta,\sigma) = (\beta R)^{\sigma} c_0^*(\theta,\sigma),$$

where $R \equiv D/p$ is the interest rate.

PROOF: For $\sigma > 0$, the first-order conditions for the first-best allocation are (dropping σ for notational simplicity):

$$\omega(\theta)U_{c_0}(\theta) = \omega(\theta)V(\theta)^{-\gamma}\left(\frac{c_0(\theta)}{V(\theta)}\right)^{-\frac{1}{\sigma}} = \eta$$

$$\omega(\theta)U_{c_1}(\theta) = \omega(\theta)\beta V(\theta)^{-\gamma}\left(\frac{c_1(\theta)}{V(\theta)}\right)^{-\frac{1}{\sigma}} = R^{-1}\eta,$$

where $\eta > 0$ is the multiplier on the resource constraint and $V(\theta) = \mathcal{C}(c_0(\theta), c_1(\theta))$. Taking the ratio and rearranging, we have:

$$c_1(\theta) = (\beta R)^{\sigma} c_0(\theta).$$

Substituting into $\mathcal{C}$, we obtain:

$$\begin{aligned} V(\theta) &= \left(c_0(\theta)^{\frac{\sigma-1}{\sigma}} + \beta\left[(\beta R)^{\sigma} c_0(\theta)\right]^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(1 + \beta\left(\beta R\right)^{\sigma-1}\right)^{\frac{\sigma}{\sigma-1}} c_0(\theta). \end{aligned}$$

The first-order condition for c_0 then becomes (after rearranging):

$$c_0(\theta) = \eta^{-\frac{1}{\gamma}} \omega(\theta)^{\frac{1}{\gamma}} \left(1 + \beta \left(\beta R\right)^{\sigma - 1}\right)^{\frac{1/\gamma - \sigma}{\sigma - 1}}.$$

We can eliminate $\eta^{-\frac{1}{\gamma}}$ using the resource condition (19), Lemma 1 and Lemma A-2 to obtain:

$$c_0(\theta) = \Omega(\theta) \left(1 + R^{-1} (\beta R)^{\sigma}\right)^{-1} \left(Y_0 + R^{-1} Y_1 + p K_0\right).$$

For $\sigma = 0$, we have $\mathcal{C}(c_0, c_1) = \min\{c_0, c_1\}$. The first-best will set $c_0 = c_1$, and hence solves the problem

$$\max_{c(\theta)} \int \omega(\theta) \frac{c(\theta)^{1-\gamma}}{1-\gamma} dF(\theta),$$

subject to

$$\int c(\theta) dF(\theta) = (1 + R^{-1})^{-1} \left(Y_0 + R^{-1} Y_1 + p K_0\right).$$

The first-order condition is $\omega(\theta) c(\theta)^{-\gamma} = \eta$, which, after substituting in the resource constraint, yields the proposed outcome with σ set to zero. *Q.E.D.*

Lemma A-3 implies that the first-best allocation $\Gamma^*(\sigma)$ is continuous in σ. Hence, in the neighborhood of $\sigma = 0$, the associated first-best allocations $\Gamma^*(\sigma)$ are also in the neighborhood of the $\sigma = 0$ first-best allocation $\Gamma^*(0)$. If our second-best allocation $\Gamma^M(\sigma)$ converges to $\Gamma^*(0)$ as $\sigma \to 0$, this means that it also converges to $\Gamma^*(\sigma)$.

The second-best allocation $\Gamma^M(\sigma)$ also needs to satisfy the incentive constraints (26) (in addition to the resource constraint (19)). With preferences (27), the incentive constraints simplify to (A-30). The next lemma shows that Assumption 1 (i) is needed for the first-best allocation $\Gamma^*(0)$ to be incentive compatible.

LEMMA A-4: *If $\Gamma^*(0)$ is incentive compatible for $\sigma = 0$, it satisfies Assumption 1 (i).*

PROOF: As $\sigma \to 0$, (A-30) can be written as

$$V'(\theta) = \frac{\xi(\theta)^{-\frac{1}{\sigma}} A(\theta) + \beta B(\theta)}{\xi(\theta)^{-\frac{1}{\sigma}} + \beta}.$$

Hence, $V'(\theta)$ must be a convex combination of $A(\theta)$ and $B(\theta)$. Moreover, when $\sigma = 0$, $V(\theta) = c_0(\theta) = c_1(\theta) \equiv c(\theta)$. *Q.E.D.*

For general $\sigma \geq 0$, consider allocations that take the following form:

$$c_0(\theta, \sigma) = e^{\sigma g(\theta)} c(\theta, \sigma) \tag{A-32}$$
$$c_1(\theta, \sigma) = c(\theta, \sigma),$$

where $c(\theta, \sigma)$ is the solution to the following linear ODE:

$$c'(\theta, \sigma) = q(\theta, \sigma) - \sigma p(\theta, \sigma) c(\theta, \sigma), \tag{A-33}$$

where

$$q(\theta,\sigma) \equiv \frac{A(\theta)+\beta e^{g(\theta)}B(\theta)}{e^{\sigma g(\theta)}+\beta e^{g(\theta)}}$$

$$p(\theta,\sigma) \equiv \frac{g'(\theta)e^{\sigma g(\theta)}}{e^{\sigma g(\theta)}+\beta e^{g(\theta)}}.$$

Note that Assumption 1 (ii) ensures p exists and is bounded. Also note that

$$q(\theta,\sigma) = \left(\frac{1+\beta e^{g(\theta)}}{e^{\sigma g(\theta)}+\beta e^{g(\theta)}}\right) c^{*\prime}(\theta). \tag{A-34}$$

We next establish that the proposed allocation is incentive compatible:

LEMMA A-5: *The proposed allocation* (A-32) *where* $c(\theta,\sigma)$ *solves* (A-33) *satisfies* (A-30).

PROOF: An allocation $\{c_0(\theta), c_1(\theta)\}$ satisfies (A-30) iff

$$c_0'(\theta) - A(\theta) + \beta\left(\frac{c_0(\theta)}{c_1(\theta)}\right)^{\frac{1}{\sigma}}(c_1'(\theta)-B(\theta)) = 0. \tag{A-35}$$

To see that the proposed allocation satisfies (A-35), note that

$$c_0'(\theta,\sigma) = e^{\sigma g(\theta)}\left[\sigma g'(\theta)c(\theta,\sigma)+c'(\theta,\sigma)\right]$$
$$c_1'(\theta,\sigma) = c'(\theta,\sigma).$$

Substituting into (A-35), we have

$$\begin{aligned}
&e^{\sigma g(\theta)}\left[\sigma g'(\theta)c(\theta,\sigma)+c'(\theta,\sigma)\right] - A(\theta) + \beta e^{g(\theta)}\left(c'(\theta,\sigma)-B(\theta)\right)\\
&= (e^{\sigma g(\theta)}+\beta e^{g(\theta)})c'(\theta,\sigma) - A(\theta) - \beta e^{g(\theta)}B(\theta) + \sigma g'(\theta)e^{\sigma g(\theta)}c(\theta,\sigma)\\
&= (e^{\sigma g(\theta)}+\beta e^{g(\theta)})\left[c'(\theta,\sigma) - q(\theta,\sigma) + \sigma p(\theta,\sigma)c(\theta,\sigma)\right].
\end{aligned}$$

Setting this equal to zero yields (A-33). *Q.E.D.*

The proposed allocation satisfies the resource constraint if

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(e^{\sigma g(\theta)}+R^{-1}\right)c(\theta,\sigma)dF(\theta) = Y_0 + R^{-1}Y_1 + pK_0. \tag{A-36}$$

Note that (A-33) is a linear first-order ODE. Hence, it can be solved in closed form up to a boundary condition $c(\underline{\theta},\sigma)$. In particular, let $P(\theta;\sigma) \equiv \int_{\underline{\theta}}^{\theta} p(\hat{\theta},\sigma)d\hat{\theta}$. From Assumption 1 (ii), P is bounded. Then,

$$c(\theta,\sigma) = e^{-\sigma P(\theta,\sigma)}\left(\int_{\underline{\theta}}^{\theta} e^{\sigma P(\hat{\theta},\sigma)}q(\hat{\theta},\sigma)d\hat{\theta} + c(\underline{\theta},\sigma)\right). \tag{A-37}$$

Substituing (A-37) into the above, the resource condition uniquely pins down $c(\underline{\theta},\sigma)$.

The final restriction on the proposed allocation is that it is weakly positive. We will verify this in the neighbourhood of $\sigma = 0$ below.

Thus a weakly positive solution to (A-33) that satisfies (A-36) is feasible and incentive compatible. We show that such a sequence of solutions converges uniformly to the first-best.

We start with:

LEMMA A-6: *Let c solve (A-33). Then $c'(\theta,\sigma)$ converges uniformly to $c^{*\prime}(\theta)$ as $\sigma \to 0$.*

PROOF: From (A-34), we have

$$c'(\theta,\sigma) - c^{*\prime}(\theta) = \left[\left(\frac{1+\beta e^{g(\theta)}}{e^{\sigma g(\theta)} + \beta e^{g(\theta)}}\right) - 1\right] c^{*\prime}(\theta).$$

Letting $\|\|$ denote the sup norm over θ, this implies

$$\|c'(\theta,\sigma) - c^{*\prime}(\theta)\| = \left\|\frac{1-e^{\sigma g(\theta)}}{e^{\sigma g(\theta)} + \beta e^{g(\theta)}} c^{*\prime}(\theta)\right\| \leq \left\|1 - e^{\sigma g(\theta)}\right\| \times \left\|\frac{c^{*\prime}(\theta)}{e^{\sigma g(\theta)} + \beta e^{g(\theta)}}\right\|.$$

As the final term on the far right is bounded, we need to show the first term in the far right expression converges to zero. To see this, note that

$$\left\|1 - e^{\sigma g(\theta)}\right\| \leq e^{\sigma\|g(\theta)\|}\left(1 - e^{-\sigma\|g(\theta)\|}\right) \to 0.$$

Q.E.D.

A corollary of this lemma is that

$$\begin{aligned}
\|c(\theta,\sigma) - c^*(\theta)\| &= \left\|\int_{\underline{\theta}}^{\theta} (c'(\hat\theta,\sigma) - c^{*\prime}(\hat\theta))d\hat\theta + c(\underline{\theta},\sigma) - c^*(\underline{\theta})\right\| \\
&\leq \left\|\int_{\underline{\theta}}^{\theta} (c'(\hat\theta,\sigma) - c^{*\prime}(\hat\theta))d\hat\theta\right\| + |c(\underline{\theta},\sigma) - c^*(\underline{\theta})| \\
&\leq \|c'(\theta,\sigma) - c^{*\prime}(\theta,\sigma)\|(\overline{\theta} - \underline{\theta}) + |c(\underline{c},\sigma) - c^*(\underline{\theta})| \to |c(\underline{\theta},\sigma) - c^*(\underline{\theta})|.
\end{aligned}$$

Hence to show uniform convergence of $c(\theta,\sigma)$ to $c^*(\theta)$ we need to show that $c(\underline{\theta},\sigma) \to c^*(\underline{\theta})$. This follows from the fact that the resource condition requires that

$$\int_{\underline{\theta}}^{\overline{\theta}} c(\theta,\sigma)dF = \int_{\underline{\theta}}^{\overline{\theta}} c^*(\theta)dF.$$

This is true for all σ. Using the Fundamental Theorem of Calculus, we can write $c(\theta,\sigma) = c(\underline{\theta},\sigma) + \int_{\underline{\theta}}^{\theta} c'(\theta',\sigma)d\theta'$ and similarly for $c^*(\theta)$. Thus, the resource constraint can be written as

$$c(\underline{\theta},\sigma) = c^*(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} (c^{*\prime}(\theta') - c(\theta',\sigma))d\theta' dF.$$

We showed above that the last term on the right converges to zero as $\sigma \to 0$. Hence, $c(\underline{\theta},\sigma) \to c^*(\underline{\theta})$. As $c(\theta,\sigma) \to c^*(\theta)$ uniformly, and $c^*(\theta) > 0$ for all θ, this also establishes that $c(\theta,\sigma) \geq 0$ in a neighborhood of $\sigma = 0$.

In sum, we have shown that there exists a sequence of feasible, incentive compatible allocations that converge uniformly to $\Gamma^*(0)$ as $\sigma \to 0$. Continuity of $\Gamma^*(\sigma)$ shown in Lemma A-3 implies that this sequence of allocations also converges uniformly to $\Gamma^*(\sigma)$ as $\sigma \to 0$. Finally, the second-best optimum $\Gamma^M(\sigma)$ must also be feasible and incentive compatible but achieve at least weakly higher welfare than the proposed allocation. Since $\Gamma^*(\sigma)$ uniquely achieves the highest welfare among all feasible allocations, we must also have $\Gamma^M(\sigma) \to \Gamma^*(\sigma)$ as $\sigma \to 0$, which is the result in Proposition 2.

C.5. *Proof of Proposition 3*

We first characterize the efficient portfolio choice. Without taxes, the investor solves

$$\max_{x,b} U(px - qb - \chi(x) + y_0, b + D(k_0 - x) + y_1)$$

with FOCs

$$U_0(p - \chi') = U_1 D$$

$$U_0 q = U_1.$$

Dividing yields

$$p - \chi'(x) = qD. \tag{A-38}$$

It is straightforward to see that the same efficiency condition would apply in the case of lump-sum taxes, a tax on net trades $z = px - qb - \chi(x)$ in period 0, or on net trades $b - Dx$ in period 1.

Turning to the Mirrlees problem with two assets, suppose both x and b are observable, so a tax $T(x,b)$ is feasible. This means that $px(\theta) - qb(\theta) - \chi(x(\theta))$ is observable. Like in Appendix C.1, we can write the allocation in terms of the observable and unobservable components of consumption, with

$$c_0^o(\theta) = px(\theta) - qb(\theta) - \chi(x(\theta)) - T(x(\theta), b(\theta))$$

$$c_1^o(\theta) = b(\theta) - Dx(\theta)$$

and $c_0^u(\theta) = y_0(\theta)$, $c_1^u(\theta) = Dk_0(\theta) + y_1(\theta)$. Thus, the incentive constraint is

$$V(\theta) \equiv U(c_0^o(\theta) + c_0^u(\theta), c_1^o(\theta) + c_1^u(\theta)) \geq U(c_0^o(\theta') + c_0^u(\theta), c_1^o(\theta') + c_1^u(\theta)) \;\; \forall \theta, \theta'$$

with the usual envelope condition

$$V'(\theta) = U_0(c_0(\theta), c_1(\theta)) c_0^{u\prime}(\theta) + U_1(c_0(\theta), c_1(\theta)) c_1^{u\prime}(\theta). \tag{A-39}$$

We can thus write the planning problem as

$$\max_{\{c_0(\theta), c_1(\theta), V(\theta)\}} \int \omega(\theta) V(\theta) dF(\theta)$$

subject to

$$V(\theta) = U(c_0(\theta), c_1(\theta)),$$

the local incentive constraint (A-39), and the resource constraint

$$\int c_0(\theta)dF(\theta) + q\int c_1(\theta)dF(\theta) = Y$$

with

$$Y \equiv \int y_0(\theta)dF(\theta) + q\int [Dk_0(\theta)+y_1(\theta)]dF(\theta) + \max_{\{x(\theta)\}}\int [px(\theta)-\chi(x(\theta))-qDx(\theta)]dF(\theta)$$

It is clear from this formulation that any second-best optimum also prescribes the efficient portfolio choice given by (A-38) for each investor.

Now suppose that what we can observe is only the net trade $z \equiv px - qb - \chi(x)$ in period 0, and thus we impose a tax $T(z)$ (a tax on the “net trade” in period 1, $b - Dx$, is equivalent). We know from above that this implements the efficient portfolio choice, which pins down $x(\theta)$. Since we know $x(\theta)$ and $z(\theta)$, we then also know $b(\theta)$. Thus, a tax $T(z)$ is equivalent to a tax $T(x, b)$.

D. APPENDIX FOR SECTION 5

D.1. *Two-period model with risk and borrowing*

To obtain additional intuition for some of the results in Section 5.1, this appendix repeats the derivations there but for the special case with two periods, $T = 1$. Denote by $m(s^1)$ the stochastic discount factor of the representative counterparty in global financial markets where $s^1 = (s_0, s_1)$. In particular, the period-0 Arrow-Debreu price of a unit of consumption delivered in state s_1 is $\pi(s_1)m(s^1)$. No arbitrage implies:

$$p(s_0) = \sum_{s_1} \pi(s_1)m(s^1)D(s^1), \qquad q(s_0) = \sum_{s_1} \pi(s_1)m(s^1). \tag{A-40}$$

The investors’ flow budget constraints (2) specialize to

$$c_0(\theta, s_0) = p(s_0)(k_0(\theta, s_{-1}) - k_1(\theta, s_0)) - q(s_0)b(\theta, s_0) + y_0(\theta) - T_0(\theta, s_0) \quad \forall s_0$$
$$c_1(\theta, s^1) = D(s^1)k_1(\theta, s_0) + b(\theta, s_0) + y_1(\theta) - T_1(\theta, s^1) \quad \forall s^1$$

where, just like in the deterministic model, we assume $D_0 = b_0 = p_1 = 0$. $p(s_0)$ denotes the period-0 price of risky capital, $q(s_0)$ the price of the bond, $b(\theta, s_0)$ the amount of the bond purchased in period 0 and $D(s^1)$ the dividends paid to capital in period 1.

We allow taxes and transfers in both periods $t = 0, 1$ to be indexed by s^t. In order to ensure that risk is relevant, we assume $\int T_0(\theta, s_0)dF(\theta) = 0$ for all s_0 and $\int T_1(\theta, s^1)dF(\theta) = 0$ for all s^1, so the economy cannot insure itself other than through trading capital and the bond with the rest of the world.

First Best. The first-best allocation is the solution to

$$\max_{\{c_0(\theta,s_0),\{c_1(\theta,s^1)\},X(s_0)\}} \mathbb{E}\int \omega(\theta)U(\{c_0(\theta, s_0), c_1(\theta, s^1)\})dF(\theta) \quad \text{s.t.}$$

$$\int c_0(\theta, s_0)dF(\theta) + q(s_0)\int c_1(\theta, s^1)dF(\theta)$$
$$= Y_0 + q(s_0)Y_1 + p(s_0)X(s_0) + q(s_0)D(s^1)(K_0 - X(s_0))$$

for all s_0, s^1 and where $X(s_0) \equiv \int (k_0(\theta) - k_1(\theta, s_0))dF(\theta)$.

As in the benchmark environment, the fact that individuals can trade assets generates an indeterminacy in the tax system that decentralizes the first-best allocation. With two assets, there are two dimensions of indeterminacy, spanned by the payoffs to the risk-free bond and risky capital. Specifically:

LEMMA A-7: *There exists a first-period tax schedule $T_0(\theta, s_0)$ that implements the first-best allocation when combined with any second-period tax schedule of the form*

$$T_1(\theta, s^1) = \alpha(\theta, s_0) + \gamma(\theta, s_0)D(s^1) \quad \forall \theta, s^1,$$

where, for any given s_0, $\alpha(\theta, s_0)$ and $\gamma(\theta, s_0)$ are arbitrary functions of θ that satisfy $\int \alpha(\theta, s_0))dF(\theta) = \int \gamma(\theta, s_0))dF(\theta) = 0$.

That is, the second-period tax schedule can be an arbitrary linear function of the payoffs to risky capital. This follows from the fact that individuals can always adjust their private holding of the two assets to account for differences in the tax system that are spanned by the payoffs to the bond and capital.

Shocks to asset prices and dividends. We now revisit how shocks to asset prices and cash flows induce changes in the optimal tax burden. In period 0, consider a baseline state $\overline{s}_0$ (with corresponding pricing kernel $m(\overline{s}_0, s_1)$ and dividends $D(\overline{s}_0, s_1)$) and compare it to another, new shock s_0 with stochastic discount factor $m(s_0, s_1)$ and dividends $D(s_0, s_1)$. For example, suppose attitudes toward risk change, or the time discounting inherent in m changes. By expression (A-40), this induces changes in the price of capital $p(s_0)$ and risk-free bonds $q(s_0)$. The next result is the special case of Proposition 4 with two time periods.

PROPOSITION A-4: *Suppose shock s_0 is realized so that the pricing kernel changes by $\Delta m(s^1) = m(s_0, s_1) - m(\overline{s}_0, s_1)$ and dividends change by $\Delta D(s^1) = D(s_0, s_1) - D(\overline{s}_0, s_1)$. Let*

$$\Delta p = \sum_{s_1} \pi(s_1) \left[m(s_0, s_1)D(s_0, s_1) - m(\overline{s}_0, s_1)D(\overline{s}_0, s_1)\right] \quad \text{and} \quad \Delta q = \sum_{s_1} \pi(s_1)\Delta m(s^1).$$

Then the following tax change $\Delta T_0(\theta, s_0) = T_0(\theta, s_0) - T_0(\theta, \overline{s}_0)$ and $\Delta T_1(\theta, s^1) = T_1(\theta, s_0, s_1) - T_1(\theta, \overline{s}_0, s_1)$ $\forall s_1$ is an optimal response:

$$\Delta T_0(\theta, s_0) = x(\theta, s_0)\Delta p - b(\theta, s_0)\Delta q - \Omega(\theta)\left[X(s_0)\Delta p - B(s_0)\Delta q\right]$$
$$\Delta T_1(\theta, s^1) = k_1(\theta, s_0)\Delta D(s^1) - \Omega(\theta)K_1(s_0)\Delta D(s^1).$$

Role of the government. We now use the two-period model to expand on the discussion of the role of the government in Section 5.1, specifically what time-zero taxation can induce investors to buy shares of the market portfolio. Lemma A-7 speaks to the fact that there are many tax schemes that implement the same optimal allocation. In fact, setting $\alpha(\theta, s_0) = \gamma(\theta, s_0) = 0$ for all θ implies that all taxation can take place in period 0. This reflects that the tax schemes are spanned by available assets. In particular, in the initial period, the government can set

$$T_0(\theta, s_0) = p(s_0)\left(k_0(\theta) - \Omega(\theta)K_0\right) + y_0(\theta) + q(s_0)y_1(\theta) - Y_0 - q(s_0)Y_1.$$

An investor of type θ then buys $k_1(\theta, s_0) = \Omega(\theta)K_1$ units of the risky asset and $\Omega(\theta)(Y_1 + B(s_0)) - y_1(\theta)$ of the risk-free bond, meaning that everyone holds shares of the market portfolio. As a result, all investors are equally affected by future shocks to prices and dividends and there is no scope for redistributive taxation going forward.

Proof of Lemma A-7. In the decentralized equilibrium, the individual's problem is

$$\max U(\{c_0(\theta,s_0),c_1(\theta,s^1)\}) \quad \text{s.t.}$$
$$c_0(\theta,s_0)=y_0(\theta)+p(s_0)x(\theta,s_0)-q(s_0)b(\theta,s_0)-T_0(\theta,s_0) \quad \forall s_0$$
$$c_1(\theta,s^1)=b(s_0)+D(s^1)(k_0(\theta)-x(\theta,s_0))-T_1(\theta,s^1) \quad \forall s^1.$$

Eliminating b, we can write the budget set as present-value constraints (suppressing θ):

$$\begin{aligned}&c_0(s_0)+q(s_0)c_1(s^1)\\ =&y_0+q(s_0)y_1+p(s_0)x(s_0)+q(s_0)D(s^1)(k_0-x(s_0))-T_0(s_0)-q(s_0)T_1(s^1)\,\forall s^1.\end{aligned}$$

The first-order conditions for this problem take the same form as the planning problem, so the first-best allocation satisfies the individual's problem as long as it satisfies the budget set. For this, we need to find a tax scheme $\{T_0(\theta,s_0),T_1(\theta,s^1)\}$ and asset positions $\{b(s_0),x(s_0)\}$ such that for all θ and s^1:

$$\Omega(\theta)C_0^*(s_0)=y_0(\theta)+p(s_0)x(\theta,s_0)-q(s_0)b(\theta,s_0)-T_0(\theta,s_0)$$
$$\Omega(\theta)C_1^*(s^1)=y_1(\theta)+D(s^1)(k_0(\theta)-x(\theta,s_0))+b(\theta,s_0)-T_1(\theta,s^1)$$

where we used Lemma 1. Using $T_1(\theta,s^1)=\alpha(\theta,s_0)+\gamma(\theta,s_0)D(s^1)$ and the aggregate resource constraint in history s^1, we have:

$$\begin{aligned}&\Omega(\theta)\left(D(s^1)K_1^*(s_0)+Y_1+B^*(s_0)\right)\\ =&y_1(\theta)+D(s^1)(k_0(\theta)-x(\theta,s_0))+b(\theta,s_0)-\alpha(\theta,s_0)-\gamma(\theta,s_0)D(s^1).\end{aligned}$$

Set

$$x(\theta,s_0)=-\gamma(\theta,s_0)-\Omega(\theta)K_1^*(s_0)+k_0(\theta)$$
$$b(\theta,s_0)=\Omega(\theta)(Y_1+B^*(s_0))-y_1(\theta)+\alpha(\theta,s_0),$$

and the second-period budget constraint is satisfied for all s^1. Setting

$$T_0(\theta,s_0)=-\Omega(\theta)C_0^*(s_0)+y_0(\theta)+p(s_0)x(\theta,s_0)-q(s_0)b(\theta,s_0),$$

the first-period budget constraint is satisfied, as well. Moreover, we have $\int T_0(\theta,s_0)dF(\theta)=0$ for all s_0. Hence, the tax system along with the proposed policies $\{b,x\}$ ensures that the household's necessary and sufficient conditions are satisfied evaluated at the first-best allocation.

D.2. *Proof of Proposition 5*

In the decentralized equilibrium, the individual's problem is $\max U(\{c_t(\theta,s^t)\})$ s.t.

$$\begin{aligned}&c_t(\theta,s^t)+p_t(s^t)(k_{t+1}(\theta,s^t)-k_t(\theta,s^{t-1}))+q_t(s^t)b_{t+1}(\theta,s^t)\\ &\qquad=y_t(\theta)+D_t(s^t)k_t(\theta,s^{t-1})+b_t(\theta,s^{t-1})-T_t(\theta,s^t) \quad \forall t,s^t.\end{aligned}$$

The first-order conditions for this problem take the same form as those for the planning problem, so the first-best allocation solves the individual's problem as long as it satisfies the budget constraints. To simplify notation, we write the prices and dividends associated with the reference state $\overline{s}^t$ as $\overline{p}_t$, $\overline{q}_t$ and $\overline{D}_t$ and analogously for the respective allocations. Similary, let p_t, q_t and D_t denote the prices and dividends under the new shock, suppressing s^t, and again we use the same convention for the allocations. From the aggregate resource condition (3) in the paper:

$$\Delta C_t = X_t \Delta p_t + \overline{p}_t \Delta X_t + \Delta B_t - B_{t+1}\Delta q_t - \overline{q}_t \Delta B_{t+1} + K_t \Delta D_t + \overline{D}_t \Delta K_t. \qquad \text{(A-41)}$$

By Lemma 1, the change in the first-best allocation is $\Delta c_t(\theta) = \Omega(\theta)\Delta C_t$. We now show that the first-best allocation is affordable for each θ under the taxes given by Proposition 4. Let individual θ set asset positions such that

$$\Delta k_t(\theta) = \Omega(\theta)\Delta K_t \text{ and } \Delta b_t(\theta) = \Omega(\theta)\Delta B_t,$$

which implies $\Delta x_t(\theta) = \Omega(\theta)\Delta X_t$. If the first-best allocation is attainable with these portfolio choices and the proposed lump-sum taxes, they are consistent with individual optimization.

We have

$$\begin{aligned}\Delta c_t(\theta) =& x_t(\theta)\Delta p_t + \overline{p}_t \Delta x_t(\theta) + \Delta b_t(\theta) - b_{t+1}(\theta)\Delta q_t - \overline{q}_t \Delta b_{t+1}(\theta) \\ & + k_t(\theta)\Delta D_t + \overline{D}_t \Delta k_t(\theta) - \Delta T_t(\theta) \\ =& \overline{p}_t \Delta x_t(\theta) + \Delta b_t(\theta) - \overline{q}_t \Delta b_{t+1}(\theta) + \overline{D}_t \Delta k_t(\theta) + \Omega(\theta)\left[X_t \Delta p_t + K_t \Delta D_t - B_{t+1}\Delta q_t\right] \\ =& \Omega(\theta)\left[\Delta C_t = X_t \Delta p_t + \overline{p}_t \Delta X_t + \Delta B_t - B_{t+1}\Delta q_t - \overline{q}_t \Delta B_{t+1} + K_t \Delta D_t + \overline{D}_t \Delta K_t\right] \\ =& \Omega(\theta)\Delta C_t,\end{aligned}$$

where the second equation used the proposed tax policy from Proposition 5, the third the proposed individual asset positions, and the last equation (A-41). Hence, the proposed taxes allow each θ to afford the change in the first-best allocation. Moreover, by construction we have $\int T_t(\theta, s^t) dF(\theta) = 0$ for all s^t. Hence, the tax system along with the proposed portfolios ensures that the household's necessary and sufficient conditions are satisfied evaluated at the first-best allocation.

D.3. *Proof of Proposition 6*

An investor's sequential budget constraint under history s^t is

$$\begin{aligned}T_t(\theta, s^t) =& y_t(\theta) + \left(D_t(s^t) + p_t(s^t)\right) k_t(\theta, s^{t-1}) - p_t(s^t) k_{t+1}(\theta, s^t) \\ & + b_t(\theta, s^{t-1}) - q_t(s^t) b_{t+1}(\theta, s^t) - c_t(\theta, s^t)\end{aligned}$$

and under any other history $\tilde{s}^t$

$$\begin{aligned}T_t(\theta, \tilde{s}^t) =& y_t(\theta) + \left(D_t(\tilde{s}^t) + p_t(\tilde{s}^t)\right) k_t(\theta, \tilde{s}^{t-1}) - p_t(\tilde{s}^t) k_{t+1}(\theta, \tilde{s}^t) \\ & + b_t(\theta, \tilde{s}^{t-1}) - q_t(\tilde{s}^t) b_{t+1}(\theta, \tilde{s}^t) - c_t(\theta, \tilde{s}^t).\end{aligned}$$

Subtracting one from the other yields:

$$\begin{aligned}\Delta T_t(\theta, s^t, \tilde{s}^t) = & k_t(\theta, s^{t-1})\Delta D_t(s^t, \tilde{s}^t) + x_t(\theta, s^t)\Delta p_t(s^t, \tilde{s}^t) - b_{t+1}(\theta, s^t)\Delta q_t(s^t, \tilde{s}^t) \\ & + \left(D_t(\tilde{s}^t) + p_t(\tilde{s}^t)\right)\Delta k_t(\theta, s^{t-1}, \tilde{s}^{t-1}) + \Delta b_t(\theta, s^{t-1}, \tilde{s}^{t-1}) \qquad \text{(A-42)} \\ & - p_t(\tilde{s}^t)\Delta k_{t+1}(\theta, s^t, \tilde{s}^t) - q_t(\tilde{s}^t)\Delta b_{t+1}(\theta, s^t, \tilde{s}^t) - \Delta c_t(\theta, s^t, \tilde{s}^t) - \Delta c_t(\theta, s^t, \tilde{s}^t).\end{aligned}$$

Multiplying the second line by $\pi(\tilde{s}^t)m_{0\to t}(\tilde{s}^t)$ and summing over t and $\tilde{s}^t$ yields

$$\begin{aligned}
&\sum_{t=0}^{T}\sum_{\tilde{s}^t}\pi(\tilde{s}^t)m_{0\to t}(\tilde{s}^t)\left[\left(D_t(\tilde{s}^t)+p_t(\tilde{s}^t)\right)\Delta k_t(\theta,s^{t-1},\tilde{s}^{t-1})+\Delta b_t(\theta,s^{t-1},\tilde{s}^{t-1})\right]\\
&=\sum_{t=1}^{T}\sum_{\tilde{s}^t}\pi(\tilde{s}^t)m_{0\to t}(\tilde{s}^t)\left[\left(D_t(\tilde{s}^t)+p_t(\tilde{s}^t)\right)\Delta k_t(\theta,s^{t-1},\tilde{s}^{t-1})+\Delta b_t(\theta,s^{t-1},\tilde{s}^{t-1})\right]\\
&=\sum_{t=1}^{T}\sum_{\tilde{s}^{t-1}}\sum_{\tilde{s}_t}\pi(\tilde{s}^{t-1},\tilde{s}_t)m_{0\to t}(\tilde{s}^{t-1},\tilde{s}_t)\left[\left(D_t(\tilde{s}^{t-1},\tilde{s}_t)+p_t(\tilde{s}^{t-1},\tilde{s}_t)\right)\Delta k_t(\theta,s^{t-1},\tilde{s}^{t-1})\right.\\
&\qquad\left.+\Delta b_t(\theta,s^{t-1},\tilde{s}^{t-1})\right]\\
&=\sum_{t=1}^{T}\sum_{\tilde{s}^{t-1}}\pi(\tilde{s}^{t-1})m_{0\to t-1}(\tilde{s}^{t-1})\sum_{\tilde{s}_t}\pi(\tilde{s}_t|\tilde{s}^{t-1})m_t(\tilde{s}_t|\tilde{s}^{t-1})\left[\left(D_t(\tilde{s}^{t-1},\tilde{s}_t)+p_t(\tilde{s}^{t-1},\tilde{s}_t)\right)\right.\\
&\qquad\left.\times\Delta k_t(\theta,s^{t-1},\tilde{s}^{t-1})+\Delta b_t(\theta,s^{t-1},\tilde{s}^{t-1})\right]\\
&=\sum_{t=1}^{T}\sum_{\tilde{s}^{t-1}}\pi(\tilde{s}^{t-1})m_{0\to t-1}(\tilde{s}^{t-1})\left[p_{t-1}(\tilde{s}^{t-1})\Delta k_t(\theta,s^{t-1},\tilde{s}^{t-1})+q_{t-1}(\tilde{s}^{t-1})\Delta b_t(\theta,s^{t-1},\tilde{s}^{t-1})\right]
\end{aligned}$$

where the first equation uses $\Delta k_0(\theta)=\Delta b_0(\theta)=0$ and the last one uses the pricing conditions (5). Similarly, multiplying the third line in (A-42) by $\pi(\tilde{s}^t)m_{0\to t}(\tilde{s}^t)$ and summing over t and $\tilde{s}^t$ yields

$$\begin{aligned}
&-\sum_{t=0}^{T}\sum_{\tilde{s}^t}\pi(\tilde{s}^t)m_{0\to t}(\tilde{s}^t)\left[p_t(\tilde{s}^t)\Delta k_{t+1}(\theta,s^t,\tilde{s}^t)+q_t(\tilde{s}^t)\Delta b_{t+1}(\theta,s^t,\tilde{s}^t)\right]\\
&=-\sum_{t=1}^{T}\sum_{\tilde{s}^{t-1}}\pi(\tilde{s}^{t-1})m_{0\to t-1}(\tilde{s}^{t-1})\left[p_{t-1}(\tilde{s}^{t-1})\Delta k_t(\theta,s^{t-1},\tilde{s}^{t-1})+q_{t-1}(\tilde{s}^{t-1})\Delta b_t(\theta,s^{t-1},\tilde{s}^{t-1})\right]
\end{aligned}$$

where we adjusted the initial date of summation and used $k_{T+1}(\theta,s^T)=b_{T+1}(\theta,s^T)=0$ for all s^T.[2] Hence, the second and third line cancel and (A-42) becomes

$$\begin{aligned}
&\sum_{t=0}^{T}\sum_{\tilde{s}^t}\pi(\tilde{s}^t)m_{0\to t}(\tilde{s}^t)\Delta T_t(\theta,s^t,\tilde{s}^t) \qquad\qquad \text{(A-43)}\\
&=\sum_{t=0}^{T}\sum_{\tilde{s}^t}\pi(\tilde{s}^t)m_{0\to t}(\tilde{s}^t)\left[k_t(\theta,s^{t-1})\Delta D_t(s^t,\tilde{s}^t)+x_t(\theta,s^t)\Delta p_t(s^t,\tilde{s}^t)-b_{t+1}(\theta,s^t)\Delta q_t(s^t,\tilde{s}^t)\right.\\
&\qquad\left.-\Omega(\theta)\Delta C_t(s^t,\tilde{s}^t)\right]
\end{aligned}$$

where we used Lemma 1. Subtracting the sequential resource constraints under histories s^t and $\tilde{s}^t$ yields

$$\begin{aligned}
\Delta C_t(s^t,\tilde{s}^t)=&\,X_t(s^t)\Delta p_t(s^t,\tilde{s}^t)+K_t(s^{t-1})\Delta D_t(s^t,\tilde{s}^t)-B_{t+1}(s^t)\Delta q_t(s^t,\tilde{s}^t)\\
&+(D_t(\tilde{s}^t)+p_t(\tilde{s}^t))\Delta K_t(s^{t-1},\tilde{s}^{t-1})-p_t(\tilde{s}^t)\Delta K_{t+1}(s^t,\tilde{s}^t)\\
&+\Delta B_t(s^{t-1},\tilde{s}^{t-1})-q_t(\tilde{s}^t)\Delta B_{t+1}(s^t,\tilde{s}^t).
\end{aligned}$$

[2]When $T=\infty$ we impose the analogous no-Ponzi and transversality conditions.

Following the same steps as before yields

$$\sum_{t=0}^{T}\sum_{\tilde{s}^t}\pi(\tilde{s}^t)m_{0\to t}(\tilde{s}^t)\Delta C_t(\theta,s^t,\tilde{s}^t)$$
$$=\sum_{t=0}^{T}\sum_{\tilde{s}^t}\pi(\tilde{s}^t)m_{0\to t}(\tilde{s}^t)\left[X_t(s^t)\Delta p_t(s^t,\tilde{s}^t)+K_t(s^{t-1})\Delta D_t(s^t,\tilde{s}^t)-B_{t+1}(s^t)\Delta q_t(s^t,\tilde{s}^t)\right].$$

Substituting back in (A-43), we obtain Proposition 6.

D.4. *Proof of Corollary 4*

Using equation (1) in the paper, returns remains unchanged when $R_{t+1}=(D_{t+1}+p_{t+1})/p_t=(\overline{D}_{t+1}+\overline{p}_{t+1})/\overline{p}_t=\overline{R}_{t+1}$. Using that $D_t=\overline{D}_t+\Delta D_t$ and $p_t=\overline{p}_t+\Delta p_t$, this happens when:

$$\frac{\Delta D_{t+1}+\Delta p_{t+1}}{\Delta p_t}=\frac{\overline{D}_{t+1}+\overline{p}_{t+1}}{\overline{p}_t}\quad\text{for all } t. \tag{A-44}$$

Under condition (A-44), we have

$$\begin{aligned}
\sum_{t=0}^{T}\overline{R}_{0\to t}^{-1}\Delta T_t(\theta)&=\sum_{t=0}^{T}\overline{R}_{0\to t}^{-1}[k_t(\theta)(\Delta p_t+\Delta D_t)-k_{t+1}(\theta)\Delta p_t-\Omega(\theta)(K_t(\Delta p_t+\Delta D_t)-K_{t+1}\Delta p_t)]\\
&=\sum_{t=0}^{T}\overline{R}_{0\to t}^{-1}[k_t(\theta)-\Omega(\theta)K_t](\Delta p_t+\Delta D_t)-\sum_{t=0}^{T}\overline{R}_{0\to t}^{-1}[k_{t+1}(\theta)-\Omega(\theta)K_{t+1}]\Delta p_t\\
&=\sum_{t=0}^{T}\overline{R}_{0\to t}^{-1}\left[k_t(\theta)-\Omega(\theta)K_t\right](\Delta p_t+\Delta D_t)\\
&\quad-\sum_{t=0}^{T}\overline{R}_{0\to t}^{-1}\left[k_{t+1}(\theta)-\Omega(\theta)K_{t+1}\right]\frac{\overline{p}_t}{\overline{D}_{t+1}+\overline{p}_{t+1}}(\Delta p_{t+1}+\Delta D_{t+1})\quad\text{by (A-44)}\\
&=\sum_{t=0}^{T}\overline{R}_{0\to t}^{-1}\left[k_t(\theta)-\Omega(\theta)K_t\right](\Delta p_t+\Delta D_t)\\
&\quad-\sum_{t=0}^{T}\overline{R}_{0\to t+1}^{-1}\left[k_{t+1}(\theta)-\Omega(\theta)K_{t+1}\right](\Delta p_{t+1}+\Delta D_{t+1})\\
&=\sum_{t=0}^{T}\overline{R}_{0\to t}^{-1}\left[k_t(\theta)-\Omega(\theta)K_t\right](\Delta p_t+\Delta D_t)-\sum_{t=1}^{T+1}\overline{R}_{0\to t}^{-1}\left[k_t(\theta)-\Omega(\theta)K_t\right](\Delta p_t+\Delta D_t)\\
&=\left[k_0(\theta)-\Omega(\theta)K_0\right](\Delta p_0+\Delta D_0)-\overline{R}_{0\to T+1}^{-1}\left[k_{T+1}(\theta)-\Omega(\theta)K_{T+1}\right](\Delta p_{T+1}+\Delta D_{T+1})\\
&=\left[k_0(\theta)-\Omega(\theta)K_0\right]\Delta p_0
\end{aligned}$$

since the last term vanishes and $\Delta D_0=0$.

D. APPENDIX FOR SECTION 6

D.1. *Proof of Equation (30)*

The investor's Euler equation under preferences (27) is:

$$c_0(\theta)^{-1/\sigma}=\beta R c_1(\theta)^{-1/\sigma}$$

with $R = D/p$. Since it holds for all investors, it aggregates to

$$C_1 = \left(\beta \frac{D}{p}\right)^{\sigma} C_0.$$

Moreover, integrating the budget constraints (14) and (15) in the paper across investors and using the market clearing condition (29), we obtain $C_0 = Y_0$ and $C_1 = Y_1 + DK$ in the closed economy. Substituting back in the aggregate Euler equation, the equilibrium asset price p^* must satisfy

$$Y_1 + DK = \left(\beta \frac{D}{p^*}\right)^{\sigma} Y_0,$$

which can be rearranged to deliver equation (30).

D.2. *Proof of Proposition 7*

First observe that, by Lemma 1, the optimal consumption allocation still satisfies $c_t(\theta) = \Omega(\theta)C_t$, $t = 0, 1$. Since $C_0 = Y_0$ and $C_1 = Y_1 + DK$ in the closed economy and we hold both dividends and the aggregate endowment fixed, this immediately implies that no investor's consumption is changing in response to the asset price change Δp^*, so $c_t(\theta) = \overline{c}_t(\theta)$ for all θ, $t = 0, 1$. By the second-period budget constraint (23) and the normalization $T_1(\theta) = 0$, this implies in turn that $x(\theta) = \overline{x}(\theta)$ for all θ. The result then follows from Proposition 1 and the fact that $\Delta D = 0$ and $X = K_0 - K_1 = 0$.

D.3. *Proof of Proposition 8*

Subtract an investor's budget constraints under the old and new prices in period 0:

$$\begin{aligned}
&c_0(\theta) - \overline{c}_0(\theta) + q(b(\theta) - \overline{b}(\theta)) \\
=&p x(\theta) - \overline{p}\,\overline{x}(\theta) - (\chi(x(\theta)) - \chi(\overline{x}(\theta))) - (T_0(\theta) - \overline{T}_0(\theta)) \\
=&(p - \overline{p})x(\theta) + \overline{p}(x(\theta) - \overline{x}(\theta)) - (\chi(x(\theta)) - \chi(\overline{x}(\theta))) - (T_0(\theta) - \overline{T}_0(\theta))
\end{aligned}$$

and in period 1:

$$c_1(\theta) - \overline{c}_1(\theta) = D(\theta)(x(\theta) - \overline{x}(\theta)) + b(\theta) - \overline{b}(\theta)$$

We eliminate $b(\theta) - \overline{b}(\theta)$ by substituting the latter into the former:

$$\begin{aligned}
&c_0(\theta) - \overline{c}_0(\theta) + q(c_1(\theta) - \overline{c}_1(\theta)) + qD(\theta)(x(\theta) - \overline{x}(\theta)) \\
&= (p - \overline{p})x(\theta) + \overline{p}(x(\theta) - \overline{x}(\theta)) - (\chi(x(\theta)) - \chi(\overline{x}(\theta))) - (T_0(\theta) - \overline{T}_0(\theta))
\end{aligned}$$

Rearranging and using equation (31) in the paper yields

$$\begin{aligned}
&c_0(\theta) - \overline{c}_0(\theta) + q(c_1(\theta) - \overline{c}_1(\theta)) - \chi'(\overline{x}(\theta))(x(\theta) - \overline{x}(\theta)) \\
&= (p - \overline{p})x(\theta) - (\chi(x(\theta)) - \chi(\overline{x}(\theta))) - (T_0(\theta) - \overline{T}_0(\theta))
\end{aligned}$$

The second-order Taylor approximation for $\chi(x)$ around the point $\overline{x}(\theta)$ is:

$$\chi(x(\theta)) - \chi(\overline{x}(\theta)) \approx \chi'(\overline{x}(\theta))(x(\theta) - \overline{x}(\theta)) + \frac{1}{2}\chi''(\overline{x}(\theta))(x(\theta) - \overline{x}(\theta))^2$$

Substituting this, we obtain

$$c_0(\theta) - \overline{c}_0(\theta) + q(c_1(\theta) - \overline{c}_1(\theta)) = x(\theta)\Delta p - \frac{1}{2}\chi''(\overline{x}(\theta))(\Delta x(\theta))^2 - (T_0(\theta) - \overline{T}_0(\theta)) \qquad \text{(A-45)}$$

where $\Delta x(\theta) \equiv x(\theta) - \overline{x}(\theta)$.

Since the aggregate resource constraint (33) takes the same form as (19), the Pareto problem (21) subject to (33) still implies $c_t(\theta) = \Omega(\theta)C_t$, $t = 0, 1$ by Lemma 1. Hence, (A-45) can be written as

$$\Delta T_0(\theta) = x(\theta)\Delta p - \frac{1}{2}\chi''(\overline{x}(\theta))(\Delta x(\theta))^2 - \Omega(\theta)\left[C_0 - \overline{C}_0 + q(C_1 - \overline{C}_1)\right]$$

Integrating (A-45) across all investors implies

$$C_0 - \overline{C}_0 + q(C_1 - \overline{C}_1) = X\Delta p - \frac{1}{2}\int \chi''(\overline{x}(\theta))\Delta x(\theta)^2 dF(\theta)$$

and substituting this back delivers Proposition 8.

F. WEALTH TAXES AS TAXES ON PRESUMPTIVE INCOME

This appendix shows why taxing fluctuating wealth market values based on an analogy to a tax on "presumptive income" is problematic. The following simple numerical example illustrates that actual and presumptive income diverge whenever asset valuations are not exclusively driven by cash flows.

Consider an investor with an asset (e.g. a private business) initially worth \$100m which generates a dividend income of \$5m and which is subject to a 2% wealth tax of \$2m. In the notation of equation (1) in the paper, D_t and p_t are initially fixed at $\overline{D}$ and $\overline{p}$ with an asset return $\overline{R} - 1 = \overline{D}/\overline{p} = 5\%$. The asset value then jumps up permanently by a factor two to $p = 2\overline{p} = \$200m$ so that also the investor's wealth tax liability doubles to \$4m. The key question is what happens to the investor's presumptive versus actual income.

Suppose first that the increased asset value is exclusively due to higher cashflows, i.e. dividend income also doubles to $D = \$10m$ (Special Case 2). From equation (1), the asset return remains constant at $D/p = \$10m/\$200m = 5\%$ and therefore the increase in presumptive income exactly matches the increase in actual income. However, in all other cases in which dividends increase by less than a factor of two, this is no longer true: actual income increases by less than presumptive income. The problem is that it is incorrect to apply the same constant 5% presumed return *to the new valuation* of $p =$\$200m because the true return to wealth D/p *falls*. In the extreme case in which dividend income remains fixed (Special Case 1), presumptive income doubles to $5\% \times \$200m = \$10m$ while actual income is unchanged at \$5m. The unchanged dividend income corresponds to a lower return to wealth of only $\overline{R} - 1 = \overline{D}/p = 2.5\%$ so the correct income calculation would have been $2.5\% \times \$200m = \$5m$ rather than the (incorrect) presumptive income calculation of $5\% \times \$200m = \$10m$. Thus "presumptive income" is overestimated and wealth taxes redistribute suboptimally away from Special Case 2.